# "Mathematics: the music of the reason, Music: mathematics of the sense"[1].

**Emilio Lluis-Puebla**

This article is dedicated to my friend
Harald Fripertinger in his
fiftieth birthday.

Mathematics is a Fine Art,
the purest of them,
who has the gift of being
the most precise
and the precision of Science.
E. Lluis-Puebla.


**Abstract**

In this conference (which aims to leave something to the attendant at any level) we will discuss about mathematics, its characteristics, research and progress in it. As an example of a mathematical theory, a brief discussion of Algebraic K-Theory is given, how was it created and its border issues will be presented. Also, we will speak of mathematics called applied and discuss how the theories of modules, categories, topos, homotopy, homology and others are used in Mathematical Music Theory, as well as its philosophical framework and mathematical objects.


## Introduction

It is an honor and a great pleasure for me to be with you. I appreciate very much the kind invitation of Prof. Dr. Jens Schwaiger to be here in this Colloquium in order to celebrate the 50$^{th}$ birthday of my admired friend Prof. Dr. Harald Fripertinger.

Many people, colleagues, students, and especially my own ones, have asked me to talk about mathematics and mathematicians and particularly my task in this discipline. I reciprocate the request hoping that this information will help you understand both mathematics and mathematicians and in particular to understand some of the areas of mathematics to which I have dedicated my life. So what I will tell you today is the story of my mathematical life.

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[1] This is the text of the invited lecture in honor of Prof. Dr. Harald Fripertinger on the occasion of his Fiftieth Anniversary Colloquium. Graz, Austria. September 26, 2015.

At this conference, due to the heterogeneity of the audience I have tried to leave something for everyone, that is, I will discuss concepts of various levels. My apologies to those who want to see (read) a long series of theorems and also my apologies to those who prefer not to see (read) a long series of theorems. I have tried, except in some specific cases, to use common words (which, by the way, is very difficult to do for mathematicians) in order to describe some of the general ideas to offer a panoramic aspect about what I have to tell you.

Mathematics has enormous applicability and is an essential language and framework for all science. This is the reason why not only a few individuals devote their lives to it, but is studied in the educational system and part of the social scene.

Perhaps, mathematical research is the farthest activity from the man in the street who do not have absolutely any idea about this scientific discipline. Mathematics is usually identified by him with the ideas that could hardly absorb (often unsuccessfully) in elementary school. Mathematics or what he thinks it is, looks cold and raw, without life (he even speaks of the coldness of the numbers). Hardly has he imagined that mathematics was created in the past and is still being created at present by some humans. It is very difficult to understand the fact that it is an abstract intellectual discipline that possess an independent and prosperous existence. One way to establish contact with it is through conferences like this (articles like this one) or through knowing and listening mathematicians working in the various disciplines of science that use mathematics. At this lecture (this article) I will discuss mainly issues related to mathematics and its creators.

**Characteristics of Mathematics**

Mathematics has several characteristics that make it different from other disciplines.

The first is that it is very difficult to describe or define its subject matter. Maybe it is easier to grasp what the subject matter of some areas of study of Astronomy or Biology, but certainly it is not from of Algebraic K-Theory. This is mainly because the objects of study are defined in the abstract concepts which are often chained to other previously defined ones. The description is reduced to formal definitions that require neural connections, which require time to take place. This, coupled with a mathematical maturity or mathematical training allows the human being to assimilate a lot of abstract ideas. For example, if you try to explain to your "inquisitive little niece" what the addition is, or what Analytic Geometry is, or what a ring is, you will require, after many intuitive explanations and formal definitions, time; lots of time.

The second characteristic is that it has a perfect logic. Euclid Mathematics is as valid today as in the time of Euclid. This contrasts with other theories, such as the flat earth or phlogiston or ether theories.

The third is the conclusiveness of mathematics, that is, different disciplines make conclusions based on mathematical manipulations.

The fourth is its independence, that is, it does not require expensive equipment, unlike the experimental sciences. Sometimes just a pencil and paper, or not even this. Archimedes drew in the sand. Leray wrote his mathematics as a prisoner of war. Despite the political regimes of all kinds, Mathematics continues to evolve. It is interesting to note that their libraries are not as big as those of other disciplines.

**What does the word mathematics mean?**

My dear friend Arrigo Coen, a philologist that already passed away told me that, mathema means scholarship, manthánein the infinitive of learning, the radical mendh means in passive, science, knowledge. Then, is the relative to learning. So in implicit sense, mathematics means: "what is worthy to be learned."

**What is Mathematics?**

There is no definition of what Mathematics is. However, it is said to be a collection of ideas and techniques to solve problems that come from all disciplines including mathematics itself.

**Some mathematical problems**

Remember the last famous Fermat theorem (which follow the Pythagorean equation $x^2+y^2=z^2$) which states that the equation $x^n+y^n=z^n$ never has positive integer solutions for any positive integer n greater than 2. Except for n = 2 these equations do not have a geometric interpretation. Apparently this problem does not seem to matter much, however, it has had an enormous influence on the development of mathematics. Fermat said he had a proof, but had no space to write it. For over 300 years, this apparently simple problem has been the reason of great efforts of many mathematicians and is precisely from these efforts, that new techniques and concepts were created, which have influence in many areas of mathematics.

The problem of the four colors says that only 4 colors are required to light or color any map of the globe with the condition that two adjacent countries should have different colors. The positive solution, over a hundred years later, was obtained by using the computer, having a very small impact on mathematics. It was the first non-trivial problem solved by the computer.

In mathematics, if a problem is solved by standard methods, the problem loses much of its interest. If it is not solved by methods known for a long time, it becomes a classic problem. A good problem is one that leads to new techniques with wide applicability to other areas.

The new ideas which are the steps for the solution of a problem constitute the progress in mathematics. Mathematicians know to appreciate the ingenious techniques.

**What is mathematical research?**

Actually, I do not like the word research, meaning to do a search again. I prefer to use the verb to create in the case of Mathematics.

There are contrasting and alternative aspects in mathematical research. Here we see some.

There are differences between what pure mathematicians and applied mathematicians do although there is also a relationship between the two.

For pure mathematics, it is the alternative on the mathematical theory and problem solving. There are a variety of problems. If some of them can be solved by clever arguments in a similar way, then we say that we have a method to solve them and if they are many, then we say that we have a "mathematical theory." So it evolves, from a collection of problems, to a "theory" which differs from the concept that it has in other scientific disciplines.

Mathematics is a human activity and therefore must be able to pass on to subsequent generations, so it is systematically organized in order that those who do study it can do it as painlessly as possible. This is the basic form of what a Mathematical Theory is.

Poincaré said that a house is made of bricks but that the bricks themselves are far from being a house.

**How innovation occurs in mathematics?**

Unlike other scientific disciplines, in mathematics, the creation of new methods or techniques, is innovation, which is vital to the progress of mathematics.

It does not require the discovery of ancient manuscripts documents or experimental work or the introduction of new technology. Innovation is given, among other things, by the creation of new techniques. For example, when Galois worked on the problem of the insolubility of the general polynomial equation of degree at least 5 he realized that the key was in the symmetries of five solutions of the equation, it provided the foundation of the general theory of symmetry, which is one of the deepest branches on the whole spectrum of mathematics called Group Theory.

There is also innovation when trying to bring cohesion to a mathematical theory, doing appropriate questions, which require a lot of intuition and insight. Innovation can also come from problems of other disciplines.

One can say that there is mathematical progress when there are continuous applications of usual methods, dramatically interspersed with new concepts and problems.

**An example: Algebraic K-theory.**

What I will mention in this example, will not necessarily be understood by non-experts in the field, but the idea is to present a brief overview of Algebraic K-theory, how was its creation and border issues.

Although since the early twentieth century was well known that a commutative monoid (a set equipped with an associative law of composition with neutral element) without zero divisors could be considered within the commutative group it generates, is until 1957 (when Grothendieck thought of it) that K-Theory began. This is just the way that negative integers are defined from the additive monoid of natural numbers and that positive rational numbers are defined by the multiplicative monoid of naturals without zero.

The idea of Grothendieck was to associate to a commutative monoid M, a commutative group K(M), unique up to isomorphism, and a canonical homomorphism of monoids $\varphi: M \to K(M)$ defined such that, for any commutative group G, any defined monoid homomorphism $f: M \to G$ is uniquely factored as $f: M \to K(M) \to G$.

Grothendieck´s group was published for the first time in 1958 in a paper by Borel and Serre. Apart from its use in the Riemann-Roch theorem, one of the most well known applications of the Grothendieck construction was realized in 1959 by Atiyah and Hirzebruch. They applied the construction to the additive monoid of the isomorphism classes of complex vector bundles with base space a CW- complex X.

They used the notation $K^0(X)$ for Grothendieck´s group. They defined $K^{-n}(X)$ using the suspension of X for $n \geq 1$. Bott´s periodicity shows that $K^n(X) \approx K^{n+2}(X)$ and was used to define $K^n(X)$ for $n \in Z$. These functors constitute a cohomology theory known as Topological K-theory which had important applications, among others, as the Atiyah-Singer theorem and the solution of the problem of getting the maximum number of linearly independent vector fields on a sphere.

One result of Serre, generalized by Swan in 1962 provided a way to translate topological concepts into algebraic concepts.

[In fact, if X is a compact Hausdorff space and C (X) is the ring of complex functions in X then there is an equivalence between the category of vector bundles B over X and category of finitely generated projective modules over C (X) given by $B \rightarrow \Gamma(B)$ where $\Gamma(B)$ denotes the sections of the bundle seen as module over C (X).]

In short, the category of vector bundles on X is equivalent to the category of projective finitely generated $\Lambda$-modules where $\Lambda$ = C (X) (the ring of complex functions in X). From here one has a definition of $K(\Lambda)$ or $K_0(\Lambda)$ that makes sense for any ring $\Lambda$, as the Grothendieck group of the category of $\Lambda$-finitely generated projective modules. So $K_0(\Lambda)$ is, among other things, a useful tool to investigate the structure of projective $\Lambda$-modules.

In 1955 Serre proposed a conjecture which in algebraic terms said "are all finitely generated projective modules over the ring $k[t_1,t_2, \cdots t_n]$ free?

There are beautiful landscapes that led mathematicians to prove this conjecture, finally, twenty years later independently by Quillen and Suslin. Attempts to solve the conjecture of Serre in the sixties gave birth to another mathematical area: Algebraic K-theory. One of the goals of Algebraic K-Theory was to provide techniques and ideas to tackle the problem of Serre. Although the final solution of the conjecture in the affirmative way did not depend on the Algebraic K-Theory, it does not diminish the great influence it had on the enormous development that has being reached more significantly than expected.

In 1964, Hymann Bass defined the functor $K_1$ using the dictionary that relates algebraic and topological concepts. It turned out that these groups were the same as those introduced by Whitehead. So we had that $K_1(\Lambda)= GL(\Lambda)/[GL(\Lambda),GL(\Lambda)]$ where $GL(\Lambda)$ denotes the infinite linear group. By results from the Homology of Groups, $K_1(\Lambda)$ equals the integral homology of $GL(\Lambda)$, that is $H_1(GL(\Lambda),Z)$. In 1950, Whitehead considered the group $K_1(Z[G])$, where Z [G] denotes the integral group ring of fundamental groups G of cell complexes, which had topological applications. A result related to this is the s-cobordism theorem.

There were several important results in the sixties taking the point of view of algebraic K-theory, among others, the finite generation of $K_1(Z[G])$ and the stability results for the solution of the congruent subgroup problem in 1967 by Bass-Milnor-Serre for $SL_n(\Lambda)$ with $\Lambda$ the ring of integers in a number field. The latter set the stage for an arithmetic issue in algebraic K-theory which has become one of its most important aspects.

During the late sixties, one of the major problems was to define functors $K_n(\Lambda)$ for $n \in Z$. This problem was suggested by analogy with Topological K-Theory.

In 1969 Milnor defined $K_2(\Lambda)$: Consider the infinite elementary group $E(\Lambda)$ generated by the elementary matrices $e_{ij}^{\Lambda}$ among which are valid certain relations. Milnor defined the Steinberg group $St(\Lambda)$ as a group with generators $x_{ij}^{\Lambda}$ subject to these relations, and defines $K_2(\Lambda)$ as the kernel of the epimorphism $St(\Lambda) \rightarrow E(\Lambda)$.

In 1970, Kervaire proved that $St(\Lambda)$ is the universal central extension of $E(\Lambda)$, and therefore $K_2(\Lambda)$ can be described as the Schur multiplier of the perfect group $E(\Lambda)$. In other words $K_2(\Lambda)=H_2(E(\Lambda),Z)$. $K_2(\Lambda)$ is extremely difficult to calculate.

In 1969 Matsumoto calculated $K_2$ of a field describing it by means of generators and relations that were used by Bass and Tate to describe $K_2$ of numeric fields.

Algebraic K-Theory is a multidisciplinary phenomenon in mathematics [and to define the higher algebraic K-groups we need the following construction due to Quillen:

THEOREM. Let X be a connected CW complex with base point p. Let N be a perfect normal subgroup $\pi_1(X,p)$. Then there exists a space $X^+$ and a transformation $f: X \rightarrow X^+$ such that

(I) $\pi_1(f)$ induces an isomorphism $\pi_1(X^+,p) \cong \pi_1(X,p)/N$;

(ii) for any $\pi_1(X^+,p)$-module A, f induces an isomorphism $H_*(X; f^{-1}A) \cong H_*(X^+,A)$;

(Iii) $(X^+,f)$ is determined, except for homotopic equivalence by (i) and (ii).

This theorem is known as Quillen´s plus construction, and was inspired by the need to find a topological interpretation of Milnor´s functor $K_2$. The idea of the proof is to attach 2-cells to destroy N and 3-cells to neutralize the effect of the 2-cells in homology.

Quillen defined in the seventies, for $i \geq 1$, the i-th algebraic K-group of $\Lambda$, as $K_i\Lambda = \pi_i(BGL\Lambda^+)$. As in the cases i = 1,2, $K_i$ is a covariant functor from the category of rings to category of groups.

One of the marks of significant progress in mathematics is the discovery of unexpected relationships between different areas. Perhaps one of the most notable examples of such development is the development of Quillen´s Algebraic K-Theory in which Algebra and Topology are related in a new and fundamental way.

On one side, Algebraic K-Theory introduced topological methods to define algebraic invariants, such as the higher K-groups of rings. On the other hand, it provides a way of translating algebraic concepts into topological concepts. Algebraic K-theory studies the properties of the groups $K_i(\Lambda)$, constructed from a ring $\Lambda$.

One of the most important problems in Algebraic K-Theory is the calculation of the $K_i$-groups for diverse rings $\Lambda$, but despite the efforts of outstanding mathematicians as Bass, Milnor, Karoubi, Quillen, Weibel, Loday, Soule, Snaith only a very small number of them is known.

Here are some:

Bass showed in 1968 that $K_1Z \cong Z/2$ and that $K_1(Z/p^2)=Z\oplus Z/p\text{-}1$, p a prime different than 2.

Milnor showed in 1971 that $K_2Z \cong Z/2$ and $K_2(Z/p^2)= 0$ for a prime p different than 2.

In 1972 Quillen calculated the algebraic K-theory of a finite field.

Lee and Sczarba found in 1976 that $K_3Z \cong Z/48$.

Let R be the ring of integers of a numeric field and I a nontrivial ideal. Calculating $K_i(R/I)$ for finite rings is an open problem $i \geq 3$. Partial results have been obtained (i) by Evens-Friedlander [EF], and (ii) by Aisbett, Lluis-Puebla, Snaith and Soule. Interestingly, since 1980, there has not been virtually any complete calculations for this problem, making it a very interesting one, as I mentioned earlier.

In my Memoir of the American Mathematical Society along with some of my other papers, $K_3$ of dual numbers and of integers modulo n appear calculated. I also have a panoramic book on Higher Algebraic K-Theory you might read if interested.

I will not talk about Milnor´s or Block-Kato´s conjectures but only say that several Fields Medals have being bestowed to distinguish famous mathematicians that worked in Algebraic K-Theory.

**Mathematics, a Fine Art.**

As I have said on many occasions, mathematics is a fine art and a science. For mathematicians, beauty and truth have equal esteem. We have much appreciation for a beautiful argument, that is, an argument that carries elegance of style, economy of effort, clarity of thought, perfection in the detail and in how to achieve a forceful and convincing deduction. Mathematicians are dedicated to one area or another depending on how beautiful we think it is relative to another. We look for elegant methods and avoid ugly arguments.

**Aesthetic characteristics of Mathematics.**

There are several aesthetic characteristics of mathematics. Universality, meaning that almost any branch of knowledge has aspects that can be analyzed mathematically. The development of simple and concise arguments are absolutely essential for the progress of mathematics. The selection and formulation of problems is an art that relies on mathematical intuition. Here, aesthetic aspects play an important role.

Poincaré writes in the early twentieth century, that a mathematical proof is not a simple juxtaposition of syllogisms, but syllogisms placed in a certain order and that the order in which they are placed is much more important than syllogisms alone. He says he is not afraid that any of them will be forgotten, because each of them will take place in the settlement without effort.

He also describes the creation process: first, conscious work is done about the problem, then one leaves the ideas to mature in the subconscious, then the solution appears, perhaps when you least expect it, and finally this one is written.

As a joke, it seems that Poincaré created Mathematics at getting on or off a tram. Hadamard recommended to take two hot baths to stimulate mathematical research. Many mathematicians drink coffee, transforming it into theorems. I have also heard that mathematics is created walking, that is when the ideas are in the "unconscious" and suddenly the "bright idea" jumps up, which is perhaps a series of neural connections that take place over time which they are best achieved when a conscious act is not strongly involved that prevents them to arise.

**Formalism and mathematical rigor.**

There is a distinction between formalism and mathematical rigor:

In formal mathematics, one creates mathematics without wondering too much about the meaning as long as this gives the correct result. One goes ahead without worrying too much about mathematical rigor hoping that in the future it is provided.

Mathematics, I must insist in order not to lose sight, is essentially a human activity and our goal is not only to invent it but to transmit it. Thus, the mathematical rigor must exist. If you do not have a solid foundation, any construction based on them could fall.

There are mathematicians who specialize as deeply as possible in an area or field and others that have a great mathematical culture, as wide as possible. The two types of mathematicians are necessary.

However, we recommend young people to begin obtaining a mathematical culture, the widest possible and then dive into a topic. This is because the essence of mathematics is to bring together seemingly disparate fields. After all, mathematics is the highest level of abstraction which has application in every discipline that claims to be called a science.

**How do we do mathematics?**

There are mathematicians working individually and others do it in a small or large groups. It is very difficult to work alone, often one does not see a triviality that stops us for a long time and which is immediately resolved by a colleague in the group. But sometimes the reverse happens, not always three heads are better than one. Sometimes, interaction with colleagues enriches both mathematics and mathematicians and precisely these interrelationships are found between seemingly disparate areas that I spoke before.

Sometimes the collaboration between mathematicians is enriching and makes mathematical creation, which is very arduous and difficult, a more human and social experience, but sometimes this cannot be, because there are no colleagues in the same field near or because of personality features of every mathematician. Working in a group does not absolve the individual hard work of meditation or thinking about mathematics.

**What kind of mathematicians are there in general?**

There are two types of mathematicians, the ones that use the brute force and elegant. That is, those who use methods or crushing techniques that lead to the resolution of the problem and those that with few arguments properly placed, get surprisingly brilliant solutions. Sometimes a single mathematician can act in the two ways on different occasions.

However, transmission of mathematics is much appreciated, by the way our brain works, when it is elegant and simple, that is, artistically. Thus, the mathematical elegance is very important. This is achieved, in general, (not in the primary source of research but) after having gone through many brilliant mathematical minds.

**How is mathematics transmitted today?**

Regarding the transmission of new mathematics, it is done after a period of time since it was created, due to reasons explained above, at levels of younger people. Very difficult mathematics has being compacted and elegantly presented facilitating its learning. This is the most effective way to transmit mathematics.

Thus, there are various types of mathematical creation and mathematicians, all indispensable.

There is an enormous amount of mathematics created during a few centuries. However, in the early 20th century only a few great mathematicians could say that covered a good part of the whole of it. Today it is almost impossible for a mathematician to cover even his own area of study. Does this mean that the great building of mathematics crush on us? It happens that as specialization is inevitable, the development of new abstract concepts hopefully absorb others created in the past. These new creations are as important as the solutions to the difficult problems or development of new techniques.

**How a problem is selected in order to do research?**

Doctoral students in mathematics, almost cannot choose the problem. Mainly the advisor will assign it. Generally it is the advisor who sees the proper technique, and puts the student to work in it.

Some mathematicians that already do research by themselves conduct their own research on a topic which occurs sometimes by itself. This arises from the communication with colleagues, curiosity, from meditation or from getting around the mathematical literature properly. The important thing is that he has the conviction of understanding mathematics.

**What is known as Applied Mathematics?**

The activity in which mathematics has applications outside their own field is called Applied Mathematics. Applied Mathematics is automatically multidisciplinary and ideally and probably should be done by someone whose primary interest is not mathematics.

However we found it is less difficult for a mathematician to go into other disciplines than the opposite. This is a great advantage for mathematics students.

**An example: Mathematical Music Theory.**

Currently it is noticeable that there was a big trend in the last three decades in mathematics to realize not only applications but do mathematics in a variety of fields of knowledge, and the field of Music has not been the exception, although this happened in music from the time of Pythagoras.

Remember that the word musicology is the adopted name of the French "musicologie" to refer to scholarly study of music. Also, from German language "Musikwissenschaft" which means "science of music". To do musicology is not easy. Musicology lacks a stable framework. It is said to be very difficult to start making musicology and navigate over in a secure framework.

In traditional musicology there is a standard problem of encapsulation. An encapsulated postulate is given and prevents any access to the (alleged) hidden complexity. In mathematics, the access to complexity and its realization may eventually give insight into the concept while in the encapsulation musicological point attempts to vacuum, usually by breaking the flow of information through a dark road which claims to be rational, ornamented with metaphors, transforming a possible profound concept in a mysterious concept, that is, transforming science into fable. In one of the most ambitious book of traditional musicology, speeches of the important parts are based on an almost endless list of external references.

I want to mention one of the most interesting projects currently being developed in this field. I refer to the Mathematical Music Theory of Guerino Mazzola.

It is largely unknown the application of mathematical concepts to Musicology and in particular Mathematical Music Theory.

It began more than three decades ago. One of the main goals of Mathematical Music Theory is to develop a scientific framework for Musicology. This framework has as foundation established scientific fields. It includes a formal language for objects and musical and musicological relations.

Music is rooted in physical, psychological and semiotic realities. But the formal description of musical instances corresponds to the mathematical formalism.

It is based on the Theories of Groups, Modules and Categories, in Algebraic Topology and Combinatorics, in Algebraic Geometry, Representation Theory, i.e. in high-level mathematics. Its purpose is to describe musical structures. The philosophy behind it is to understand the aspects of music that are subject to reason in the same way that physics do with the natural phenomena in a scientific way. This theory is based: in language suited to handle the relevant concepts of musical structures, in a set of postulates and theorems about musical structures subject to the conditions defined and functionality for the composition and analysis with or without a computer.

Mazzola, in a magnificent panoramic article, "Towards Big Science ..." cites Pierre Boulez elements of a program of the sixties where he intended that the arts and science are reconciled. (I would say that artists and scientists).

Mazzola continues: "Music is a central creation of life and human thought. Acting on another layer of reality than physics. We believe that the attempt to understand or to compose a major work in music is as important and difficult as the attempt to unify gravity, electromagnetism, the weak and strong forces. Surely, the ambitions are comparable, and therefore, tools should be comparable."

In the eighties, Mazzola observed that global musical structures are held together with local data structures. These are the concepts studied in what is now known as Classical Mathematical Music Theory.

Later in the nineties, Mazzola mentioned three major paradigms of Mathematics and Musicology that have occurred during the 150 years that have been parallel in the evolution of both and the growing presence of mathematics in Music. These are: global structures, symmetries and the Philosophy of Yoneda.

The first means, in words, that the locally trivial structures can be put together in aesthetic configurations valid if they are glued in a nontrivial way.

The second, symmetries and (fractals) are used in composition, also they appear in nature and in mathematics play a crucial role as well in physics.

As for the third, the Philosophy of Yoneda, in words says that to understand an object go around it. This means understanding by changing perspectives. In mathematics, this Yoneda´s Lemma has important applications in Homological Algebra, in Algebraic Topology and Algebraic Geometry just to name a few. It says that a mathematical object can be classified by its functor up to isomorphism. In Music, the score is only the first glance and with all its interpretations constitute its identity. What a wonderful point of view for both performer and audience. It leaves aside the sterile competition in art and in science, as if they were Olympic Games.

"Classification means the task to fully understand an object. This is the Lemma of Yoneda in its full philosophical implications." "Understanding art means synthesizing all their interpretive perspectives.”

Fifteen years ago, Mazzola in his article "Status Quo 2000" (which we very much appreciated it was presented to the world in Mexico in a splendid plenary exposition in Saltillo in the Congress of the Mathematical Society of Mexico), explains why the approach by a geometric theoretical model of that time developed into a framework that is appropriate for many musical problems. This new framework is based on more sophisticated mathematics such as Category and Topos Theory.

Within this area, high-level mathematics is carried out, not just applications, that is, new mathematics is created, and mathematical results are tested within the defined objects.

One of the main purposes of Mathematical Music Theory is to establish a stable framework, defining the concepts in a precise way.

The word gesture means hand movement or of other parts of the body or face in which various affections of the mind are expressed.

Musical performance can be defined as a transformation of the mental level of the score into a set of sound events.

The musical performance consists of the score, possibly its analysis, the thaw of the symbols of the score to gestures that are then transformed into sounds by the instrument.

This concept excludes other types of musical execution not because they are not important, but because the chosen type is the perspective that has undergone the most intense and elaborate scientific research.

The thawing of the score to gestures that act on the instrument interface and generate sounds, play an important role, but this is still not a relevant issue (unfortunately) in performance theory. Just the transformation P of the score to the sounds is.

As a commentary, note that there is also the reverse process of freezing the gestures, ending with the modern abstract musical notation.

Musical Performance Theory and its practice, does not focus on P. The focus is in the research and understanding of the structures that are behind. In the research of the theory of musical performance, what is behind is called "expressive playing." This concept, somewhat ambiguous, refers to the communication process leading to P. As such, it starts from the creative side of the composer and performer, and addresses the audience and the analyst. This is mediated by acoustic and gestural performance of music.

There are many more interesting topics regarding musical performance. I invite you to enjoy the book “Musical Performance” of Guerino Mazzola where I have taken this explanation.

Mazzola, 8 years ago, (in 2007) presents a new categorical programmatically oriented framework for describing the relationship between music and mathematical activities. This relationship can be described in terms of adjoint functors, which extend the functorial presentation described in his fundamental book of 2002 “The Topos of Music”, (in which Harald and I are contributors). Therefore, in a meta-level, relations between music and mathematical activities are investigated from a mathematical point of view.

Far from being isomorphic, Music and Mathematics seem to have some structures that can be related by one of the most powerful concepts of category theory: the concept of adjoint functor (see my book in Homological Algebra for a definition of adjoint functor). This construction, proposed by Daniel Kan in 1958 as a technical device for the study of combinatorial properties in Homotopy Theory turns out to be the most appropriate tool to link three main categories:

equations or formulas (category of spectroids)
diagram schemes (directed graphs category) and
gestures (category of diagrams of curves in topological spaces).

The category of digraphs or directed graphs, which has recently been proposed as a foundational concept of mathematics for both classical and categorical set theory, seems to provide a musically interesting mediating structure between the other two categories, on which Music and Mathematics act in adjoint positions.

Through diagrams, mathematics turn gestures into formulas. In fact, a diagram is a system of transformational arrows. In such a system one can follow different paths or trajectories starting and ending at the same two points. These paths can be seen as gestures. If two such paths commute, that is, they produce the same composite transformation, then we have exactly what is called a formula or equation: two expressions give the same result. Very generally speaking, the formulas are commutative relations of gestures of paths or trajectories. Conversely, the musical activity "thaws" formulas and makes them gestures that can be described as formula development in space-time.

Mazzola uses the category of directed graphs of curves in topological spaces as the theoretical framework of Gesture Theory. From a purely theoretical aspect, "gestures of gestures" (or hypergestures) and "natural gestures" are canonically defined, as discussed by exemplifying the case of the gesture of a finger of a pianist and its hypergesture generalizations. As in the case of development of Category Theory and Topos Theory, as Mac Lane did, the notion of “gesture” as suggested in the work of Mazzola offers a good example of the "collision" between algebraic and topological methods.

Mazzola´s and Andreatta´s paper of 2007 suggests that the mathematical structuralism could be taken as a philosophical position for theoretical musical activity once it is accepted that Mathematical Music Theory deals with Music as a structured system.

The gesture is a morphism, where the relationship is a real movement, not just a symbolic arrow without substance as bridge. The arrow is a symbol of category theory that suggests a bridge between the domain and the codomain that points to a metaphor for carrying performance. However, according to Jean Cavaillé: "Understanding is to catch the gesture and be able to continue". This means that human and operational competence evidence are intimately linked to performance, and this is a gesture, not the abstract arrow.

In the article mentioned, the mathematical gestures category is introduced in terms of digraphs in topological spaces.

Then it is shown that the space of gestures in a topological space is a topological space. Therefore, iteration construction of gestures is possible and naturally leads to the notion of hypergestures. Hypergestures generalize the homotopies between continuous curves.

Mathematically, a gesture is defined as a digraph or directed graph D, called the skeleton of the gesture. It is also required that a transformation g that associates with each arrow *a* of D a continuous curve $g(a): I \to X$ defined on the unit interval $I = [0, 1]$ of real numbers so that the arrows that correspond go to continuous curves that correspond. The system of these curves is called the *body of the gesture*.

In this example, X is the coordinate space used for the positions of the tips of the fingers of a pianist in a given key (pitch), a level above the key (position) and the time in the event.

One of the most obvious expression gesture is the movement of the body of a music performer.

For example, it is easy to recognize the gestural power of pianists. It is logical that one must therefore try to model the gestures of musicians. In collaboration with his Ph.D. student Stefan Müller, Mazzola modeled in 2001-2003 the movements of a pianist hand at the level of computer graphics. The idea was not only to model the hand movements, but also implement software that will transform the abstract symbols of a score in hand movements that were suitable for the interpretation of the music on a piano keyboard.

In the traditional theory of musical performance one looks at the transformation P (score) symbol in the score to sound events. This is shown in the bottom of the rectangular diagram. In the gestural extension of this process without corporatization, we need to create the sound through gestural actions. Sounds are the result of natural gestural curves interacting with the keyboard; these curves are shown in the upper right of the diagram.

In order to generate these physical curves, one has first to thaw the symbols of notes and then transform them into gestural symbols. This thaw is displayed in the upper-left bottom half of the diagram.

The most powerful device in the Mathematical Theory of Gestures is the concept of a hypergesture. Renate Wieland mentioned the mysterious idea of a "gesture within a gesture". This construction is as follows: it can be shown that the set of gestures of a fixed skeleton D to a topological space X is itself a topological space, denoted with $D\underline{@}X$. Therefore, we can consider gestures $h: F \to D\underline{@}X$. These gestures are called hypergestures: that is, gestures whose body is a system of curves of gestures. While this sounds complicated, it's pretty intuitive. For example, if we have a gesture with a loop as a skeleton, and then a hypergesture again with the loop as a skeleton, then a hypergesture is a loop of loops, in fact a closed tube.

A closed tube is a hypergesture, namely, a loop of loops.

Such hypergestures are very useful in the generation of gestural interpretations of classical compositions. For example, using tonal modulation of B flat major to G major in the beginning of the Allegro of Beethoven´s Op.106, it can be described by such hypergestures, see Mazzola´s paper of 2009.

You can read Mazzola´s papers I have shown you and from which I have taken some of the concepts in order to motivate you in this fascinating subject. In them you will see the mathematics concepts he uses.

Again let me recommend you to read the recently appeared book "Musical Performance" by Guerino Mazzola for those who want to venture into this fascinating field where I also took some passages to provide to you a motivation for this theme.

Let me say that I dedicated myself in the late 70s and to date to Homotopy Theory and Cohomology Theory, that is, to Algebraic Topology, to Homological Algebra, etc. This mathematics were considered very "pure" mathematics in the seventies. But over thirty years later, those mathematics turned into “applied mathematics” but with an extra: you can do new mathematics there. All this happened in my other field of passion: Music! This is the surprising story of my mathematical life!

The Mathematical Music Theory of Guerino Mazzola is the most interesting project currently being developed in this field. Music is a central creation of life and human thought. Mathematics provides the scientific basis for understanding Music and Musicology.

We are currently experiencing a radical change in Musicology as the one experienced in Physics 500 years ago. It is a wonderful moment.

Let me mention the following extraordinary well written articles by Harald Fripertinger. Some of them I have used for my own students to work for their thesis. They are not only very interesting on the intelligent ways of enumerating and counting that Harald realized, but also the clarity, excellence and elegance in the exposition of his astonishing results. Last year Harald offered us a stupendous lecture at our International Music and Mathematics Congress in Vallarta, Mexico. Thank you, Harald, for setting a high level mark in the writing of mathematical creativity in our field. Congratulations on at least your first third of your life.

By the way, during that International Congress it was announced that the next Mathematics and Computation in Music Congress (MCM 2017) which takes place every two years will take place in Mexico City at UNAM in 2017. That is the place where the Congress will be, exactly where I work at the Faculty of Science at UNAM. You are most welcome to assist.

Sir James Joseph Sylvester wrote in 1864: "May not Music be described as the mathematics of Sense, Mathematics as music of the reason? The soul of each the same?" Both are created, recreated, we can appreciate and enjoy it. An advantage or disadvantage, as you want to see, is that for mathematics there is no musical instrument where you can play it, it remains at the level score, you might say, it goes directly from brain to brain.

For me, the most important relationship between mathematics and music is that both are "Fine Arts". They possess similar characteristics. They are related in the sense that Mathematics provides the scientific basis for understanding Music and Musicology and that the latter can be considered a science.

Finally, I again express that mathematics is one of the Fine Arts, the purest of them, who has the gift of being the most precise, and the precision of Science.

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