Aequationes Math. XX (2007) 1–38
0001-9054/07/010001-38
DOI 10.1007/s00010-007-2903-9

Aequationes Mathematicae

Formal translation equation and formal cocycle equations for iteration groups of type I

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Herrn Professor Walter Benz in Dankbarkeit und Verehrung gewidmet

Summary. We investigate the translation equation

$$F(s+t,x) = F(s,F(t,x)), \qquad s,t \in \mathbb{C}, \tag{T}$$

and the cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha(t,F(s,x)), \qquad s,t \in \mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha(t,F(s,x)) + \beta(t,F(s,x)), \qquad s,t \in \mathbb{C} \tag{Co2}$$

in $\mathbb{C}[\![x]\!]$, the ring of formal power series over $\mathbb{C}$. Here we restrict ourselves to iteration groups $(F(s,x))_{s\in\mathbb{C}}$ of type I, i.e. to solutions of (T) of the form $F(s,x) = c_1(s)x + \cdots$, where $c_1 \neq 1$ is a generalized exponential function. It is easy to prove that the coefficient functions $c_n(s)$, $\alpha_n(s)$, $\beta_n(s)$ of

$$F(s,x) = \sum_{n\geq 1} c_n(s)x^n, \quad \alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \quad \beta(s,x) = \sum_{n\geq 0} \beta_n(s)x^n$$

are polynomials in $c_1(s)$, polynomials or rational functions in one more generalized exponential function and, in a special case, polynomials in an additive function. For example, we obtain $c_n(s) = P_n(c_1(s))$, $P_n(y) \in \mathbb{C}[y]$, $n \geq 1$. For this we do not need any detailed information about the polynomials P_n.

Under some conditions on the exponential and additive functions it is possible to replace the exponential and additive functions by independent indeterminates. In this way we obtain formal versions of the translation equation and the cocycle equations in rings of the form $(\mathbb{C}[y])[\![x]\!]$, $(\mathbb{C}[S,\sigma])[\![x]\!]$, $(\mathbb{C}(S))[\![x]\!]$, $(\mathbb{C}(S)[U])[\![x]\!]$, and $(\mathbb{C}(S)[\sigma,U])[\![x]\!]$. We solve these equations in a completely algebraic way, by deriving formal differential equations or Aczél–Jabotinsky type equations for these problems. It is possible to get the structure of the coefficients in great detail which are now polynomials or rational functions. We prove the universal character (depending on certain parameters) of these polynomials or rational functions. And we deduce the canonical form $S^{-1}\big(yS(x)\big)$ for iteration groups of type I. This approach seems to be more clear and more general than the original one. Some simple substitutions allow us to solve these problems in rings of the form $(\mathbb{C}[\![u]\!])[\![x]\!]$, i. e. where the coefficient functions are formal series.

Mathematics Subject Classification (2000). Primary 39B12, 39B50; Secondary 13F25.

Keywords. Translation equation, cocycle equations, formal functional equations, formal group law, Aczél–Jabotinsky type equations, iteration groups of type I, ring of formal power series over $\mathbb{C}$.

1. Introduction

The problem to describe the one-parameter groups (also called iteration groups or flows) in the group of invertible formal power series in one indeterminate with complex coefficients and, more generally, to describe one-parameter groups of invertible formal power series transformations ("formally biholomorphic mappings") was studied by several authors, mainly in connection with the embedding problem, that is, whether a given formal power series (a formally biholomorphic mapping) can be embedded in such an iteration group. We mention D. N. Lewis [18], S. Sternberg [27], K. T. Chen [1], E. Peschl and L. Reich [22], L. Reich and J. Schwaiger [25], G. Mehring [20], and C. Praagman [23].

Motivated by this question of embeddability the problem arises to find the structure and the explicit form of these iteration groups in detail, not necessarily as a part of the embedding problem. We will now formulate this problem more precisely in the case of invertible formal series in one indeterminate. Denote by $\mathbb{C}[\![x]\!]$ the ring of formal power series in x over $\mathbb{C}$, and by $(\Gamma, \circ)$ the group of invertible series with respect to the substitution $\circ$.

By an iteration group (a one-parameter group) in Γ we understand a homomorphism

$$\theta\colon (\mathbb{C}, +) \to \Gamma.$$

We use the notation $\theta(t) = F(t, x) = F_t(x)$, $t \in \mathbb{C}$, where

$$F(t,x) = \sum_{n=1}^{\infty} c_n(t) x^n, \qquad t \in \mathbb{C},$$

$c_n\colon \mathbb{C} \to \mathbb{C}$, $n \geq 1$, are the coefficient functions. The name iteration group will also be used for the family $(F_t)_{t\in\mathbb{C}}$ of formal series. The condition that θ is a homomorphism is equivalent to the relation

$$F(s+t, x) = F\bigl(s, F(t,x)\bigr), \qquad s, t \in \mathbb{C}, \tag{T}$$

which is the well-known translation equation in our situation. It also follows from (T) that c_1 is a generalized exponential function. (Let us mention here that, more generally, one may study homomorphisms θ from an abelian group $(G, +)$ into Γ. This point of view is taken by W. Jabłoński and L. Reich in [15], [16]. In the present paper we will only deal with the situation $G = \mathbb{C}$.)

The *main problem* is then *to find the detailed structure and explicit form of the coefficient functions* c_n $(n \geq 2)$ *of the solutions* $(F(t,x))_{t\in\mathbb{C}}$ *of* (T).

If we assume that the coefficient functions c_n are everywhere holomorphic, i. e. entire functions, then it is known that the functions c_n are, also in the general situation of formal power series transformations, polynomials in t and in certain exponential functions $t \mapsto e^{\lambda_1 t}, \ldots, t \mapsto e^{\lambda_r t}$ (see e. g. [25]). If no regularity condition is imposed, then G. Mehring has proved that the coefficient functions are polynomials in finitely many additive functions and in generalized exponential functions (see [19, pp. 42ff]). However, the explicit form and the universal character of these polynomials, but also possible relations among the involved additive and generalized exponential functions, remain still open. Only recently, the detailed structure of the coefficient functions was given in [15], [16] in the case when θ is a homomorphism from $(G,+)$ into Γ_1, that means if $c_1 = 1$, and in forthcoming papers for homomorphisms θ for which $c_1 \neq 1$.

In our present paper we apply a somewhat different approach to obtain the structure of the coefficient functions in the situation when $\theta\colon (\mathbb{C},+) \to \Gamma$ with $c_1 \neq 1$. We call our approach the *method of formal functional equations*. The reason for this terminology is the following. While iteration problems in $\mathbb{C}[\![x]\!]$ are always formal in the sense that x is an indeterminate, the coefficients c_n under investigation are functions in the usual sense. Now we will show that under the afore mentioned hypothesis, (T) is equivalent to the problem

$$G(y\cdot z,x) = G\bigl(y,G(z,x)\bigr), \tag{$\mathrm{T}_{\mathrm{formal}}$}$$

$$G(1,x) = x \tag{B}$$

for a formal power series

$$G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!],$$

that is for a formal power series in x whose coefficients are polynomials in the indeterminate y over $\mathbb{C}$.

By this equivalence, our problem is completely formalized. $(\mathrm{T}_{\mathrm{formal}})$ is the first example of what we call a *formal functional equation*. The reduction of (T) to $(\mathrm{T}_{\mathrm{formal}})$ comes from the easy to prove fact that the coefficient functions c_n are, under the assumption $c_1 \neq 1$, polynomials $P_n(c_1)$ in c_1. For this we do not need any detailed information about the polynomials P_n.

We solve $(\mathrm{T}_{\mathrm{formal}})$ by three different methods, based upon the formal derivations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ in $(\mathbb{C}[y])[\![x]\!]$. Each of these approaches yields an ordinary or partial differential equation in $(\mathbb{C}[y])[\![x]\!]$, together with boundary conditions. According to the different ways in which the differentiations can be applied to $(\mathrm{T}_{\mathrm{formal}})$, we obtain an autonomous differential equation (Section 2.2), a partial differential equation (Section 2.1), and an Aczél–Jabotinsky type differential equation (Section 2.3), which can be solved in such a way that we get the structure of the coefficients in great detail. We have to show that all solutions of any of these formal differential equations satisfy $(\mathrm{T}_{\mathrm{formal}})$. The main results about the solutions of $(\mathrm{T}_{\mathrm{formal}})$ are contained in Theorems 4, 9 and 12. We also find the form $S^{-1}\bigl(yS(x)\bigr)$, $S\in\Gamma_1$, for the solutions of $(\mathrm{T}_{\mathrm{formal}})$ (cf. Theorem 7) using a technique of rearranging the solution of $(\mathrm{T}_{\mathrm{formal}})$ as $\sum_{n\geq 1}\phi_n(x)y^n$ and considering certain summable families of formal series.

The simple substitution $y = 1+u$ (just after the proof of Theorem 7) transforms $(\mathrm{T}_{\mathrm{formal}})$ to another, also purely algebraic problem which can be seen as a formal functional equation for objects from $(\mathbb{C}[\![u]\!])[\![x]\!]$. It has the form

$$K(u+v+uv,x) = K\bigl(u,K(v,x)\bigr) \tag{13}$$

in $(\mathbb{C}[\![u,v]\!])[\![x]\!]$, for

$$K(u,x) = (1+u)x + \sum_{n\geq 2} Q_n(u)x^n \in (\mathbb{C}[\![u]\!])[\![x]\!].$$

We notice that here the well-known formal group law (cf. [11, (1.1.7) page 2]) $u+v+uv$ plays a role, so that (13) is a sort of representation of this formal group law on $\mathbb{C}[\![x]\!]$.

The formal functional equation ($\mathrm{T}_{\mathrm{formal}}$) was already solved by D. Gronau in [6, Theorem 3] under even weaker assumptions than made in the present paper. Gronau's approach is different from the method of differentiation and from the method of rearranging a solution $G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n$ of ($\mathrm{T}_{\mathrm{formal}}$) as $G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n$ (cf. (10)), since he constructs the power series transformation S in the representation $G(y,x) = S^{-1}(yS(x))$ in our Theorem 7 as a limit $\lim_{n\to\infty}(T_2 \circ \ldots \circ T_n)$ in the order topology in Γ_1. His research also covers the situation related to iteration groups of type II which is not treated in our paper. Gronau mentions in [6] that the idea of introducing formal functional equations goes back to [24]. A generalization to higher dimensions, also by D. Gronau, can be found in [7].

In a similar way other functional equation problems in $\mathbb{C}[\![x]\!]$ can be treated by our method of formal functional equations. Here we will do this for the system of cocycle equations ((Co1), (Co2)) which appears in the problem of covariant embeddings of a linear functional equation with respect to an iteration group $(F(t,x))_{t\in\mathbb{C}}$, $F(t,x) = c_1(t)x + \cdots$, (see [2], [3], [5], [21], [8], [9], [10]). Again we will consider here only the case where the given iteration group is an iteration group of type I, that is $c_1 \neq 1$.

Also in the problem of ((Co1), (Co2)) we can relatively easily observe that the coefficient functions of the solutions (α,β) of ((Co1), (Co2)) are polynomials in c_1 and polynomials or rational functions in one more generalized exponential function, and, in a special case, polynomials in an additive function. Before now replacing these functions by appropriate indeterminates (in order to get formal functional equations), one has to look for possible "algebraic relations" which may exist between the given generalized exponential functions. This technical problem can be solved by using the general results of [26] and by some additional lemmas referring to the special situation in our paper (see Lemma 16). After introducing appropriate indeterminates we obtain purely algebraic problems, like ($\mathrm{Co1}_{\mathrm{formal}}$) and ($\mathrm{Co2}_{\mathrm{formal}_1}$), ($\mathrm{Co2}_{\mathrm{formal}_2}$), or ($\mathrm{Co2}_{\mathrm{formal}_3}$).

Also here, using the possible formal differentiations with respect to x and with respect to the indeterminates introduced above, we obtain in several ways systems of differential equations which are solved in Sections 3.1, 3.2 and 3.3. This leads to the main results on the solutions of the system of formal cocycle equations presented in Theorems 20, 22 and 23. However it should be mentioned that the system of differential and functional equations studied in Section 3.3 (of Aczél–Jabotinsky type) has solutions which do not satisfy ((Co1), (Co2)). These solutions are excluded by the condition (27). A similar phenomenon is already known for the original Aczél–Jabotinsky equation related to iteration groups.

By the substitution $y = 1 + u$ in the formal translation equation ($\mathrm{T}_{\mathrm{formal}}$) we embed this problem into a problem related to the ring $(\mathbb{C}[\![u]\!])[\![x]\!]$ instead of $(\mathbb{C}[y])[\![x]\!]$ without changing the set of solutions of ($\mathrm{T}_{\mathrm{formal}}$) essentially. Similar substitutions can be found for the equation ($\mathrm{Co2}_{\mathrm{formal}}$) in Section 3.4 which transform this equation to a problem for formal power series in x whose coefficients belong to rings which are power series in certain indeterminates. The transformed equation ($\mathrm{Co2}^*_{\mathrm{formal}}$) can be solved completely by applying differentiation techniques. Here ($\mathrm{Co2}^*_{\mathrm{formal}}$) may have solutions which do not come from solutions of ($\mathrm{Co2}_{\mathrm{formal}}$) (see Theorem 24).

2. The translation equation in $\mathbb{C}[\![x]\!]$

We study the translation equation

$$F(s+t,x) = F\big(s, F(t,x)\big), \qquad s,t \in \mathbb{C}, \tag{T}$$

where

$$F(s,x) = \sum_{n\geq 1} c_n(s)x^n \in \mathbb{C}[\![x]\!], \qquad s \in \mathbb{C},$$

with $\operatorname{ord}(F(s,x)) = 1$ for all $s \in \mathbb{C}$, whence $c_1(s) \neq 0$ for all $s \in \mathbb{C}$.

The family $F = (F(s,x))_{s\in\mathbb{C}}$ satisfies (T) if and only if the coefficient functions c_n satisfy a system of functional equations of the form

$$\begin{aligned}
c_1(s+t) &= c_1(s)c_1(t),\\
c_2(s+t) &= c_1(s)c_2(t) + c_2(s)c_1(t)^2,\\
&\;\;\dots\\
c_n(s+t) &= c_1(s)c_n(t) + c_n(s)c_1(t)^n + \tilde{P}_n\big(c_2(s),\dots,c_{n-1}(s),c_1(t),\dots,c_{n-1}(t)\big)
\end{aligned} \tag{1}$$

for all $s,t \in \mathbb{C}$, where $\tilde{P}_n$ are universal polynomials which are linear in $c_j(s)$ for $2 \leq j \leq n-1$.

Therefore, c_1 is a generalized exponential function. If $F = (F(s,x))_{s\in\mathbb{C}}$ is a solution of (T) with $c_1 \neq 1$, then F is called an iteration group of type I. If $c_1 = 1$, then F is an iteration group of type II.

Let F be a solution of (T). From $F(0+0,x) = F(0,F(0,x))$ we obtain $F(0,x) = x$, i. e. $c_1(0) = 1$ and $c_n(0) = 0$ for $n \geq 2$.

In the present manuscript we restrict ourselves to iteration groups of type I.

Lemma 1. *Let e be a generalized exponential function, $e \neq 1$. Then the following hold true:*

1. *e takes infinitely many values.*
2. *Consider $P(x_1,x_2) \in \mathbb{C}[x_1,x_2]$. If $P(e(s),e(t)) = 0$ for all $s,t \in \mathbb{C}$, then $P = 0$.*

Proof. 1. Let e be a generalized exponential function which takes only finitely many values. Then all these values are roots of unity: If on the contrary $e(t)$ were not a root of unity for some $t \in \mathbb{C}^*$, then $e(nt) = e(t)^n$ for $n \in \mathbb{N}$ are infinitely many values in $e(\mathbb{C})$. Let $e(\mathbb{C}) = \{c_1, \ldots, c_r\}$, where c_i is primitive of order n_i, $1 \le i \le r$, and let $N = \operatorname{lcm}(n_1, \ldots, n_r)$. Then $c_i^N = 1$, $1 \le i \le r$, and for any $t \in \mathbb{C}$ we have $e(t) = e(Nt/N) = e(t/N)^N = 1$.

2. Since $e(s)$ and $e(t)$ run with s and t independently through infinitely many values, it follows from $P(e(s), e(t)) = 0$ for all $s, t \in \mathbb{C}$ by standard arguments that $P(x_1, x_2) = 0$. □

Lemma 2. *If F is an iteration group of type I, then there exist polynomials $P_n(y) \in \mathbb{C}[y]$, such that*

$$c_n(s) = P_n\big(c_1(s)\big), \qquad s \in \mathbb{C},\ n \ge 1.$$

Proof. The assertion is trivial for $n = 1$. Assume that $n \ge 2$. Using (1) and induction we obtain

$$\begin{aligned} c_n(s+t) &= c_1(s)c_n(t) + c_n(s)c_1(t)^n \\ &\qquad + \tilde{P}_n\big(P_2(c_1(s)), \ldots, P_{n-1}(c_1(s)), P_1(c_1(t)), \ldots, P_{n-1}(c_1(t))\big) \\ &= c_1(s)c_n(t) + c_n(s)c_1(t)^n + \hat{P}_n\big(c_1(s), c_1(t)\big) \end{aligned}$$

with a suitable polynomial $\hat{P}_n$. Since c_1 is a generalized exponential function which takes infinitely many values (see Lemma 1), there exist $t_m \in \mathbb{C}$, $m \ge 2$, so that $c_1(t_m)^m - c_1(t_m) \ne 0$. As a consequence of $c_n(s+t) = c_n(t+s)$ for all $s, t \in \mathbb{C}$ we have

$$\begin{aligned} &c_1(s)c_n(t) + c_n(s)c_1(t)^n + \hat{P}_n\big(c_1(s), c_1(t)\big) \\ &\qquad = c_1(t)c_n(s) + c_n(t)c_1(s)^n + \hat{P}_n\big(c_1(t), c_1(s)\big) \end{aligned} \tag{2}$$

and we derive that

$$\begin{aligned} c_n(s) &= \frac{c_n(t_n)\big(c_1(s)^n - c_1(s)\big) + \hat{P}_n\big(c_1(t_n), c_1(s)\big) - \hat{P}_n\big(c_1(s), c_1(t_n)\big)}{c_1(t_n)^n - c_1(t_n)} \\ &= P_n\big(c_1(s)\big), \qquad s \in \mathbb{C},\ n \ge 2. \end{aligned}$$

Thus $c_n(s)$ is a polynomial in $c_1(s)$. □

From (1) and Lemma 2 and we deduce for $n \ge 2$ that

$$\begin{aligned} P_n\big(c_1(s) \cdot c_1(t)\big) &= P_n\big(c_1(s+t)\big) = c_n(s+t) \\ &= c_1(s)P_n(c_1(t)) + P_n(c_1(s))c_1(t)^n \\ &\quad + \tilde{P}_n\big(P_2(c_1(s)), \ldots, P_{n-1}(c_1(s)), c_1(t), \ldots, P_{n-1}(c_1(t))\big). \end{aligned} \tag{3}$$

Since $c_1 \ne 1$, from (3) and Lemma 1 we obtain the polynomial identity

$$P_n(y \cdot z) = yP_n(z) + P_n(y)z^n + \tilde{P}_n\big(P_2(y), \ldots, P_{n-1}(y), z, \ldots, P_{n-1}(z)\big) \tag{4}$$

for all $n \geq 2$. This system of identities is equivalent to the formal translation equation

$$G(y \cdot z, x) = G\big(y, G(z,x)\big) \tag{$\mathrm{T}_{\mathrm{formal}}$}$$

in $(\mathbb{C}[y,z])[\![x]\!]$ for

$$G(y,x) = yx + \sum_{n \geq 2} P_n(y)x^n,$$

$P_n(y) \in \mathbb{C}[y]$, $n \geq 2$, and the boundary condition

$$G(1,x) = x. \tag{B}$$

Hence we obtain

Theorem 3. *$F(s,x) = c_1(s)x + \sum_{n\geq 2} P_n(c_1(s))x^n$ is a solution of* (T) *if and only if $G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n$ is a solution of* ($\mathrm{T}_{\mathrm{formal}}$) *and* (B). □

In the sequel we solve the system consisting of the formal translation equation ($\mathrm{T}_{\mathrm{formal}}$) and the boundary condition (B). In $\mathbb{C}[y]$ we have the formal derivation with respect to y. In $(\mathbb{C}[y])[\![x]\!]$ we have the formal derivation with respect to x. Moreover the mixed chain rule is valid for formal derivations.

Usually

$$\frac{\partial}{\partial y} G(y,x)|_{y=1} = x + \sum_{n \geq 2} h_n x^n = H(x)$$

is called the infinitesimal generator of G.

Differentiation of ($\mathrm{T}_{\mathrm{formal}}$) with respect to y yields

$$z \frac{\partial}{\partial t} G(t,x)|_{t=yz} = \frac{\partial}{\partial y} G\big(y, G(z,x)\big).$$

For $y = 1$ we get

$$z \frac{\partial}{\partial z} G(z,x) = H(G(z,x)). \tag{$\mathrm{D}_{\mathrm{formal}}$}$$

Differentiation of ($\mathrm{T}_{\mathrm{formal}}$) with respect to z and application of the mixed chain rule yields

$$y \frac{\partial}{\partial t} G(t,x)|_{t=yz} = \frac{\partial}{\partial t} G(y,t)|_{t=G(z,x)} \frac{\partial}{\partial z} G(z,x).$$

For $z = 1$ we get

$$y \frac{\partial}{\partial y} G(y,x) = H(x) \frac{\partial}{\partial x} G(y,x). \tag{$\mathrm{PD}_{\mathrm{formal}}$}$$

Combining ($\mathrm{D}_{\mathrm{formal}}$) and ($\mathrm{PD}_{\mathrm{formal}}$), we obtain an Aczél–Jabotinsky differential equation of the form

$$H(x) \frac{\partial}{\partial x} G(y,x) = H(G(y,x)). \tag{$\mathrm{AJ}_{\mathrm{formal}}$}$$

2.1. The differential equation ($\mathrm{PD}_{\mathrm{formal}}$)

Now we solve ($\mathrm{PD}_{\mathrm{formal}}$) together with (B) in order to solve ($\mathrm{T}_{\mathrm{formal}}$). The advantage of this procedure lies in the circumstance that no substitution of the unknown series $G(y,x)$ is needed and that ($\mathrm{PD}_{\mathrm{formal}}$) is a linear equation. In a completely algebraic way it is possible to prove

Theorem 4. 1. *For any generator $H(x) = x + h_2x^2 + \cdots$ the differential equation ($\mathrm{PD}_{\mathrm{formal}}$) together with (B) has exactly one solution. It is given by*

$$G(y,x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

2. *The polynomials P_n, $n \geq 2$, are of formal degree n, they satisfy $P_n(0) = 0$, and they are of the form*

$$P_n(y) = \frac{h_n}{n-1}(y^n - y) + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\dots,h_{n-1})}{n-j}(y^n - y^j), \tag{5}$$

where the polynomials $\Phi_j^{(n)}$, $1 \leq j \leq n-1$, are (recursively) determined by

$$\sum_{r=2}^{n-1} h_r(n-r+1)P_{n-r+1}(y) = \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\dots,h_{n-1})y^j. \tag{6}$$

Proof. Writing $G(y,x)$ as $\sum_{n\geq 0} P_n(y)x^n$, and using $h_1 = 1$, ($\mathrm{PD}_{\mathrm{formal}}$) reads as

$$y\sum_{n\geq 0} P_n'(y)x^n = \left(\sum_{n\geq 1} h_nx^n\right)\left(\sum_{n\geq 0} nP_n(y)x^{n-1}\right).$$

We compare coefficients of x^n. Together with (B) we obtain $P_0(y) = 0$ and $P_1(y) = y$. For $n = 2$ we have

$$yP_2'(y) = 2P_2(y) + h_2y$$

which yields $P_2(y) = -h_2y + ay^2$, for $a \in \mathbb{C}$. From (B) we deduce that $P_2(1) = 0$ and therefore $P_2(y) = h_2(y^2 - y)$. Thus, P_2 is uniquely determined and has formal degree 2.

Assume that $n > 2$ and that the assertions (5) and (6) are valid for all P_j for $j < n$. From ($\mathrm{PD}_{\mathrm{formal}}$) we derive

$$yP_n'(y) = h_ny + nP_n(y) + \sum_{r=2}^{n-1} h_r(n-r+1)P_{n-r+1}(y). \tag{7}$$

From the induction hypothesis we deduce that $\sum_{r=2}^{n-1} h_r(n-r+1)P_{n-r+1}(y)$ is a polynomial of formal degree equal to $n-1$. Hence we have

$$\sum_{r=2}^{n-1} h_r(n-r+1)P_{n-r+1}(y) = \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\dots,h_{n-1})y^j, \tag{6}$$

where the coefficients $\Phi_j^{(n)}$ are polynomials in $h_2, \dots, h_{n-1}$. Expressing the polynomial $P_n(y)$ as $\sum_{j=0}^{d} a_j y^j$ for some $d \in \mathbb{N}$, we derive from (7) that

$$\sum_{j=0}^{d} j a_j y^j = h_n y + n \sum_{j=0}^{d} a_j y^j + \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2, \dots, h_{n-1}) y^j.$$

This allows the determination of the coefficients a_j, $j \neq n$, as

$$a_j = \begin{cases} 0, & \text{if } j = 0, \\ \frac{h_n + \Phi_1^{(n)}(h_2, \dots, h_{n-1})}{1-n}, & \text{if } j = 1, \\ \frac{\Phi_j^{(n)}(h_2, \dots, h_{n-1})}{j-n}, & \text{if } 2 \leq j \leq n-1, \\ \text{not determined}, & \text{if } j = n, \\ 0, & \text{if } j > n. \end{cases}$$

From (B) it follows that $P_n(1) = 0$, whence

$$a_n = -\sum_{j=0}^{n-1} a_j = \frac{h_n}{n-1} + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2, \dots, h_{n-1})}{n-j}$$

and P_n has the desired representation (5). □

Theorem 5. *Each solution $G(y, x)$ of* ($\mathrm{PD}_{\mathrm{formal}}$) *and* (B) *is a solution of the formal translation equation* ($\mathrm{T}_{\mathrm{formal}}$).

Proof. Let z be an indeterminate. We prove that

$$U(y, z, x) := G(yz, x),$$
$$V(y, z, x) := G(z, G(y, x))$$

satisfy the system

$$y \frac{\partial}{\partial y} f(y, z, x) = H(x) \frac{\partial}{\partial x} f(y, z, x), \tag{8}$$

$$f(1, z, x) = G(z, x). \tag{9}$$

If we further prove that the system ((8),(9)) has a unique solution in $(\mathbb{C}[y, z])[\![x]\!]$, then we have shown that G satisfies ($\mathrm{T}_{\mathrm{formal}}$). First we demonstrate that U satisfies (8) and (9).

$$\begin{aligned} y \frac{\partial}{\partial y} U(y, z, x) &= y \frac{\partial}{\partial y} G(yz, x) = yz \frac{\partial}{\partial t} G(t, x)|_{t=yz} \overset{(\mathrm{PD}_{\mathrm{formal}})}{=} H(x) \frac{\partial}{\partial x} G(yz, x) \\ &= H(x) \frac{\partial}{\partial x} U(y, z, x) \end{aligned}$$

and

$$U(1, z, x) = G(z, x).$$

Similarly we prove that V satisfies (8) and (9).

$$\begin{aligned} y\frac{\partial}{\partial y}V(y,z,x) &= y\frac{\partial}{\partial y}G\bigl(z,G(y,x)\bigr) = y\frac{\partial}{\partial y}G(y,x)\frac{\partial}{\partial t}G(z,t)|_{t=G(y,x)} \\ &\overset{(\mathrm{PD}_{\mathrm{formal}})}{=} H(x)\frac{\partial}{\partial x}G(y,x)\frac{\partial}{\partial t}G(z,t)|_{t=G(y,x)} \\ &= H(x)\frac{\partial}{\partial x}G\bigl(z,G(y,x)\bigr) \\ &= H(x)\frac{\partial}{\partial x}V(y,z,x) \end{aligned}$$

and

$$V(1,z,x) = G(z,x).$$

In order to show that there is a unique solution of (8) and (9) we write $f(y,z,x)$ as $\sum_{n\geq 0} f_n(y,z)x^n$. Comparison of coefficients in (8) together with (9) shows that

$$f_0(y,z) = 0, \quad f_1(yz) = yz,$$

and

$$f_n(y,z) = P_n(yz), \qquad n \geq 2,$$

with exactly the same polynomials P_n as in Theorem 4. Hence the solution f of (8) and (9) is uniquely determined. $\square$

Let Γ_1 be the set of all formal power series $S(x) \in \mathbb{C}[\![x]\!]$ with $S(x) \equiv x \bmod x^2$. Finally we prove that there exist simple normal forms of solutions of $(\mathrm{T}_{\mathrm{formal}})$ with respect to substitutional conjugation with elements from Γ_1. More precisely, we prove that each solution $G(y,x)$ of $(\mathrm{T}_{\mathrm{formal}})$ can be represented as $G(y,x) = S^{-1}(yS(x))$ for some $S(x) \in \Gamma_1$, where S^{-1} is the inverse of S with respect to substitution.

From the particular representation of $G(y,x)$ in Theorem 4 we deduce that

$$G(y,x) = \sum_{n\geq 1}\phi_n(x)y^n, \tag{10}$$

where $\phi_1 \in \Gamma_1$, and both $(\phi_n(x))_{n\geq 1}$ and $(\phi_n(x)y^n)_{n\geq 1}$ are summable families in $(\mathbb{C}[y])[\![x]\!]$. This allows us to rewrite $(\mathrm{PD}_{\mathrm{formal}})$ and (B) as

$$\sum_{n\geq 1} n\phi_n(x)y^n = H(x)\sum_{n\geq 1}\phi_n'(x)y^n, \tag{11}$$

$$\sum_{n\geq 1}\phi_n(x) = x, \tag{12}$$

where $(\phi_n'(x))_{n\geq 1}$ is also a summable family. We note that (11) is satisfied if and only if

$$n\phi_n(x) = H(x)\phi_n'(x) \tag{11_n}$$

holds true for all $n \geq 1$.

(11_n) is a Briot–Bouquet differential equation (cf. [17, section 5.2] [13, section 11.1], [14, section 12.6]). Hence the following theorem could be derived from the formal part of the theory of these equations. We will present a direct proof.

Theorem 6. *Consider $H(x) = x + h_2x^2 + \cdots$.*

1. *Let $n \geq 1$. For any $\varphi_n^{(n)} \in \mathbb{C}$, there is exactly one solution ϕ_n of (11_n), so that*
$$\phi_n(x) \equiv \varphi_n^{(n)} x^n \bmod x^{n+1}.$$
Denote by $\phi_{n,0}$ the solution corresponding to $\varphi_n^{(n)} = 1$.
2. *Let $n \geq 1$. If $\phi_n(x) \equiv \varphi_n^{(n)} x^n \bmod x^{n+1}$ is a solution of (11_n), then*
$$\phi_n(x) = \varphi_n^{(n)} \phi_{n,0}(x).$$
3. *$\phi_{n,0}(x) = [\phi_{1,0}(x)]^n$ for $n \geq 1$.*

Proof. 1. We determine all solutions $\psi(x) = \sum_{\nu \geq n} \psi_\nu x^\nu$ of (11_n). From
$$n \sum_{\nu \geq n} \psi_\nu x^\nu = \left(\sum_{\nu \geq 1} h_\nu x^\nu \right) \left(\sum_{\nu \geq n} \nu \psi_\nu x^{\nu - 1} \right)$$
we obtain by comparison of coefficients that
$$\begin{aligned} n\psi_n &= n\psi_n, \\ n\psi_{n+1} &= (n+1)\psi_{n+1} + h_2 n \psi_n, \\ &\cdots \\ n\psi_\ell &= \ell\psi_\ell + \Psi_\ell(h_2, \ldots, h_{\ell-(n-1)}), \end{aligned}$$
which shows that the polynomials Ψ_ℓ, $\ell > n$, are uniquely determined. Therefore, for any choice of $\psi_n \in \mathbb{C}$ there exists a unique solution $\psi(x) = \sum_{\nu \geq n} \psi_\nu x^\nu$ of (11_n).

2. Simple computations show that $\psi(x) := \varphi_n^{(n)} \phi_{n,0}(x)$ is a solution of (11_n) and that $\psi(x) \equiv \varphi_n^{(n)} x^n \bmod x^{n+1}$.

3. Let $\psi(x) := [\phi_{1,0}(x)]^n$, then $\psi(x) \equiv x^n \bmod x^{n+1}$ and
$$\begin{aligned} H(x)\psi'(x) &= nH(x)[\phi_{1,0}(x)]^{n-1}\phi'_{1,0}(x) \\ &\overset{(11_n)}{=} n[\phi_{1,0}(x)]^{n-1}\phi_{1,0}(x) \\ &= n\psi(x), \end{aligned}$$
which shows that $\psi(x)$ is a solution of (11_n). By the first assertion $\psi(x) = \phi_{n,0}(x)$ which finishes the proof. □

Since $\operatorname{ord}(\phi_{n,0}(x)) = n$ the families $(\phi_{n,0}(x))_{n \geq 1}$ and $(\varphi_n^{(n)} \phi_{n,0}(x))_{n \geq 1}$ for $\varphi_n^{(n)} \in \mathbb{C}$ are summable.

Theorem 7. 1. *If* $G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n$ *is a solution of* $(\mathrm{T}_{\mathrm{formal}})$ *and* (B), *then*

$$G(y,x) = S^{-1}\big(yS(x)\big)$$

for some $S \in \Gamma_1$.

2. *Conversely, every* $S \in \Gamma_1$ *allows the construction of a solution*

$$G(y,x) = S^{-1}\big(yS(x)\big)$$

of $(\mathrm{T}_{\mathrm{formal}})$ *and* (B).

Proof. 1. We have

$$G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n = \sum_{n\geq 1} \varphi_n^{(n)}[y\phi_{1,0}(x)]^n.$$

If we set $S(x) := \phi_{1,0}(x)$, then $S(x) \in \Gamma_1$ and

$$x = G(1,x) = \sum_{n\geq 1} \varphi_n^{(n)}[\phi_{1,0}(x)]^n = \sum_{n\geq 1} \varphi_n^{(n)}[S(x)]^n.$$

Moreover $\varphi_1^{(1)} = 1$, $\Sigma(x) := \sum_{n\geq 1} \varphi_n^{(n)} x^n$ is in Γ_1 and $\Sigma(S(x)) = x$. Hence, $\Sigma(x) = S^{-1}(x)$ and $G(y,x) = S^{-1}\big(yS(x)\big)$.

2. Simple computations show that $G(y,x) = S^{-1}(yS(x))$ satisfies $(\mathrm{T}_{\mathrm{formal}})$ and (B) for every $S \in \Gamma_1$. □

Now we replace y by $1+u$ and z by $1+v$ in $(\mathrm{T}_{\mathrm{formal}})$ and obtain

$$G(1+u+v+uv,x) = G\big(1+u, G(1+v,x)\big).$$

Denoting $G(y,x)$ which is $G(1+u,x)$ by $K(u,x)$ yields

$$K(u+v+uv,x) = K\big(u, K(v,x)\big) \tag{13}$$

in $(\mathbb{C}[u,v])[\![x]\!]$ for

$$K(u,x) = (1+u)x + \sum_{n\geq 2} Q_n(u)x^n,$$

$Q_n(u) \in \mathbb{C}[u]$, $n \geq 2$. The boundary condition (B) is replaced by

$$K(0,x) = x. \tag{B$'$}$$

After this transformation the boundary condition requires evaluation of $K(u,x)$ for $u = 0$. Since substitution of 0 into a formal power series is possible, we are able to determine the solutions of (13) and (B$'$) even in the ring $(\mathbb{C}[\![u,v]\!])[\![x]\!]$, which contains $(\mathbb{C}[u,v])[\![x]\!]$ as a subring. Here we consider

$$K(u,x) = \sum_{n\geq 0} Q_n(u)x^n,$$

$Q_n(u) \in \mathbb{C}[\![u]\!]$, $n \geq 0$, i. e. Q_n are not necessarily polynomials but formal series.

Similarly as with ($\mathrm{PD}_{\mathrm{formal}}$), differentiation of (13) with respect to v yields

$$\frac{\partial}{\partial v} K(u+v+uv,x) = \frac{\partial}{\partial v} K\big(u, K(v,x)\big)$$

which is equivalent to

$$(1+u)\frac{\partial}{\partial t} K(t,x)|_{t=u+v+uv} = \frac{\partial}{\partial t} K(u,t)|_{t=K(v,x)} \frac{\partial}{\partial v} K(v,x).$$

For $v = 0$ we get

$$(1+u)\frac{\partial}{\partial u} K(u,x) = H(x)\frac{\partial}{\partial x} K(u,x), \tag{14}$$

where

$$H(x) = x + \sum_{n\geq 2} h_n x^n = \frac{\partial}{\partial y} G(y,x)|_{y=1} = \frac{\partial}{\partial u} G(1+u,x)|_{u=0} = \frac{\partial}{\partial u} K(u,x)|_{u=0}$$

is the infinitesimal generator of G and K.

Theorem 8. 1. *For any generator $H(x) = x + h_2x^2 + \cdots$ the differential equation (14) together with* (B′) *has exactly one solution*

$$K(u,x) = (1+u)x + \sum_{n\geq 2} Q_n(u)x^n \in (\mathbb{C}[\![u]\!])[\![x]\!].$$

2. *The formal series Q_n, $n \geq 2$, are polynomials of formal degree n, they satisfy $Q_n(0) = 0$, and they are of the form*

$$Q_n(u) = \frac{h_n}{n-1}\big((1+u)^n - (1+u)\big) + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\ldots,h_{n-1})}{n-j}\big((1+u)^n - (1+u)^j\big),$$

where the polynomials $\Phi_j^{(n)}$, $1 \leq j \leq n-1$, are (recursively) determined by

$$\sum_{r=2}^{n-1} h_r(n-r+1)Q_{n-r+1}(u) = \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\ldots,h_{n-1})(1+u)^j.$$

Proof. Introducing coefficients in (14) and using $h_1 = 1$ we get

$$(1+u)\left(\sum_{n\geq 0} Q_n'(u)x^n\right) = \left(\sum_{n\geq 1} h_n x^n\right)\left(\sum_{n\geq 0} nQ_n(u)x^{n-1}\right).$$

From (B′) we deduce that $Q_0(u) = 0$ and $Q_1(u) = 1+u$. For $n \geq 2$ we have to solve the differential equation

$$Q_n'(u) = (1+u)^{-1}\left(nQ_n(u) + \sum_{r=2}^{n-1} h_r(n-r+1)Q_{n-r+1}(u) + h_n(1+u)\right) \tag{15}$$

in $\mathbb{C}[\![u]\!]$ together with the boundary condition $Q_n(0) = 0$. By the formal part of Cauchy's theorem on the existence and uniqueness of solutions of analytic differential equations (see e.g. [12, section 2.5]), for every $n \geq 2$ there exists exactly one solution $Q_n(u) \in \mathbb{C}[\![u]\!]$. The method of *variation of the integration constant* allows the computation of these Q_n.

First we determine the solutions of the homogeneous differential equation

$$Q_n^{(h)\prime}(u) = (1+u)^{-1} n Q_n^{(h)}(u).$$

The general solution of this equation is $Q_n^{(h)} = \gamma_n \cdot (1+u)^n$ for $\gamma_n \in \mathbb{C}$. Replacing the constant γ_n by a function $\gamma_n(u)$ we try to express the solution of (15) in the form

$$Q_n(u) = \gamma_n(u)(1+u)^n. \tag{16}$$

For $n = 2$ we obtain $Q_2(u) = h_2\left((1+u)^2 - (1+u)\right)$. Now we assume that the assertion is valid for Q_r, $1 \leq r < n$. Then we obtain from (15) that

$$Q_n'(u) = (1+u)^{-1}\left(nQ_n(u) + \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\ldots,h_{n-1})(1+u)^j + h_n(1+u)\right).$$

Together with (16) we have

$$\begin{aligned}
\gamma_n'(u) &= (1+u)^{-n}\left(Q_n'(u) - n\gamma_n(u)(1+u)^{n-1}\right)\\
&= (1+u)^{-(n+1)}\left(\sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\ldots,h_{n-1})(1+u)^j + h_n(1+u)\right)\\
&= \sum_{j=1}^{n-1} \Phi_j^{(n)}(h_2,\ldots,h_{n-1})(1+u)^{j-n-1} + h_n(1+u)^{-n}.
\end{aligned}$$

Therefore,

$$\gamma_n(u) = \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\ldots,h_{n-1})}{j-n}(1+u)^{j-n} + \frac{h_n}{1-n}(1+u)^{1-n} + c_n.$$

From $Q_n(0) = 0$ we deduce $\gamma_n(0) = 0$ and consequently

$$c_n = \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\ldots,h_{n-1})}{n-j} + \frac{h_n}{n-1}.$$

Summarizing

$$\gamma_n(u) = \frac{h_n}{n-1}\left(1-(1+u)^{1-n}\right) + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\ldots,h_{n-1})}{n-j}\left(1-(1+u)^{j-n}\right)$$

and

$$Q_n(u) = \frac{h_n}{n-1}\big((1+u)^n - (1+u)\big) + \sum_{j=1}^{n-1} \frac{\Phi_j^{(n)}(h_2,\dots,h_{n-1})}{n-j}\left((1+u)^n - (1+u)^j\right). \qquad \square$$

Now we show that similar methods can be applied in order to determine the solutions of ($\mathrm{T}_{\mathrm{formal}}$) and (B) using the differential equations ($\mathrm{D}_{\mathrm{formal}}$) or ($\mathrm{AJ}_{\mathrm{formal}}$).

2.2. The differential equation ($\mathrm{D}_{\mathrm{formal}}$)

Since in ($\mathrm{D}_{\mathrm{formal}}$) the series $G(z,x)$ is substituted into $H(x)$ we have to assume that $G(z,x) = \sum_{n\geq 1} P_n(z)x^n$. Concerning the computation of $H(G(z,x))$ we just mention that the multiplicative powers $[G(z,x)]^\nu$ of $G(z,x)$, where ν is a positive integer, are of the form

$$[G(z,x)]^\nu = \sum_{n\geq\nu} \sum_{\substack{r_1+\cdots+r_\nu=n\\ r_j\geq 1}} \left(\prod_{j=1}^{\nu} P_{r_j}(z)\right) x^n$$

and therefore

$$H(G(z,x)) = \sum_{n\geq 1} \left(\sum_{\nu=1}^{n} h_\nu \sum_{\substack{r_1+\cdots+r_\nu=n\\ r_j\geq 1}} \left(\prod_{j=1}^{\nu} P_{r_j}(z)\right)\right) x^n.$$

Theorem 9. 1. *For any generator $H(x) = x + h_2x^2 + \cdots$ the differential equation* ($\mathrm{D}_{\mathrm{formal}}$) *together with* (B) *has exactly one solution. It is given by*

$$G(z,x) = zx + \sum_{n\geq 2} P_n(z)x^n \in (\mathbb{C}[z])[\![x]\!].$$

2. *The polynomials P_n, $n\geq 2$, are of formal degree n, they satisfy $P_n(0)=0$, and they are of the form*

$$P_n(z) = \frac{h_n}{n-1}(z^n - z) + \sum_{j=2}^{n} \frac{\Psi_j^{(n)}(h_2,\dots,h_{n-1})}{j-1}(z^j - z),$$

where the polynomials $\Psi_j^{(n)}$, $2\leq j\leq n$, are (recursively) determined by

$$\sum_{\nu=2}^{n-1} h_\nu \sum_{\substack{r_1+\cdots+r_\nu=n\\ r_j\geq 1}} \left(\prod_{j=1}^{\nu} P_{r_j}(z)\right) = \sum_{j=2}^{n} \Psi_j^{(n)}(h_2,\dots,h_{n-1})z^j. \qquad \square$$

We omit here a detailed proof of Theorem 9.

Theorem 10. *Each solution $G(z,x)$ of* ($\mathrm{D}_{\mathrm{formal}}$) *and* (B) *is a solution of the formal translation equation* ($\mathrm{T}_{\mathrm{formal}}$).

For proving this theorem we show that

$$U(y,z,x) := G(yz,x),$$
$$V(y,z,x) := G\big(z, G(y,x)\big)$$

both satisfy the system

$$z\frac{\partial}{\partial z} f(y,z,x) = H\big(f(y,z,x)\big),$$
$$f(y,1,x) = G(y,x),$$

where $\operatorname{ord}(f(y,z,x)) \geq 1$. This system has the unique solution $f(y,z,x) = yzx + \sum_{n\geq 2} P_n(yz)x^n$ with exactly the same polynomials P_n as in Theorem 9. □

If we write $G(z,x)$ in the form

$$G(z,x) = \sum_{n\geq 1} \phi_n(x) z^n,$$

then $\phi_1(x) \in \Gamma_1$, and both $(\phi_n(x))_{n\geq 1}$ and $(\phi_n(x)z^n)_{n\geq 1}$ are summable families in $(\mathbb{C}[z])[\![x]\!]$. The differential equation ($\mathrm{D}_{\mathrm{formal}}$) together with (B) is equivalent to the system

$$\sum_{n\geq 1} n\phi_n(x) z^n = \sum_{\nu\geq 1} h_\nu \left[\sum_{n\geq 1} \phi_n(x) z^n\right]^\nu \tag{17}$$

$$\sum_{n\geq 1} \phi_n(x) = x. \tag{18}$$

We note that (17) is satisfied if and only if

$$n\phi_n(x) = \sum_{\nu=1}^{n} h_\nu \sum_{\substack{r_1+\cdots+r_\nu=n \\ r_j\geq 1}} \left(\prod_{j=1}^{\nu} \phi_{r_j}(x)\right) \tag{17_n}$$

holds true for all $n \geq 1$.

Theorem 11. *Consider $H(x) = x + h_2x^2 + \cdots$.*

1. *Every $\phi_1(x) \in \mathbb{C}[\![x]\!]$ satisfies* (17_1).
2. *Let $\phi_1 \in \mathbb{C}[\![x]\!] \setminus \{0\}$. For each $n \geq 2$ there exists exactly one solution ϕ_n of* (17_n)*, depending on ϕ_1. It is given by $\phi_n(x) := \varphi_n[\phi_1(x)]^n$, where $\varphi_1 = 1$ and*

$$\varphi_n = \sum_{\nu=2}^{n} \frac{h_\nu}{n-1} \sum_{r_1+\cdots+r_\nu=n} \prod_{j=1}^{\nu} \varphi_{r_j}, \qquad n \geq 2.$$

   Consequently, φ_n does not depend on the choice of ϕ_1.

3. *The system* (17) *and* (18) *has a unique solution. It is given by*

$$\sum_{n\geq 1} \varphi_n [\phi_1(x)]^n z^n$$

for $\varphi_1 = 1$, φ_n *for* $n \geq 2$ *given as above, and* $\phi_1(x) = (x + \sum_{n\geq 2} \varphi_n x^n)^{-1}$ *which is an element of* Γ_1. □

Based on these results it is possible to give another simple proof of Theorem 7.

2.3. The differential equation ($\mathrm{AJ}_{\mathrm{formal}}$)

Now we turn our attention to the Aczél–Jabotinsky differential equation. Again we have to assume that $G(z, x) = \sum_{n\geq 1} P_n(z)x^n$.

Theorem 12. 1. *For any generator* $H(x) = x + h_2x^2 + \cdots$ *the differential equation* ($\mathrm{AJ}_{\mathrm{formal}}$) *has exactly one solution of the form*

$$G(y, x) = yx + \sum_{n\geq 2} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

Moreover, for every $\tilde{P}_1(y) \in \mathbb{C}[y]$ *there exist uniquely determined polynomials* $\tilde{P}_n(y) \in \mathbb{C}[y]$, $n \geq 2$, *so that*

$$G(y, x) = \tilde{P}_1(y)x + \sum_{n\geq 2} \tilde{P}_n(y)x^n \in (\mathbb{C}[y])[\![x]\!]$$

is a solution of ($\mathrm{AJ}_{\mathrm{formal}}$) *as well.*

2. *The polynomials* P_n, $n \geq 2$, *(from the unique solution* $G(y, x) \equiv yx \bmod x^2$*) are of formal degree* n, *they satisfy* $P_n(0) = 0$, *and they are of the form*

$$P_n(y) = \frac{h_n}{n-1}(y^n - y) + \sum_{j=2}^{n} \frac{\Theta_j^{(n)}(h_2, \dots, h_{n-1})}{n-1}(y^j - y),$$

where the polynomials $\Theta_j^{(n)}$, $2 \leq j \leq n$, *are (recursively) determined by*

$$\sum_{\nu=2}^{n-1} h_\nu \left(\sum_{\substack{r_1+\cdots+r_\nu=n \\ r_j\geq 1}} \left(\prod_{j=1}^{\nu} P_{r_j}(y) \right) - (n-\nu+1)P_{n-\nu+1}(y) \right)$$
$$= \sum_{j=2}^{n} \Theta_j^{(n)}(h_2, \dots, h_{n-1})y^j.$$ □

Theorem 13. *Each solution* $G(y, x)$ *of* ($\mathrm{AJ}_{\mathrm{formal}}$) *with* $G(y, x) \equiv yx \bmod x^2$ *is a solution of* ($\mathrm{T}_{\mathrm{formal}}$).

For proving this theorem we show that

$$U(y,z,x) := G(yz,x),$$
$$V(y,z,x) := G\big(z, G(y,x)\big)$$

both satisfy the system

$$H(x)\frac{\partial}{\partial x} f(y,z,x) = H\big(f(y,z,x)\big),$$
$$f(y,z,x) = yzx + \sum_{n\geq 2} f_n(y,z)x^n,$$

where $\operatorname{ord} f(y,z,x) \geq 1$. This system has the unique solution $f(y,z,x) = yzx + \sum_{n\geq 2} P_n(yz)x^n$ with exactly the same polynomials P_n as in Theorem 12. □

If we write $G(y,x)$ in the form $G(y,x) = \sum_{n\geq 1} \phi_n(x)y^n$, then $\phi_1(x) \in \Gamma_1$, and both $(\phi_n(x))_{n\geq 1}$ and $(\phi_n(x)y^n)_{n\geq 1}$ are summable families in $(\mathbb{C}[y])[\![x]\!]$. The formal Aczél–Jabotinsky equation implies that

$$H(x)\sum_{n\geq 1} \phi_n'(x)y^n = \sum_{\nu\geq 1} h_\nu \left[\sum_{n\geq 1} \phi_n(x)y^n\right]^\nu. \tag{19}$$

We note that (19) is satisfied if and only if

$$H(x)\phi_n'(x) = \phi_n(x) + \sum_{\nu=2}^{n} h_\nu \sum_{\substack{r_1+\cdots+r_\nu=n\\ r_j\geq 1}} \left(\prod_{j=1}^{\nu} \phi_{r_j}(x)\right) \tag{19_n}$$

holds true for all $n \geq 1$. These are Briot–Bouquet differential equations (cf. [17, section 5.2] [13, section 11.1], [14, section 12.6]). The following theorem can be proved by direct calculations similar to Theorem 6 but would also follow from the theory of Briot–Bouquet equations:

Theorem 14. *Consider* $H(x) = x + h_2x^2 + \cdots$.

1. *For every* $\varphi_1 \in \mathbb{C}$*, there is exactly one solution*

$$\phi_1(x) \equiv \varphi_1 x \bmod x^2$$

of (19_1).

2. *If* $\phi_1 = \varphi_1 x + \cdots$, $\varphi_1 \neq 0$*, is a solution of* (19_1)*, then for each* $n \geq 2$ *there exists exactly one solution* $\phi_n(x)$ *of* (19_n)*. It is given by* $\phi_n(x) = \varphi_n[\phi_1(x)]^n$*, where* $\varphi_1 = 1$ *and*

$$\varphi_n = \sum_{\nu=2}^{n} \frac{h_\nu}{n-1} \sum_{r_1+\cdots+r_\nu=n} \prod_{j=1}^{\nu} \varphi_{r_j}, \qquad n \geq 2.$$

Consequently, φ_n *does not depend on the choice of* ϕ_1. □

Based on these results it is possible to give another simple proof of Theorem 7.

3. The cocycle equations

In [2], [3], [4] and [5], we were studying the problem of a covariant embedding of the linear functional equation $\varphi(p(x)) = a(x)\varphi(x) + b(x)$ for suitable formal power series $a(x)$, $b(x)$ and $p(x)$, with respect to an iteration group $F = (F(s,x))_{s\in\mathbb{C}}$. In this context, we always assume that $\operatorname{ord}(a(x)) = 0$ and $\operatorname{ord}(p(x)) = 1$. For describing the covariant embeddings, we have to solve the two cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha\big(t, F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha\big(t, F(s,x)\big) + \beta\big(t, F(s,x)\big), \qquad s,t\in\mathbb{C}, \tag{Co2}$$

under the boundary conditions

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

for

$$\alpha(s,x) = \sum_{n\geq 0}\alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0}\beta_n(s)x^n.$$

Again we restrict ourselves to iteration groups F of type I. Moreover, it is possible to consider just the normal forms $F(s,x) = c_1(s)x$ since the next theorem holds true:

Theorem 15. *Consider the iteration group $F(s,x) = S^{-1}(c_1(s)S(x))$ for $c_1 \neq 1$ and $S(x)\in\Gamma_1$ and the normal form $\tilde{F}(s,x) = c_1(s)x$. The system* ((Co1), (Co2), (B1)) *is equivalent to the system*

$$\tilde{\alpha}(s+t,y) = \tilde{\alpha}(s,y)\tilde{\alpha}\big(t, c_1(s)y\big), \qquad s,t\in\mathbb{C}, \tag{$\tilde{\text{C}}$o1}$$

$$\tilde{\beta}(s+t,y) = \tilde{\beta}(s,y)\tilde{\alpha}\big(t, c_1(s)y\big) + \tilde{\beta}\big(t, c_1(s)y\big), \quad s,t\in\mathbb{C}, \tag{$\tilde{\text{C}}$o2}$$

and

$$\tilde{\alpha}(0,y) = 1, \qquad \tilde{\beta}(0,y) = 0, \tag{$\tilde{\text{B}}$1}$$

where $\tilde{\alpha}(s,y) = \alpha\big(s, S^{-1}(y)\big)$ and $\tilde{\beta}(s,y) = \beta\big(s, S^{-1}(y)\big)$.

Proof. The boundary condition ($\tilde{\text{B}}$1) is clearly equivalent to (B1). Let α be a solution of (Co1), then

$$\begin{aligned}\tilde{\alpha}(s+t,y) &= \alpha\big(s+t, S^{-1}(y)\big) = \alpha\big(s, S^{-1}(y)\big)\alpha\big(t, F(s, S^{-1}(y))\big)\\ &= \tilde{\alpha}(s,y)\alpha\big(t, S^{-1}(c_1(s)S(S^{-1}(y)))\big) = \tilde{\alpha}(s,y)\alpha\big(t, S^{-1}(c_1(s)y)\big)\\ &= \tilde{\alpha}(s,y)\tilde{\alpha}\big(t, c_1(s)y\big).\end{aligned}$$

Hence, $\tilde{\alpha}$ satisfies ($\tilde{\text{C}}$o1). Conversely, if $\tilde{\alpha}$ satisfies ($\tilde{\text{C}}$o1), then similar computations prove that $\alpha(s,x) = \tilde{\alpha}(s, S(x))$ satisfies (Co1).

If β is a solution of (Co2), then

$$\begin{aligned}\tilde\beta(s+t,y) &= \beta\big(s+t,S^{-1}(y)\big)\\ &= \beta\big(s,S^{-1}(y)\big)\alpha\big(t,F(s,S^{-1}(y))\big)+\beta\big(t,F(s,S^{-1}(y))\big)\\ &= \tilde\beta(s,y)\alpha\big(t,S^{-1}(c_1(s)S(S^{-1}(y)))\big)+\beta\big(t,S^{-1}(c_1(s)S(S^{-1}(y)))\big)\\ &= \tilde\beta(s,y)\alpha\big(t,S^{-1}(c_1(s)y)\big)+\beta\big(t,S^{-1}(c_1(s)y)\big)\\ &= \tilde\beta(s,y)\tilde\alpha\big(t,c_1(s)y\big)+\tilde\beta\big(t,c_1(s)y\big).\end{aligned}$$

Similarly we show that if $\tilde\beta$ satisfies ($\tilde{\mathrm{C}}$o2), then $\beta(s,x)=\tilde\beta(s,S(x))$ satisfies (Co2). $\square$

The solutions of the first cocycle equation (Co1) under (B1) are easily obtained. By comparison of coefficients we derive that α_0 is a generalized exponential function and $\hat\alpha(s,x):=\frac{\alpha(s,x)}{\alpha_0(s))}$ is a solution of (Co1) and (B1). Conversely if α_0 is a generalized exponential function and $\hat\alpha(s,x)=1+\sum_{n\geq 1}\hat\alpha_n(s)x^n$ is a solution of (Co1) and (B1), then $\alpha(s,x):=\alpha_0(s)\hat\alpha(s,x)$ satisfies ($\bar{\mathrm{C}}$o1) and (B1).

Let $\gamma(s,x)=\sum_{n\geq 1}\gamma_n(s)x^n$ be the formal logarithm $\ln(\hat\alpha(s,x))$. Then $\hat\alpha(s,x)$ is a solution of $\overline{(\mathrm{Co1})}$ and (B1) if and only if γ satisfies

$$\gamma(s+t,x)=\gamma(s,x)+\gamma\big(t,c_1(s)x\big) \qquad (\mathrm{Co1}')$$

and

$$\gamma(0,x)=0. \qquad (\mathrm{B1}')$$

By [4, Corollary 3] this means that $\gamma(s,x)=\tilde E(c_1(s)x)-\tilde E(x)$ for some $\tilde E(x)$ with $\operatorname{ord}(\tilde E(x))\geq 1$. Therefore $\hat\alpha(s,x)=\exp(\gamma(s,x))=\frac{E(c_1(s)x)}{E(x)}$ with $E(x)=\exp(\tilde E(x))\equiv 1 \bmod x$ which is a multiplicative unit in $\mathbb{C}[\![x]\!]$. Summarizing, the general solution of (Co1) and (B1) is

$$\alpha(s,x)=\alpha_0(s)\frac{E(c_1(s)x)}{E(x)} \qquad (20)$$

for a generalized exponential function $\alpha_0(s)$ and a formal series $E(x)$ with $E(x)\equiv 1 \bmod x$. The transformation of (Co1) to (Co1$'$) allows us to find the solution by direct computation, and it is not necessary to study differential equations derived from (Co1). The formal version of (Co1$'$) is

$$\Gamma(ST,x)=\Gamma(S,x)+\Gamma(T,Sx) \qquad (\mathrm{Co1}_{\mathrm{formal}})$$

for

$$\Gamma(S,x)=\sum_{n\geq 1}P_n(S)x^n\in(\mathbb{C}[S])[\![x]\!]$$

with $\Gamma(1,x)=0$. It follows immediately that $P_n(S)=\tilde E_n\cdot(S^n-1)$, $n\geq 1$, with an arbitrary $\tilde E_n\in\mathbb{C}$. If we introduce by $S=1+U$ a new indeterminate U, then the transformed equation (Co1$_{\mathrm{formal}}$) may be considered in the ring $\mathbb{C}[\![U,x]\!]$. As in

Theorem 8 it turns out that there exist only the polynomial solutions which have already occurred above.

The situation of the second cocycle equations is again more complicated. Before we describe the formal version of (Co2) we collect some properties of generalized exponential functions.

Two generalized exponential functions $e_1, e_2 \colon \mathbb{C} \to \mathbb{C}^*$ are called algebraically independent if each polynomial $P(y,z) \in \mathbb{C}[y,z]$ such that $P(e_1(s), e_2(s)) = 0$ for all $s \in \mathbb{C}$ vanishes identically. Otherwise they are called algebraically dependent.

Two generalized exponential functions $e_1, e_2 \colon \mathbb{C} \to \mathbb{C}^*$ are called multiplicatively independent if a relation $e_1^{r_1} e_2^{r_2} = 1$ with $(r_1, r_2) \in \mathbb{Z}^2$ is only possible for $(r_1, r_2) = (0,0)$. Otherwise they are called multiplicatively dependent.

Lemma 16. 1. *Two generalized exponential functions are algebraically independent if and only if they are multiplicatively independent.*

2. *Assume that e_1 and e_2 are algebraically independent generalized exponential functions and consider a polynomial $P(x_1, x_2, x_3, x_4)$ over $\mathbb{C}$. If*

$$P(e_1(s), e_1(t), e_2(s), e_2(t)) = 0 \quad \textit{for all} \ \ s, t \in \mathbb{C},$$

then $P = 0$.

3. *Let e be a generalized exponential function. For each integer $n \in \mathbb{Z} \setminus \{0\}$ there exists exactly one generalized exponential function f so that $f^n = e$.*
4. *Assume that e_1 and e_2 are multiplicatively dependent generalized exponential functions. Then there exists a generalized exponential function f and $s_1, s_2 \in \mathbb{Z}$ so that $e_1 = f^{s_1}$ and $e_2 = f^{s_2}$.*
5. *Assume that $a \neq 0$ is an additive function and $e \neq 1$ is a generalized exponential function. Then a and e are algebraically independent, i. e. if there exists a polynomial $P(x_1, x_2) \in \mathbb{C}[x_1, x_2]$ so that $P(e(s), a(s)) = 0$ for all $s \in \mathbb{C}$, then $P = 0$.*
6. *Assume that $a \neq 0$ is an additive function and $e \neq 1$ is a generalized exponential function, and let $P(x_1, x_2, x_3, x_4) \in \mathbb{C}[x_1, x_2, x_3, x_4]$. If*

$$P(e(s), e(t), a(s), a(t)) = 0 \quad \textit{for all} \ \ s, t \in \mathbb{C},$$

then $P = 0$.

7. *Assume that $a \neq 0$ is an additive function and $e \neq 1$ is a generalized exponential function, and let $P(x_1, x_2) \in \mathbb{C}(x_1)[x_2]$ be a rational function in x_1 and a polynomial in x_2. If $P(e(s), a(s)) = 0$ for all $s, t \in \mathbb{C}$, then $P = 0$.*
8. *Assume that $a \neq 0$ is an additive function and $e \neq 1$ is a generalized exponential function, and let $P(x_1, x_2, x_3, x_4) \in \mathbb{C}(x_1, x_2)[x_3, x_4]$ be a rational function in x_1, x_2 and a polynomial in x_3, x_4. If $P(e(s), e(t), a(s), a(t)) = 0$ for all $s, t \in \mathbb{C}$, then $P = 0$.*

Proof. 1. See Theorem 4 of [26].

2. Consider a polynomial P so that $P(e_1(s), e_1(t), e_2(s), e_2(t)) = 0$. There exist

integers k, ℓ and polynomials $P_{ij}(x_1, x_3)$, $0 \le i \le k$, $0 \le j \le \ell$, so that

$$P(x_1, x_2, x_3, x_4) = \sum_{i=0}^{k} \sum_{j=0}^{\ell} P_{ij}(x_1, x_3) x_2^i x_4^j.$$

Consider some $s_0 \in \mathbb{C}$. Then, for all $t \in \mathbb{C}$,

$$0 = P(e_1(s_0), e_1(t), e_2(s_0), e_2(t)) = \sum_{i=0}^{k} \sum_{j=0}^{\ell} P_{ij}\big(e_1(s_0), e_2(s_0)\big) e_1(t)^i e_2(t)^j.$$

Since e_1 and e_2 are algebraically independent, we have $P_{ij}(e_1(s_0), e_2(s_0)) = 0$ for $0 \le i \le k$, $0 \le j \le \ell$. These relations hold true for every $s_0 \in \mathbb{C}$, whence $P_{ij} = 0$ for $0 \le i \le k$, $0 \le j \le \ell$. Consequently $P = 0$.

3. Consider $n \neq 0$ and let $f(t) := e(t/n)$. Then f is a generalized exponential function and it is easy to prove that $e(t) = f(nt) = f(t)^n$ for all $t \in \mathbb{C}$. If h is a generalized exponential function satisfying $h^n = e = f^n$, then $h(nt) = h(t)^n = e(t) = f(t)^n = f(nt)$ for all $t \in \mathbb{C}$. Thus $h = f$.

4. Assume that e_1 and e_2 are multiplicatively dependent. Then there exists $(r_1, r_2) \in \mathbb{Z} \setminus \{(0,0)\}$ such that $e_1^{r_1} e_2^{r_2} = 1$. Then $e_1^{r_1} = e_2^{-r_2}$.

If $r_1 = 0$, then $e_2^{-r_2} = 1$, thus $e_2 = 1$. Hence let $f = e_1$, $s_1 = 0$ and $s_2 = 1$. If $r_2 = 0$, then $e_1^{r_1} = 1$, thus $e_1 = 1$. Hence let $f = e_2$, $s_1 = 1$ and $s_2 = 0$. If $r_1 r_2 \neq 0$, then there exists a unique generalized exponential function f so that $f^{r_1 r_2} = e_1^{r_1} = e_2^{-r_2}$. Hence $f^{r_2} = e_1$ and $f^{-r_1} = e_2$.

5. See Theorem 7 of [26].

6. The proof is similar to the proof of assertion 2.

7. The assertion follows from assertion 5.

8. The proof follows from 7. as assertion 6 follows from assertion 5. □

Assume that $\alpha(s, x)$, given by (20), satisfies (Co1) and (B1). Let

$$\delta(s, x) := \frac{\beta(s, x)}{\alpha(s, x)},$$

then $\beta(s, x)$ is a solution of (Co2) and (B1) if and only if

$$\delta(s + t, x) = \delta(s, x) + \frac{E(x)}{\alpha_0(s) E\big(c_1(s)x\big)}\, \delta\big(t, c_1(s)x\big) \tag{Co2$'$}$$

and $\delta(0, x) = 0$. For

$$\Delta(s, x) := \frac{\delta(s, x)}{E(x)},$$

we deduce that $\delta(s, x)$ is a solution of (Co2$'$) and $\delta(0, x) = 0$ if and only if $\Delta(s, x)$ satisfies

$$\Delta(s + t, x) = \Delta(s, x) + \frac{1}{\alpha_0(s)} \Delta(t, c_1(s)x) \tag{Co2$''$}$$

and

$$\Delta(0,x) = 0. \tag{B1$''$}$$

If we write $\Delta(s,x)$ as $\sum_{n\geq 0}\Delta_n x^n$, then $\Delta(s,x)$ satisfies (Co2$''$) if and only if

$$\Delta_n(s+t) = \Delta_n(s) + \alpha_0(s)^{-1}\Delta_n(t)c_1(s)^n, \qquad n \geq 0. \tag{21}$$

Assume that $\Delta(s,x)$ satisfies (Co2$''$). Since $\Delta(s+t) = \Delta(t+s)$ we obtain

$$\Delta_n(t)(1-\alpha_0(s)^{-1}c_1(s)^n) = \Delta_n(s)\big(1-\alpha_0(t)^{-1}c_1(t)^n\big), \qquad n \geq 0.$$

If α_0 and c_1 are algebraically independent, then $\alpha_0 \neq c_1^n$ for all $n \in \mathbb{Z}$, whence there exists a sequence $(t_n)_{n\geq 0}$ with $t_n \in \mathbb{C}$ so that $1-\alpha_0(t_n)^{-1}c_1(t_n)^n \neq 0$ for $n \geq 0$. Consequently

$$\Delta_n(s) = \frac{\Delta_n(t_n)}{1-\alpha_0(t_n)^{-1}c_1(t_n)^n}\big(1-\alpha_0(s)^{-1}c_1(s)^n\big), \qquad n \geq 0. \tag{22}$$

For $n \geq 0$, the coefficients $\Delta_n(s)$ are polynomials $D_n(c_1(s), \alpha_0(s)^{-1})$ in c_1 and in α_0^{-1}, thus for all $s,t \in \mathbb{C}$ we have

$$\begin{aligned}
D_n\big(c_1(s)c_1(t), \alpha_0(s)^{-1}\alpha_0(t)^{-1}\big) &= D_n\big(c_1(s+t), \alpha_0(s+t)^{-1}\big)\\
&= \Delta_n(s+t)\\
&\overset{(21)}{=} D_n\big(c_1(s), \alpha_0(s)^{-1}\big)\\
&\quad + \alpha_0(s)^{-1}D_n\big(c_1(t), \alpha_0(t)^{-1}\big)c_1(s)^n.
\end{aligned}$$

Since α_0 and c_1 are algebraically independent generalized exponential functions, by Lemma 16 we obtain the polynomial identities

$$D_n(ST, \sigma\tau) = D_n(S,\sigma) + \sigma D_n(T,\tau)S^n, \qquad n \geq 0,$$

in independent variables S, T, σ and τ. This system of identities and $\Delta(0,x) = 0$ is equivalent to the formal cocycle equation

$$D(ST, \sigma\tau, x) = D(S,\sigma,x) + \sigma D(T,\tau,Sx) \tag{Co2$_{\mathrm{formal}_1}$}$$

in $(\mathbb{C}[S,T,\sigma,\tau])[\![x]\!]$ for

$$D(S,\sigma,x) = \sum_{n\geq 0} D_n(S,\sigma)x^n,$$

$D_n(S,\sigma) \in \mathbb{C}[S,\sigma]$, $n \geq 0$, and the boundary condition

$$D(1,1,x) = 0. \tag{B2$_1$}$$

Theorem 17. *Let α_0 and c_1 be algebraically independent generalized exponential functions. If $\Delta(s,x)$ is a solution of* (Co2$''$)*, then its coefficient functions are polynomials $D_n\big(c_1(s), \alpha_0(s)^{-1}\big)$ in $c_1(s)$ and $\alpha_0(s)^{-1}$. Moreover, $\Delta(s,x) = \sum_{n\geq 0} D_n\big(c_1(s), \alpha_0(s)^{-1}\big)x^n$ is a solution of* (Co2$''$) *and* (B1$''$) *if and only if $D(S,\sigma,x) = \sum_{n\geq 0} D_n(S,\sigma)x^n$ satisfies* (Co2$_{\mathrm{formal}_1}$) *and* (B2$_1$) *in* $(\mathbb{C}[S,\sigma])[\![x]\!]$. $\square$

If α_0^{-1} and c_1 are algebraically dependent, then they are multiplicatively dependent and there exist a generalized exponential function $e \neq 1$ and integers $r_0, r_1 \in \mathbb{Z}$, so that $\alpha_0^{-1} = e^{r_0}$ and $c_1 = e^{r_1}$. Without loss of generality $r_1 > 0$.

Assume that $\Delta(s,x)$ is a solution of (Co2$''$). If $\alpha_0^{-1} c_1^n = e^{r_0+nr_1} \neq 1$ for all $n \geq 0$, then there exists a sequence $(t_n)_{n\geq 0}$ of complex numbers so that $1 - \alpha_0(t_n)^{-1} c_1(t_n)^n \neq 0$ for $n \geq 0$ and $\Delta_n(s)$ is given by (22). For $n \geq 0$, the coefficients $\Delta_n(s)$ are polynomials in c_1 and in α_0^{-1}, thus

$$\begin{aligned} D_n\big(e^{r_1}(s)e^{r_1}(t), e^{r_0}(s)e^{r_0}(t)\big) &= D_n\big(c_1(s)c_1(t), \alpha_0(s)^{-1}\alpha_0(t)^{-1}\big) \\ &= D_n\big(c_1(s+t), \alpha_0(s+t)^{-1}\big) \\ &= \Delta_n(s+t) \\ &\overset{(21)}{=} D_n\big(c_1(s), \alpha_0(s)^{-1}\big) \\ &\quad + \alpha_0(s)^{-1} D_n\big(c_1(t), \alpha_0(t)^{-1}\big) c_1(s)^n \\ &= D_n\big(e^{r_1}(s), e^{r_0}(s)\big) \\ &\quad + e^{r_0}(s) D_n\big(e^{r_1}(t), e^{r_0}(t)\big) e^{nr_1}(s). \end{aligned}$$

When we replace $e(s)$ by the indeterminate S and $e(t)$ by T, we obtain for $n \geq 0$ the identities

$$D_n\big((ST)^{r_1}, (ST)^{r_0}\big) = D_n(S^{r_1}, S^{r_0}) + S^{r_0+nr_1} D_n(T^{r_1}, T^{r_0}).$$

Since it is possible that r_0 is a negative integer $\tilde{D}_n(S) := D_n(S^{r_1}, S^{r_0})$ is a rational function in S which satisfies

$$\tilde{D}_n(ST) = \tilde{D}_n(S) + S^{r_0+nr_1} \tilde{D}_n(T).$$

This system of identities and $\Delta(0,x) = 0$ is equivalent to the formal cocycle equation

$$D(ST,x) = D(S,x) + S^{r_0} D(T, S^{r_1}x) \qquad (\mathrm{Co2}_{\mathrm{formal}_2})$$

in $(\mathbb{C}(S,T))[\![x]\!]$ for

$$D(S,x) = \sum_{n\geq 0} \tilde{D}_n(S) x^n,$$

$\tilde{D}_n(S) \in \mathbb{C}(S)$, $n \geq 0$, and the boundary condition

$$D(1,x) = 0. \qquad (\mathrm{B2}_2)$$

Theorem 18. *Assume that $\alpha_0^{-1} = e^{r_0}$ and $c_1 = e^{r_1}$ for a generalized exponential function $e \neq 1$, $r_0, r_1 \in \mathbb{Z}$, and $e^{r_0+nr_1} \neq 1$ for all $n \geq 0$. If $\Delta(s,x)$ is a solution of* (Co2$''$)*, then its coefficient functions are rational functions $\tilde{D}_n(e(s))$ in $e(s)$. Moreover, $\Delta(s,x) = \sum_{n\geq 0} \tilde{D}_n\big(e(s)\big)x^n$ is a solution of* (Co2$''$) *and* (B1$''$) *if and only if $D(S,x) = \sum_{n\geq 0} \tilde{D}_n(S)x^n$ satisfies* ($\mathrm{Co2}_{\mathrm{formal}_2}$) *and* ($\mathrm{B2}_2$) *in* $(\mathbb{C}(S))[\![x]\!]$. $\square$

Assume that $\Delta(s,x)$ is a solution of (Co2$''$). If $\alpha_0 = c_1^{n_0}$ for a nonnegative integer n_0, then the coefficient functions Δ_n are described by (22) for $n \neq n_0$. For $n = n_0$ we obtain from (21) that Δ_{n_0} is additive. Therefore, in this situation the coefficient functions Δ_n, $n \geq 0$, are polynomials in c_1, α_0^{-1} and in an additive function A. Similarly as above there exist a nontrivial generalized exponential function e and integers $r_0, r_1 \in \mathbb{Z}$ such that $\alpha_0^{-1} = e^{r_0}$ and $c_1 = e^{r_1}$. Without loss of generality we assume that $r_1 > 0$. Then $\Delta_n(s)$, $n \geq 0$, is a polynomial in $e^{r_0}(s)$, $e^{r_1}(s)$ and $A(s)$. Hence $\Delta_n(s) = D_n(e(s), A(s))$ which is a rational function in $e(s)$ and a polynomial in $A(s)$. If $A \neq 0$ it is possible to replace $e(s)$, $e(t)$, $A(s)$ and $A(t)$ by the independent variables S, T, U and V, respectively, and we obtain

$$D_n(ST, U+V) = D_n(S,U) + S^{r_0+nr_1} D_n(T,V), \qquad n \geq 0,$$

which is a system of identities in $\mathbb{C}(S,T)[U,V]$. This system and $\Delta(0,x) = 0$ is equivalent to

$$D(ST, U+V, x) = D(S,U,x) + S^{r_0} D(T,V,S^{r_1}x) \qquad (\mathrm{Co2}_{\mathrm{formal}_3})$$

in $(\mathbb{C}(S,T)[U,V])[\![x]\!]$ for

$$D(S,U,x) = \sum_{n\geq 0} D_n(S,U)x^n,$$

$D_n(S,U) \in \mathbb{C}(S)[U]$, $n \geq 0$, and the boundary condition

$$D(1,0,x) = 0. \qquad (\mathrm{B2}_3)$$

If $A = 0$, then $\Delta_n(s) = D_n(e(s))$ which is a rational function in $e(s)$. Now it is possible to replace $e(s)$ and $e(t)$ by the independent variables S and T and we obtain again the formal cocycle equation ($\mathrm{Co2}_{\mathrm{formal}_2}$) and the boundary condition ($\mathrm{B2}_2$).

Theorem 19. *Assume that $\alpha_0^{-1} = e^{r_0}$ and $c_1 = e^{r_1}$ for a generalized exponential function $e \neq 1$, $r_0, r_1 \in \mathbb{Z}$, and $e^{r_0+n_0r_1} = 1$ for some $n_0 \geq 0$. If $\Delta(s,x)$ is a solution of* (Co2$''$)*, then there exists an additive function A so that the coefficient functions $D_n\big(e(s), A(s)\big)$ of $\Delta(s,x)$ are rational functions in $e(s)$ and polynomials in $A(s)$.*

If $A \neq 0$, then $\Delta(s,x) = \sum_{n\geq 0} D_n\big(e(s), A(s)\big)x^n$ is a solution of (Co2$''$) *and* (B1$''$) *if and only if $D(S,U,x) = \sum_{n\geq 0} D_n(S,U)x^n$ satisfies* ($\mathrm{Co2}_{\mathrm{formal}_3}$) *and* ($\mathrm{B2}_3$) *in $(\mathbb{C}(S)[U])[\![x]\!]$.*

If $A = 0$, let $\tilde{D}_n(e(s)) := D_n(e(s), 0)$. Then $\Delta(s,x) = \sum_{n\geq 0} \tilde{D}_n\big(e(s)\big)x^n$ is a solution of (Co2$''$) *and* (B1$''$) *if and only if $D(S,x) = \sum_{n\geq 0} \tilde{D}_n(S)x^n$ satisfies* ($\mathrm{Co2}_{\mathrm{formal}_2}$) *and* ($\mathrm{B2}_2$) *in $(\mathbb{C}(S))[\![x]\!]$.* □

The three formal equations ($\mathrm{Co2}_{\mathrm{formal}_1}$), ($\mathrm{Co2}_{\mathrm{formal}_2}$) and ($\mathrm{Co2}_{\mathrm{formal}_3}$) together with ($\mathrm{B2}_1$), ($\mathrm{B2}_2$) and ($\mathrm{B2}_3$) motivate the study of the following system

$$D(ST, \sigma\tau, U+V, x) = D(S,\sigma,U,x) + \sigma^{\lambda} S^{\mu} D(T,\tau,V,S^{\nu}x) \qquad (\mathrm{Co2}_{\mathrm{formal}})$$

for $\lambda, \mu, \nu \in \mathbb{Z}$, $\nu > 0$, where

$$D(S,\sigma,U,x) = \sum_{n\geq 0} D_n(S,\sigma,U)x^n \in (\mathbb{C}(S)[\sigma,U])[\![x]\!],$$

and the boundary condition

$$D(1,1,0,x) = 0. \tag{B2}$$

3.1. A system of formal differential equations related to $(\mathrm{Co2}_{\mathrm{formal}})$

Differentiation of $(\mathrm{Co2}_{\mathrm{formal}})$ with respect to T and substituting $T = 1$, $\tau = 1$, $V = 0$ yields

$$S\frac{\partial}{\partial S}\, D(S,\sigma,U,x) = \sigma^\lambda S^\mu K(S^\nu x), \tag{$\mathrm{Co2D1}_{\mathrm{formal}}$}$$

where $K(x) = \frac{\partial}{\partial T}\, D(T,1,0,x)|_{T=1}$. Similarly, differentiation with respect to τ or V, respectively, and substituting $T = 1$, $\tau = 1$, $V = 0$ yields

$$\sigma\frac{\partial}{\partial \sigma}\, D(S,\sigma,U,x) = \sigma^\lambda S^\mu L(S^\nu x) \tag{$\mathrm{Co2D2}_{\mathrm{formal}}$}$$

and

$$\frac{\partial}{\partial U}\, D(S,\sigma,U,x) = \sigma^\lambda S^\mu M(S^\nu x), \tag{$\mathrm{Co2D3}_{\mathrm{formal}}$}$$

where $L(x) = \frac{\partial}{\partial \tau}\, D(1,\tau,0,x)|_{\tau=1}$ and $M(x) = \frac{\partial}{\partial V}\, D(1,1,V,x)|_{V=0}$.

From $(\mathrm{Co2D3}_{\mathrm{formal}})$ and (B2) we immediately obtain

$$D(S,\sigma,U,x) = \sigma^\lambda S^\mu M(S^\nu x)U + \tilde{D}(S,\sigma,x)$$

with $\tilde{D}(S,\sigma,x) \in (\mathbb{C}(S)[\sigma])[\![x]\!]$ and $\tilde{D}(1,1,x) = 0$. Inserting this representation of $D(S,\sigma,U,x)$ into $(\mathrm{Co2D2}_{\mathrm{formal}})$ we get

$$\frac{\partial}{\partial \sigma}\, \tilde{D}(S,\sigma,x) = \sigma^{\lambda-1} S^\mu \Big(L(S^\nu x) - \lambda M(S^\nu x)U\Big).$$

Now we consider three different cases.

Case 1. Assume that $\lambda > 0$. Since the left hand side does not depend on U, necessarily $M(S^\nu x) = 0$ and

$$\frac{\partial}{\partial \sigma}\, \tilde{D}(S,\sigma,x) = \sigma^{\lambda-1} S^\mu L(S^\nu x).$$

Therefore

$$D(S,\sigma,U,x) = \tilde{D}(S,\sigma,x) = \frac{1}{\lambda}\sigma^\lambda S^\mu L(S^\nu x) + \hat{D}(S,x)$$

with $\hat{D}(S,x) \in (\mathbb{C}(S))[\![x]\!]$ and $\hat{D}(1,x) = -L(x)/\lambda$.

Case 2. If $\lambda < 0$, then necessarily $M(S^\nu x) = 0$ and, moreover, $L(S^\nu x) = 0$ since the coefficients of $\tilde{D}(S, \sigma, x)$ are supposed to be polynomials in σ. Thus

$$\frac{\partial}{\partial \sigma}\, \tilde{D}(S, \sigma, x) = 0$$

and

$$D(S, \sigma, U, x) = \tilde{D}(S, \sigma, x) = \hat{D}(S, x)$$

with $\hat{D}(S, x) \in (\mathbb{C}(S))[\![x]\!]$ and $\hat{D}(1, x) = 0$.

Case 3. If $\lambda = 0$, then $L(S^\nu x) = 0$ since the coefficients of $\tilde{D}(S, \sigma, x)$ are supposed to be polynomials in σ. Thus

$$\frac{\partial}{\partial \sigma}\, \tilde{D}(S, \sigma, x) = 0,$$

whence $\tilde{D}(S, \sigma, x) = \hat{D}(S, x)$, with $\hat{D}(S, x) \in (\mathbb{C}(S))[\![x]\!]$, and

$$D(S, \sigma, U, x) = S^\mu M(S^\nu x)U + \hat{D}(S, x)$$

with $\hat{D}(1, x) = 0$. Finally we have to insert these three different representations of D into $(\mathrm{Co2D1}_{\mathrm{formal}})$.

In Case 1 we obtain the differential equation

$$\frac{\partial}{\partial S}\, \hat{D}(S, x) = \sigma^\lambda S^{\mu-1}\Big(-\frac{\mu}{\lambda} L(S^\nu x) - \frac{\nu}{\lambda} S^\nu x L'(S^\nu x) + K(S^\nu x)\Big). \tag{23}$$

Since the left hand side is independent of σ, we have

$$K(S^\nu x) = \frac{1}{\lambda}\Big(\mu L(S^\nu x) + \nu S^\nu x L'(S^\nu x)\Big).$$

Hence we deduce the relation

$$K_n = \frac{\mu + n\nu}{\lambda} L_n, \qquad n \geq 0,$$

between the coefficients of $K(x) = \sum_{n\geq 0} K_n x^n$ and $L(x) = \sum_{n\geq 0} L_n x^n$. If $\mu + n\nu \neq 0$ for all $n \geq 0$, then $L(x)$ is uniquely determined by $K(x)$,

$$L(x) = \sum_{n\geq 0} \frac{\lambda K_n}{\mu + n\nu} x^n.$$

If $\mu + n_0\nu = 0$ for some $n_0 \geq 0$, then

$$L(x) = \sum_{\substack{n\geq 0\\ n\neq n_0}} \frac{\lambda K_n}{\mu + n\nu} x^n + L_{n_0} x^{n_0}.$$

As explained above, (23) is of the form $\frac{\partial}{\partial S}\, \hat{D}(S, x) = 0$, thus $\hat{D}(S, x) = \bar{D}(x) \in \mathbb{C}[\![x]\!]$ and $\bar{D}(x) = \hat{D}(1, x) = -\frac{1}{\lambda} L(x)$. Consequently, if $\mu + n\nu \neq 0$ for all $n \geq 0$, then

$$D(S, \sigma, U, x) = \frac{1}{\lambda}\sigma^\lambda S^\mu L(S^\nu x) - \frac{1}{\lambda} L(x) = \sum_{n\geq 0} \frac{K_n}{\mu + n\nu}(\sigma^\lambda S^{\mu+n\nu} - 1)x^n.$$

If $\mu + n_0\nu = 0$, then

$$\begin{aligned} D(S,\sigma,U,x) &= \frac{1}{\lambda}\sigma^\lambda S^\mu L(S^\nu x) - \frac{1}{\lambda}L(x) \\ &= \sum_{n\neq n_0} \frac{K_n}{\mu+n\nu}(\sigma^\lambda S^{\mu+n\nu}-1)x^n + \frac{L_{n_0}}{\lambda}(\sigma^\lambda - 1)x^{n_0}. \end{aligned}$$

In Case 2 we derive from ($\mathrm{Co2D1}_{\mathrm{formal}}$) that

$$\frac{\partial}{\partial S}\,\hat{D}(S,x) = \sigma^\lambda S^{\mu-1}K(S^\nu x).$$

Since the left hand side does not depend on σ we have $K(S^\nu x) = 0$, whence $\frac{\partial}{\partial S}\,\hat{D}(S,x) = 0$ and $\hat{D}(S,x) = \bar{D}(x) \in \mathbb{C}[\![x]\!]$. Together with $\hat{D}(1,x) = 0$ it follows that

$$D(S,\sigma,U,x) = 0.$$

In Case 3 we derive from ($\mathrm{Co2D1}_{\mathrm{formal}}$) that

$$\frac{\partial}{\partial S}\,\hat{D}(S,x) = S^{\mu-1}K(S^\nu x) - S^{\mu-1}U\Big(\mu M(S^\nu x) + \nu S^\nu x M'(S^\nu x)\Big).$$

This leads to the two equations

$$\frac{\partial}{\partial S}\,\hat{D}(S,x) = S^{\mu-1}K(S^\nu x), \tag{24}$$

$$0 = \mu M(S^\nu x) + \nu S^\nu x M'(S^\nu x). \tag{25}$$

According to (24) the coefficient functions $\hat{D}_n(S)$ of $\hat{D}(S,x)$ satisfy the differential equation

$$\hat{D}_n'(S) = K_n S^{\mu+n\nu-1}. \tag{26}$$

From the boundary condition $\hat{D}(1,x) = 0$ we deduce that

$$D_n(S) = \frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)$$

if $\mu + n\nu \neq 0$. If $\mu + n_0\nu = 0$, then $K_{n_0} = 0$ since otherwise $\hat{D}_{n_0}(S)$ as a solution of (26) is not a rational function in S.

If $M(x) = \sum_{n\geq 0} M_n x^n$ satisfies (25), then $M_n = 0$ if $\mu + n\nu \neq 0$, and M_{n_0} is not determined by (25) if $\mu + n_0\nu = 0$.

Summarizing, we have shown that $D(S,\sigma,U,x)$ is of the form

$$\begin{cases} \displaystyle\sum_{n\geq 0} \frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n, & \text{if } \mu+n\nu \neq 0 \text{ for all } n \geq 0, \\ \displaystyle\sum_{n\neq n_0} \frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n + M_{n_0}Ux^{n_0}, & \text{if } \mu + n_0\nu = 0. \end{cases}$$

Theorem 20. *The unique solution $D(S,\sigma,U,x)$ of the three differential equations* ($\mathrm{Co2D1_{formal}}$)*,* ($\mathrm{Co2D2_{formal}}$)*,* ($\mathrm{Co2D3_{formal}}$)*, and the boundary condition* (B2) *is of the form*

$$\begin{cases} \displaystyle\sum_{n\geq 0}\frac{K_n}{\mu+n\nu}(\sigma^\lambda S^{\mu+n\nu}-1)x^n & \lambda>0,\ \mu+n\nu\neq 0\ \forall n\geq 0,\\ \displaystyle\sum_{n\neq n_0}\frac{K_n}{\mu+n\nu}(\sigma^\lambda S^{\mu+n\nu}-1)x^n+\frac{L_{n_0}}{\lambda}(\sigma^\lambda-1)x^{n_0} & \lambda>0,\ \mu+n_0\nu=0,\ n_0\geq 0,\\ \displaystyle\sum_{n\geq 0}\frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n & \lambda=0,\ \mu+n\nu\neq 0\ \forall n\geq 0,\\ \displaystyle\sum_{n\neq n_0}\frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n+M_{n_0}Ux^{n_0} & \lambda=0,\ \mu+n_0\nu=0,\ n_0\geq 0,\\ 0 & \lambda<0. \end{cases}$$

Additionally, $D(S,\sigma,U,x)$ is a solution of ($\mathrm{Co2_{formal}}$)*.*

Proof. From the representation of the solution $D(S,\sigma,U,x)$ of ($\mathrm{Co2D1_{formal}}$), ($\mathrm{Co2D2_{formal}}$), ($\mathrm{Co2D3_{formal}}$), and (B2) it is straightforward to prove that $D(S,\sigma,U,x)$ satisfies ($\mathrm{Co2_{formal}}$). □

Remark 21. The solutions of ($\mathrm{Co2_{formal_1}}$), ($\mathrm{Co2_{formal_2}}$) and ($\mathrm{Co2_{formal_3}}$) can be obtained as special cases of the solutions in Theorem 20. First we specialize ($\mathrm{Co2_{formal}}$) by choosing appropriate values for λ, μ and ν, then we solve this equation and finally in the solution we replace certain indeterminates by 1 or 0. In order to get solutions of ($\mathrm{Co2_{formal_1}}$) we have to set $\lambda=1$, $\mu=0$, and $\nu=1$ in ($\mathrm{Co2_{formal}}$), then for $U=V=0$ we get

$$D(S,\sigma,x)=\sum_{n\geq 0}K_n(\sigma S^n-1)x^n.$$

For ($\mathrm{Co2_{formal_2}}$) we have to set $\lambda=0$, $\mu=r_0$, and $\nu=r_1$ in ($\mathrm{Co2_{formal}}$), then for $U=V=0$ and $\sigma=\tau=1$ we get

$$D(S,x)=\sum_{\substack{n\geq 0\\ r_0+nr_1\neq 0}}\frac{K_n}{r_0+nr_1}(S^{r_0+nr_1}-1)x^n.$$

Eventually, for ($\mathrm{Co2_{formal_3}}$) we have to set $\lambda=0$, $\mu=r_0$, and $\nu=r_1$ in ($\mathrm{Co2_{formal}}$), then for $\sigma=\tau=1$ we get that $D(S,U,x)$ is equal to

$$\begin{cases} \displaystyle\sum_{n\geq 0}\frac{K_n}{r_0+nr_1}(S^{r_0+nr_1}-1)x^n, & \text{if } r_0+nr_1\neq 0\ \forall n\geq 0,\\ \displaystyle\sum_{n\neq n_0}\frac{K_n}{r_0+nr_1}(S^{r_0+nr_1}-1)x^n+M_{n_0}Ux^{n_0}, & \text{if } r_0+n_0r_1=0. \end{cases}$$

These solutions can easily be transformed to get the solutions of the cocycle equation (Co2) described in [4].

3.2. Another system of formal differential equations related to ($\mathrm{Co2}_{\mathrm{formal}}$)

Differentiation of ($\mathrm{Co2}_{\mathrm{formal}}$) with respect to S, σ or U, respectively, and substituting $S=1$, $\sigma=1$, $U=0$ yields the equations

$$T\frac{\partial}{\partial T}D(T,\tau,V,x)=K(x)+\mu D(T,\tau,V,x)+\nu x\frac{\partial}{\partial x}D(T,\tau,V,x), \tag{$\mathrm{Co2PD1}_{\mathrm{formal}}$}$$

$$\tau\frac{\partial}{\partial \tau}D(T,\tau,V,x)=L(x)+\lambda D(T,\tau,V,x), \tag{$\mathrm{Co2PD2}_{\mathrm{formal}}$}$$

$$\frac{\partial}{\partial V}D(T,\tau,V,x)=M(x). \tag{$\mathrm{Co2PD3}_{\mathrm{formal}}$}$$

It is possible to prove the following result:

Theorem 22. *The unique solution $D(T,\tau,V,x)$ of the three differential equations* ($\mathrm{Co2PD1}_{\mathrm{formal}}$), ($\mathrm{Co2PD2}_{\mathrm{formal}}$), ($\mathrm{Co2PD3}_{\mathrm{formal}}$), *and the boundary condition* (B2) *is of the form*

$$\begin{cases}
\sum_{n\geq 0}\frac{K_n}{\mu+n\nu}(\tau^\lambda T^{\mu+n\nu}-1)x^n & \lambda>0,\ \mu+n\nu\neq 0\ \forall n\geq 0,\\
\sum_{n\neq n_0}\frac{K_n}{\mu+n\nu}(\tau^\lambda T^{\mu+n\nu}-1)x^n+\frac{L_{n_0}}{\lambda}(\tau^\lambda-1)x^{n_0} & \lambda>0,\ \mu+n_0\nu=0,\ n_0\geq 0,\\
\sum_{n\geq 0}\frac{K_n}{\mu+n\nu}(T^{\mu+n\nu}-1)x^n & \lambda=0,\ \mu+n\nu\neq 0\ \forall n\geq 0,\\
\sum_{n\neq n_0}\frac{K_n}{\mu+n\nu}(T^{\mu+n\nu}-1)x^n+M_{n_0}Vx^{n_0} & \lambda=0,\ \mu+n_0\nu=0,\ n_0\geq 0,\\
0 & \lambda<0.
\end{cases}$$

Additionally, $D(T,\tau,V,x)$ is a solution of ($\mathrm{Co2}_{\mathrm{formal}}$). □

3.3. A system of formal Aczél–Jabotinsky equations related to ($\mathrm{Co2}_{\mathrm{formal}}$)

Combining the three equations ($\mathrm{Co2D1}_{\mathrm{formal}}$), ($\mathrm{Co2D2}_{\mathrm{formal}}$), ($\mathrm{Co2D3}_{\mathrm{formal}}$) together with ($\mathrm{Co2PD1}_{\mathrm{formal}}$), ($\mathrm{Co2PD2}_{\mathrm{formal}}$), ($\mathrm{Co2PD3}_{\mathrm{formal}}$) we obtain

$$\sigma^\lambda S^\mu K(S^\nu x)=K(x)+\mu D(S,\sigma,U,x)+\nu x\frac{\partial}{\partial x}D(S,\sigma,U,x), \tag{$\mathrm{Co2AJ1}_{\mathrm{formal}}$}$$

$$\sigma^\lambda S^\mu L(S^\nu x)=L(x)+\lambda D(S,\sigma,U,x), \tag{$\mathrm{Co2AJ2}_{\mathrm{formal}}$}$$

$$\sigma^\lambda S^\mu M(S^\nu x)=M(x). \tag{$\mathrm{Co2AJ3}_{\mathrm{formal}}$}$$

We call this the system of Aczél–Jabotinsky equations related to ($\mathrm{Co2}_{\mathrm{formal}}$). Here only the first equation is a differential equation with respect to x. Let us also note that this Aczél–Jabotinsky system has a somewhat simpler structure than the Aczél–Jabotinsky equation related to an iteration group since no substitution into D appears.

Theorem 23. *If $\lambda \neq 0$ or $\mu + n\nu \neq 0$ for all $n \geq 0$, then the system of the Aczél–Jabotinsky equations* (Co2AJ1$_{\mathrm{formal}}$), (Co2AJ2$_{\mathrm{formal}}$), (Co2AJ3$_{\mathrm{formal}}$) *has a unique solution $D(S,\sigma,U,x) \in (\mathbb{C}(S)[\sigma,U])[\![x]\!]$ given by*

$$\begin{cases} \displaystyle\sum_{n\geq 0} \frac{K_n}{\mu+n\nu}(\sigma^\lambda S^{\mu+n\nu}-1)x^n & \lambda > 0,\ \mu+n\nu \neq 0\ \forall n \geq 0, \\ \displaystyle\sum_{n\neq n_0} \frac{K_n}{\mu+n\nu}(\sigma^\lambda S^{\mu+n\nu}-1)x^n + \frac{L_{n_0}}{\lambda}(\sigma^\lambda-1)x^{n_0} & \lambda > 0,\ \mu+n_0\nu = 0,\ n_0 \geq 0, \\ \displaystyle\sum_{n\geq 0} \frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n & \lambda = 0,\ \mu+n\nu \neq 0\ \forall n \geq 0, \\ 0 & \lambda < 0. \end{cases}$$

Additionally, $D(S,\sigma,U,x)$ is a solution of (Co2$_{\mathrm{formal}}$).

If $\lambda = 0$ and $\mu + n_0\nu = 0$ for some $n_0 \geq 0$, then the three Aczél–Jabotinsky equations together with

$$D_{n_0}(ST,\sigma\tau,U+V) = D_{n_0}(S,\sigma,U) + D_{n_0}(T,\tau,V) \tag{27}$$

have a unique solution

$$D(S,\sigma,U,x) = \sum_{n\neq n_0} \frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n + M_{n_0}Ux^{n_0}.$$

Additionally, $D(S,\sigma,U,x)$ is a solution of (Co2$_{\mathrm{formal}}$).

Proof. From (Co2AJ3$_{\mathrm{formal}}$) we deduce that

$$M(x) = \begin{cases} 0, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0 \text{ and } \mu+n\nu \neq 0 \text{ for all } n \geq 0, \\ M_{n_0}x^{n_0}, & \text{if } \lambda = 0 \text{ and } \mu+n_0\nu = 0. \end{cases}$$

If $\lambda \neq 0$ the second equation (Co2AJ2$_{\mathrm{formal}}$) implies

$$D(S,\sigma,U,x) = \frac{1}{\lambda}\left(\sigma^\lambda S^\mu L(S^\nu x) - L(x)\right). \tag{28}$$

If $\lambda < 0$, then $L(x) = 0$, whence $D(S,\sigma,U,x) = 0$.
If $\lambda = 0$, then $S^\mu L(S^\nu x) = L(x)$, whence

$$L(x) = \begin{cases} 0, & \text{if } \mu+n\nu \neq 0 \text{ for all } n \geq 0, \\ L_{n_0}x^{n_0}, & \text{if } \mu+n_0\nu = 0. \end{cases}$$

From (28) and (Co2AJ1$_{\mathrm{formal}}$) we deduce that

$$\frac{\nu x}{\lambda}\left(\sigma^\lambda S^{\mu+\nu}L'(S^\nu x) - L'(x)\right) = \sigma^\lambda S^\mu K(S^\nu x) - K(x) - \frac{\mu}{\lambda}\left(\sigma^\lambda S^\mu L(S^\nu x) - L(x)\right).$$

This means

$$\sum_{n\geq 0}\frac{\mu+n\nu}{\lambda}L_n(\sigma^\lambda S^{\mu+n\nu}-1)x^n=\sum_{n\geq 0}K_n(\sigma^\lambda S^{\mu+n\nu}-1)x^n.$$

If $\mu+n\nu\neq 0$ for all $n\geq 0$, then $L_n=\lambda K_n/(\mu+n\nu)$, $n\geq 0$, and

$$D(S,\sigma,U,x)=\sum_{n\geq 0}\frac{K_n}{\mu+n\nu}(\sigma^\lambda S^{\mu+n\nu}-1)x^n.$$

If $\mu+n_0\nu=0$ for some $n_0\geq 0$, then $L_n=\lambda K_n/(\mu+n\nu)$, $n\neq n_0$, and

$$D(S,\sigma,U,x)=\sum_{n\neq n_0}\frac{K_n}{\mu+n\nu}(\sigma^\lambda S^{\mu+n\nu}-1)x^n+\frac{L_{n_0}}{\lambda}(\sigma^\lambda-1)x^{n_0}.$$

If $\lambda=0$ then (Co2AJ1$_{\mathrm{formal}}$) reduces to

$$S^\mu K(S^\nu x)-K(x)=\mu D(S,\sigma,U,x)+\nu x\frac{\partial}{\partial x}D(S,\sigma,U,x),$$

which yields

$$\sum_{n\geq 0}K_n(S^{\mu+n\nu}-1)x^n=\sum_{n\geq 0}(\mu+n\nu)D_n(S,\sigma,U)x^n.$$

If $\mu+n\nu\neq 0$ for all $n\geq 0$, then

$$D_n(S,\sigma,U)=\frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1),\qquad n\geq 0,$$

and

$$D(S,\sigma,U,x)=\sum_{n\geq 0}\frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n.$$

If $\mu+n_0\nu=0$ for some $n_0\geq 0$, then

$$D(S,\sigma,U,x)=\sum_{n\neq n_0}\frac{K_n}{\mu+n\nu}(S^{\mu+n\nu}-1)x^n+D_{n_0}(S,\sigma,U)x^{n_0}.\tag{29}$$

Now we determine $D_{n_0}(S,\sigma,U)\in\mathbb{C}(S)[\sigma,U]$ so that (27) is satisfied. Writing $D_{n_0}(S,\sigma,U)=\sum_{j=0}^d D_{n_0,j}(S,\sigma)U^j$ as a polynomial in U, we derive that

$$\sum_{j=0}^d D_{n_0,j}(ST,\sigma\tau)(U+V)^j=\sum_{j=0}^d D_{n_0,j}(S,\sigma)U^j+\sum_{j=0}^d D_{n_0,j}(T,\tau)V^j.$$

From $V=0$ or $U=0$ we immediately obtain $D_{n_0,j}(ST,\sigma\tau)=D_{n_0,j}(S,\sigma)=D_{n_0,j}(T,\tau)$ for $j\geq 1$. Therefore $D_{n_0,j}(S,\sigma)=D_{n_0,j}(T,\tau)=D_{n_0,j}\in\mathbb{C}(S)[\sigma]\cap\mathbb{C}(T)[\tau]=\mathbb{C}$ for $j\geq 1$. For $U=V$ and $j\geq 1$ we deduce $D_{n_0,j}2^jU^j=D_{n_0,j}2U^j$, thus $D_{n_0,j}=0$ for $j\geq 2$. So far we have shown that

$$D_{n_0}(S,\sigma,U)=D_{n_0,0}(S,\sigma)+D_{n_0,1}U$$

for some $D_{n_0,1} \in \mathbb{C}$. Now we prove that $D_{n_0,0}(S,\sigma) = 0$. According to (27) we have

$$D_{n_0,0}(ST,\sigma\tau) = D_{n_0,0}(S,\sigma) + D_{n_0,0}(T,\tau).$$

Differentiation with respect to σ yields

$$\tau \frac{\partial}{\partial z} D_{n_0,0}(ST,z)|_{z=\sigma\tau} = \frac{\partial}{\partial \sigma} D_{n_0,0}(S,\sigma),$$

where the left hand side is either 0 or a polynomial in τ. Hence $\frac{\partial}{\partial\sigma} D_{n_0,0}(S,\sigma) = 0$ and $D_{n_0,0}(S,\sigma) = \hat{D}_{n_0,0}(S) \in \mathbb{C}(S)$. By (27) we have $\hat{D}_{n_0,0}(ST) = \hat{D}_{n_0,0}(S) + \hat{D}_{n_0,0}(T)$. Differentiation with respect to S or T yields $T\hat{D}'_{n_0,0}(ST) = \hat{D}'_{n_0,0}(S)$ or $S\hat{D}'_{n_0,0}(ST) = \hat{D}'_{n_0,0}(T)$. Therefore $T\hat{D}'_{n_0,0}(T) = S\hat{D}'_{n_0,0}(S) = c \in \mathbb{C}(S) \cap \mathbb{C}(T) = \mathbb{C}$. Thus $\hat{D}'_{n_0,0}(S) = c/S$ and consequently $c = 0$ since otherwise $\hat{D}_{n_0,0}(S)$ were not a rational function in S. Hence $\hat{D}'_{n_0,0}(S) = 0$, which means that $\hat{D}_{n_0,0}(S)$ is constant and due to (27) it is equal to 0. Summarizing we have deduced that $D_{n_0}(S,\sigma,U) = D_{n_0,1}U$, for some $D_{n_0,1} \in \mathbb{C}$. □

We see that the system ($\mathrm{Co2AJ1}_{\mathrm{formal}}$), ($\mathrm{Co2AJ2}_{\mathrm{formal}}$), ($\mathrm{Co2AJ3}_{\mathrm{formal}}$) can have more solutions than ($\mathrm{Co2}_{\mathrm{formal}}$) from which we derived it. This situation occurs if $\mu + n_0\nu = 0$ for some $n_0 \geq 0$, confer (29), where for each $D_{n_0}(S,\sigma,U)$ we get a solution $D(S,\sigma,U,x)$ of the system of Aczél–Jabotinsky equations but in general not of ($\mathrm{Co2}_{\mathrm{formal}}$).

3.4. Further generalization of ($\mathrm{Co2}_{\mathrm{formal}}$)

Now we introduce four independent indeterminates A, B, α and β by setting $S = 1 + A$, $T = 1 + B$, $\sigma = 1 + \alpha$ and $\tau = 1 + \beta$. Then we get from ($\mathrm{Co2}_{\mathrm{formal}}$) the equation

$$\begin{aligned} &D(1 + A + B + AB, 1 + \alpha + \beta + \alpha\beta, U + V, x) \\ &\qquad = D(1 + A, 1 + \alpha, U, x) + (1 + \alpha)^{\lambda}(1 + A)^{\mu} D\big(1 + B, 1 + \beta, V, (1 + A)^{\nu} x\big). \end{aligned}$$

Let $E(r,s,t,x) = D(1 + r, 1 + s, t, x)$, then we study the formal equation

$$\begin{aligned} &E(A + B + AB, \alpha + \beta + \alpha\beta, U + V, x) \\ &\qquad = E(A,\alpha,U,x) + (1 + \alpha)^{\lambda}(1 + A)^{\mu} E\big(B,\beta,V,(1 + A)^{\nu} x\big). \end{aligned} \tag{$\mathrm{Co2}^*_{\mathrm{formal}}$}$$

The boundary condition (B2) is replaced by

$$E(0,0,0,x) = 0. \tag{B2*}$$

Now it is possible to consider $E(A,\alpha,U,x)$ as an element of $(\mathbb{C}(\!(A)\!)[\![\alpha,U]\!])[\![x]\!]$. I.e. the coefficient functions $E_n(A,\alpha,U)$ of $E(A,\alpha,U,x) = \sum_{n\geq 0} E_n(A,\alpha,U)x^n$ are formal Laurent series in A and formal power series in α and U. However, since we suppose that 0 can be inserted for the variable A, the coefficient functions

$E_n(A,\alpha,U)$ belong to the ring $\mathbb{C}[\![A,\alpha,U]\!]$. In this setting it is even possible to consider arbitrary complex numbers λ,μ and $\nu\neq 0$, since the formal binomial series yields

$$(1+A)^\lambda = \sum_{n\geq 0}\binom{\lambda}{n}A^n \in \mathbb{C}[\![A]\!],$$

where

$$\binom{\lambda}{n} := \prod_{i=0}^{n-1}\frac{\lambda-i}{i}.$$

Differentiation of ($\mathrm{Co2}^*_{\mathrm{formal}}$) with respect to B and substituting $B=0$, $\beta=0$, $V=0$ yields

$$(1+A)\frac{\partial}{\partial A}E(A,\alpha,U,x) = (1+\alpha)^\lambda(1+A)^\mu K\big((1+A)^\nu x\big), \qquad (\mathrm{Co2D1}^*_{\mathrm{formal}})$$

where $K(x)=\frac{\partial}{\partial A}E(A,0,0,x)|_{A=0}$. Similarly, differentiation with respect to β or V, respectively, and substituting $B=0$, $\beta=0$, $V=0$ yields

$$(1+\alpha)\frac{\partial}{\partial \alpha}E(A,\alpha,U,x) = (1+\alpha)^\lambda(1+A)^\mu L\big((1+A)^\nu x\big) \qquad (\mathrm{Co2D2}^*_{\mathrm{formal}})$$

and

$$\frac{\partial}{\partial U}E(A,\alpha,U,x) = (1+\alpha)^\lambda(1+A)^\mu M\big((1+A)^\nu x\big), \qquad (\mathrm{Co2D3}^*_{\mathrm{formal}})$$

where $L(x)=\frac{\partial}{\partial\alpha}E(0,\alpha,0,x)|_{\alpha=0}$ and $M(x)=\frac{\partial}{\partial U}E(0,0,U,x)|_{U=0}$.

From ($\mathrm{Co2D3}^*_{\mathrm{formal}}$) and ($\mathrm{B2}^*$) we immediately obtain

$$E(A,\alpha,U,x) = (1+\alpha)^\lambda(1+A)^\mu M\big((1+A)^\nu x\big)U + \tilde{E}(A,\alpha,x)$$

with $\tilde{E}(A,\alpha,x)\in(\mathbb{C}[\![A,\alpha]\!])[\![x]\!]$ and $\tilde{E}(0,0,x)=0$. Inserting this representation of $E(A,\alpha,U,x)$ into ($\mathrm{Co2D2}^*_{\mathrm{formal}}$) we get

$$\frac{\partial}{\partial\alpha}\tilde{E}(A,\alpha,x) = (1+\alpha)^{\lambda-1}(1+A)^\mu\Big(L\big((1+A)^\nu x\big) - \lambda M\big((1+A)^\nu x\big)U\Big).$$

Now we consider two different cases.

Case 1. If $\lambda\neq 0$, then necessarily $M((1+A)^\nu x)=0$ and

$$\frac{\partial}{\partial\alpha}\tilde{E}(A,\alpha,x) = (1+\alpha)^{\lambda-1}(1+A)^\mu L\big((1+A)^\nu x\big).$$

Therefore

$$E(A,\alpha,U,x) = \tilde{E}(A,\alpha,x) = \frac{(1+\alpha)^\lambda}{\lambda}(1+A)^\mu L\big((1+A)^\nu x\big) + \hat{E}(A,x),$$

where $\hat{E}(A,x)\in(\mathbb{C}[\![A]\!])[\![x]\!]$ and $\hat{E}(0,x)=-L(x)/\lambda$.

Case 2. If $\lambda=0$, then we have

$$\frac{\partial}{\partial\alpha}\tilde{E}(A,\alpha,x) = (1+\alpha)^{-1}(1+A)^\mu L\big((1+A)^\nu x\big)$$

and therefore

$$\tilde{E}(A,\alpha,x) = \log(1+\alpha)(1+A)^{\mu}L\big((1+A)^{\nu}x\big) + \hat{E}(A,x),$$

where $\hat{E}(A,x) \in (\mathbb{C}[\![A]\!])[\![x]\!]$ and $\hat{E}(0,x) = 0$. Thus $E(A,\alpha,U,x)$ is equal to

$$(1+A)^{\mu}M\big((1+A)^{\nu}x\big)U + \log(1+\alpha)(1+A)^{\mu}L\big((1+A)^{\nu}x\big) + \hat{E}(A,x).$$

Finally we have to insert these two different representations of E into the formal equation ($\mathrm{Co2D1}^{*}_{\mathrm{formal}}$).

In Case 1 we obtain the differential equation

$$\begin{aligned}\frac{\partial}{\partial A}\,\hat{E}(A,x) = (1+\alpha)^{\lambda}(1+A)^{\mu-1}\Big(-\frac{\mu}{\lambda}L\big((1+A)^{\nu}x\big) \\ -\frac{\nu}{\lambda}(1+A)^{\nu}xL'\big((1+A)^{\nu}x\big) + K\big((1+A)^{\nu}x\big)\Big).\end{aligned} \tag{30}$$

Since the left hand side is independent of α, we have

$$K\big((1+A)^{\nu}x\big) = \frac{1}{\lambda}\Big(\mu L\big((1+A)^{\nu}x\big) + \nu(1+A)^{\nu}xL'\big((1+A)^{\nu}x\big)\Big).$$

Hence we deduce the relation

$$K_n = \frac{\mu+n\nu}{\lambda}L_n, \qquad n \geq 0,$$

between the coefficients of $K(x) = \sum_{n\geq 0} K_n x^n$ and $L(x) = \sum_{n\geq 0} L_n x^n$. If $\mu + n\nu \neq 0$ for all $n \geq 0$, then $L(x)$ is uniquely determined by $K(x)$,

$$L(x) = \sum_{n\geq 0}\frac{\lambda K_n}{\mu+n\nu}x^n.$$

If $\mu + n_0\nu = 0$ for some $n_0 \geq 0$, then

$$L(x) = \sum_{\substack{n\geq 0\\ n\neq n_0}}\frac{\lambda K_n}{\mu+n\nu}x^n + L_{n_0}x^{n_0}.$$

As explained above, (30) is of the form $\frac{\partial}{\partial A}\,\hat{E}(A,x) = 0$, thus $\hat{E}(A,x) = \bar{E}(x) \in \mathbb{C}[\![x]\!]$ and $\bar{E}(x) = \hat{E}(1,x) = -\frac{1}{\lambda}L(x)$. Consequently, if $\mu + n\nu \neq 0$ for all $n \geq 0$, then

$$\begin{aligned}E(A,\alpha,U,x) &= \frac{(1+\alpha)^{\lambda}}{\lambda}(1+A)^{\mu}L\big((1+A)^{\nu}x\big) - \frac{1}{\lambda}L(x) \\ &= \sum_{n\geq 0}\frac{K_n}{\mu+n\nu}\big((1+\alpha)^{\lambda}(1+A)^{\mu+n\nu} - 1\big)x^n.\end{aligned}$$

If $\mu + n_0\nu = 0$, then

$$E(A,\alpha,U,x) = \frac{(1+\alpha)^{\lambda}}{\lambda}(1+A)^{\mu}L\big((1+A)^{\nu}x\big) - \frac{1}{\lambda}L(x)$$

$$= \sum_{n\neq n_0} \frac{K_n}{\mu+n\nu}\big((1+\alpha)^\lambda(1+A)^{\mu+n\nu}-1\big)x^n + \frac{L_{n_0}}{\lambda}\big((1+\alpha)^\lambda-1\big)x^{n_0}.$$

In Case 2 we derive from (Co2D1$^*_{\text{formal}}$) that

$$\begin{aligned}\frac{\partial}{\partial A}\hat{E}(A,x) = (1+A)^{\mu-1}\Big(&K\big((1+A)^\nu x\big) - \mu M\big((1+A)^\nu x\big)U \\ &- \nu(1+A)^\nu x M'\big((1+A)^\nu x\big)U \\ &- \mu\log(1+\alpha)L\big((1+A)^\nu x\big) \\ &- \nu(1+A)^\nu x\log(1+\alpha)L'\big((1+A)^\nu x\big)\Big).\end{aligned}$$

This leads to the three equations

$$\frac{\partial}{\partial A}\hat{E}(A,x) = (1+A)^{\mu-1}K\big((1+A)^\nu x\big), \tag{31}$$

$$0 = \mu M\big((1+A)^\nu x\big) + \nu(1+A)^\nu x M'\big((1+A)^\nu x\big), \tag{32}$$

$$0 = \mu L\big((1+A)^\nu x\big) + \nu(1+A)^\nu x L'\big((1+A)^\nu x\big). \tag{33}$$

According to (31) the coefficient functions $\hat{E}_n(A)$ of $\hat{E}(A,x)$ satisfy the differential equation

$$\hat{E}_n'(A) = K_n(1+A)^{\mu+n\nu-1}, \qquad n\geq 0. \tag{34}$$

From the boundary condition $\hat{E}(0,x)=0$ we deduce that

$$E_n(A) = \frac{K_n}{\mu+n\nu}\big((1+A)^{\mu+n\nu}-1\big)$$

if $\mu+n\nu\neq 0$. If $\mu+n_0\nu=0$, then from (31) and the boundary condition we obtain

$$E_{n_0}(A) = K_{n_0}\log(1+A).$$

If $M(x)=\sum_{n\geq 0} M_n x^n$ satisfies (32), then $M_n=0$ for all n satisfying $\mu+n\nu\neq 0$ and M_{n_0} is not determined by (32) for $\mu+n_0\nu=0$.

If $L(x)=\sum_{n\geq 0} L_n x^n$ satisfies (33), then $L_n=0$ for all n satisfying $\mu+n\nu\neq 0$ and L_{n_0} is not determined by (33) for $\mu+n_0\nu=0$.

Summarizing, for $\lambda=0$ we have shown that $E(A,\alpha,U,x)$ is equal to

$$\begin{cases}\displaystyle\sum_{n\geq 0}\frac{K_n}{\mu+n\nu}\big((1+A)^{\mu+n\nu}-1\big)x^n, & \text{if } \mu+n\nu\neq 0 \text{ for all } n\geq 0,\\ \displaystyle\sum_{n\neq n_0}\frac{K_n}{\mu+n\nu}\big((1+A)^{\mu+n\nu}-1\big)x^n & \\ \quad + M_{n_0}Ux^{n_0} + L_{n_0}\log(1+\alpha)x^{n_0} & \\ \quad + K_{n_0}\log(1+A)x^{n_0}, & \text{if } \mu+n_0\nu=0.\end{cases}$$

Theorem 24. *The unique solution $E(A,\alpha,U,x)$ of the three differential equations* $(\mathrm{Co2D1}^*_{\mathrm{formal}})$, $(\mathrm{Co2D2}^*_{\mathrm{formal}})$, $(\mathrm{Co2D3}^*_{\mathrm{formal}})$, *and the boundary condition* $(\mathrm{B2}^*)$ *is of the form*

$$\begin{cases} \sum_{n\geq 0} \frac{K_n}{\mu+n\nu}\big((1+\alpha)^\lambda (1+A)^{\mu+n\nu}-1\big)x^n & \lambda\neq 0,\ \mu+n\nu\neq 0\ \forall n\geq 0,\\ \sum_{n\neq n_0} \frac{K_n}{\mu+n\nu}\big((1+\alpha)^\lambda (1+A)^{\mu+n\nu}-1\big)x^n & \\ \quad +\frac{L_{n_0}}{\lambda}\big((1+\alpha)^\lambda-1\big)x^{n_0} & \lambda\neq 0,\ \mu+n_0\nu=0,\ n_0\geq 0,\\ \sum_{n\geq 0} \frac{K_n}{\mu+n\nu}\big((1+A)^{\mu+n\nu}-1\big)x^n & \lambda=0,\ \mu+n\nu\neq 0\ \forall n\geq 0,\\ \sum_{n\neq n_0} \frac{K_n}{\mu+n\nu}\big((1+A)^{\mu+n\nu}-1\big)x^n & \\ \quad + M_{n_0}Ux^{n_0}+L_{n_0}\log(1+\alpha)x^{n_0} & \\ \quad + K_{n_0}\log(1+A)x^{n_0} & \lambda=0,\ \mu+n_0\nu=0,\ n_0\geq 0. \end{cases}$$

Additionally, $E(A,\alpha,U,x)$ is a solution of $(\mathrm{Co2}^*_{\mathrm{formal}})$. □

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Manuscript received: October 4, 2006 and, in final form, February 6, 2007.