Aequat. Math. 79 (2010), 111–156

0001-9054/10/010111-46
published online April 21, 2010
DOI 10.1007/s00010-010-0003-8

Aequationes Mathematicae

The formal translation equation for iteration groups of type II

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Abstract. We investigate the translation equation

$$F(s+t,x) = F(s,F(t,x)), \qquad s,t \in \mathbb{C}, \tag{T}$$

in $\mathbb{C}[\![x]\!]$, the ring of formal power series over $\mathbb{C}$. Here we restrict ourselves to iteration groups of type II, i.e. to solutions of (T) of the form $F(s,x) \equiv x + c_k(s)x^k \mod x^{k+1}$, where $k \geq 2$ and $c_k \neq 0$ is necessarily an additive function. It is easy to prove that the coefficient functions $c_n(s)$ of

$$F(s,x) = x + \sum_{n \geq k} c_n(s)x^n$$

are polynomials in $c_k(s)$. It is possible to replace this additive function c_k by an indeterminate. In this way we obtain a formal version of the translation equation in the ring $(\mathbb{C}[y])[\![x]\!]$. We solve this equation in a completely algebraic way, by deriving formal differential equations or an Aczél–Jabotinsky type equation. This way it is possible to get the structure of the coefficients in great detail which are now polynomials. We prove the universal character (depending on certain parameters, the coefficients of the infinitesimal generator H of an iteration group of type II) of these polynomials. Rewriting the solutions $G(y,x)$ of the formal translation equation in the form $\sum_{n\geq 0} \phi_n(x)y^n$ as elements of $(\mathbb{C}[\![x]\!])[\![y]\!]$, we obtain explicit formulas for ϕ_n in terms of the derivatives $H^{(j)}(x)$ of the generator H and also a representation of $G(y,x)$ as a Lie–Gröbner series. Eventually, we deduce the canonical form (with respect to conjugation) of the infinitesimal generator H as $x^k + hx^{2k-1}$ and find expansions of the solutions $G(y,x) = \sum_{r\geq 0} G_r(y,x)h^r$ of the above mentioned differential equations in powers of the parameter h.

Mathematics Subject Classification (2000). Primary 39B12, 39B50; Secondary 13F25.

Keywords. Translation equation, formal functional equations, Aczél–Jabotinsky type equations, iteration groups of type II, ring of formal power series over $\mathbb{C}$.

1. Introduction

In [1] we introduce the method of "formal functional equations" to solve the translation equation (and the associated system of cocycle equations) in rings of formal power series over $\mathbb{C}$ in the case of iteration groups of type I. Let $\mathbb{C}[\![x]\!]$ be the ring of formal power series $F(x) = \sum_{\nu\geq 0} c_\nu x^\nu$ over $\mathbb{C}$ in the indeterminate x, and denote by $(\Gamma, \circ)$ the group of formal series which are invertible with respect to substitution $\circ$. We consider the translation equation

$$F(s+t,x) = F\left(s, F(t,x)\right), \qquad s,t \in \mathbb{C}, \tag{T}$$

for $F_t(x) = F(t,x) = \sum_{\nu\geq 1} c_\nu(t)x^\nu \in \Gamma$, $t \in \mathbb{C}$. (Cf. the introduction of [1] for the motivation to study (T) and basic results on its solutions $(F_t)_{t\in\mathbb{C}}$.) A family $(F_t)_{t\in\mathbb{C}}$ which satisfies (T) is called iteration group, and neglecting the trivial iteration group, there are two types of such groups, namely iteration groups of type I where the coefficient c_1 is a generalized exponential function different from 1, and iteration groups of type II, where $c_1 = 1$.

In the present paper we will only treat iteration groups of type II, and then it is known (see Sect. 2) that for each iteration group of this type there exists an integer $k \geq 2$ such that

$$F_t(x) = x + c_k(t)x^k + \cdots, \qquad t \in \mathbb{C},$$

where $c_k \colon \mathbb{C} \to \mathbb{C}$ is an additive function different from 0.

Similarly as in [1], it is our main purpose to get *detailed information on the coefficient functions c_ν of a solution of* (T) *as polynomials in c_k*, but also to study the dependence of iteration groups of type II on parameters h which are related to the infinitesimal generator H of an iteration group of type II when the generator is in the uniquely determined normal form $H(x) = x^k + hx^{2k-1}$. Furthermore we obtain several other representations of iteration groups of type II, i.e. Lie–Gröbner series expansions and representations involving the iterates $H^{(j)}(x)$, $j \geq 0$, of the infinitesimal generator.

Using the formulation of (T) in the case of iteration groups of type II as an infinite system of functional equations (2) (in fact of inhomogeneous Cauchy equations) and using some basic properties of additive functions we see in Sect. 2 that (T) may be replaced by a "formal translation equation" of type II, namely

$$G(y+z,x) = G\left(y, G(z,x)\right) \tag{$\mathrm{T}_{\mathrm{formal}}$}$$

$$G(0,x) = x \tag{B}$$

where $G(y,x) \in (\mathbb{C}[y])[\![x]\!]$ is a formal series in x whose coefficients are polynomials in y, and where ($\mathrm{T}_{\mathrm{formal}}$) is an identity in $(\mathbb{C}[y,z])[\![x]\!]$. We obtain the form of iteration groups of type II by replacing y by an arbitrary additive function $c_k \neq 0$. It should be mentioned that the underlying infinite system of functional equations (2) is much more complicate and interesting than the corresponding system for iteration groups of type I. (See [7].)

($\mathrm{T}_{\mathrm{formal}}$) can be solved by using various differentiation processes which are, however, purely algebraic operations. Hence we may again speak of a "method of formal functional equations" to solve (T) in the case of iteration groups of type II. This approach yields three types of differential equations, namely (D_{formal}), (PD_{formal}) and (AJ_{formal}).

In Sect. 3 (PD_{formal}) is solved in such a way that it yields rather detailed information on the coefficient functions of an iteration group of type II (Theorem 4). In particular, it gives universal representations of the coefficient functions as polynomials in c_k (replacing y) where the coefficients of the generator $H(x) = x^k + h_{k+1}x^{k+1} + \cdots$ of the iteration groups serve as parameters (Theorem 4). (PD_{formal}) has the advantage that it is a linear (partial) differential relation where no substitution of the unknown $G(y, x)$ is needed.

In Sect. 4 we use the formal differential equation (D_{formal}) to construct all iteration groups of type II. The result (Theorem 9) is essentially the same as Theorem 4. In Sect. 5 a formal analogue to the (third) Aczél–Jabotinsky differential equation, (AJ_{formal}), in connection with the appropriate initial condition $G(y, x) \equiv x + yx^k \bmod x^{k+1}$, is applied to obtain the structure of iteration groups of type II, again (Theorem 13). Here the calculations are more delicate. But since Aczél–Jabotinsky type equations play a critical role in several problems of iteration theory, e.g. the description of all maximal families of commuting (invertible) formal series (see [8,10]), it would be helpful and interesting to apply the formal Aczél–Jabotinsky equation also to obtain solutions of such problems which would mean to study them under a weaker initial condition.

Since $G(y, x)$, a solution of (T_{formal}), belongs to $(\mathbb{C}[y])[\![x]\!]$, it is also an element of $\mathbb{C}[\![y, x]\!]$, the ring of formal power series in two indeterminates over $\mathbb{C}$, and hence it may be seen as an element $G(y, x) = \sum_{n\geq 0} \phi_n(x)y^n$ of $(\mathbb{C}[\![x]\!])[\![y]\!]$. This approach is worked out in Sect. 6. It follows from (PD_{formal}) that the family $(\phi_n(x))_{n\geq 0}$, a summable family of formal power series, is the solution of the rather simple recursion (14_n), (15). We obtain explicit formulas for ϕ_n in terms of the derivatives $H^{(j)}(x)$ of the generator H (Theorem 22) and also a representation of $G(y, x)$ as a Lie–Gröbner series (Theorem 24).

In Sect. 7 we deduce in Theorem 28 as basis for Sects. 8 and 9, the existence and uniqueness of the binomial normal forms $x^k + hx^{2k-1}$ for generators of iteration groups of type II (see also e.g. [7]), with respect to simultaneous conjugation of iteration groups.

In Sects. 8 and 9 we follow the classical ideas of the theory of differential equations in the complex domain to find expansions of the solutions $G(y, x) = \sum_{r\geq 0} G_r(y, x)h^r$ of (PD_{formal}) and (D_{formal}) in powers of the parameter h, provided the generator is already in normal form $H(x) = x^k + hx^{2k-1}$. We present such expansions for (PD_{formal}) in Theorem 30 in Sect. 8.

In Sect. 9 we use (D_{formal}) to give another solution of the above mentioned problem (Theorem 33). Interestingly, we obtain for $G_r(y, x)$ the structure

$$x^{[r]}(1 - (k - 1)yx^{k-1})^{-[r]/(k-1)}P_r(\ln(1 - (k - 1)yx^{k-1})), \qquad r \geq 0,$$

where $[r] := r(k-1)+1$ and P_r is a polynomial of degree r, so that only powers x^α, the binomial (algebraic) series $(1-(k-1)yx^{k-1})^{-1/(k-1)}$ and polynomials in $\ln(1-(k-1)yx^{k-1})$ are involved. $G_0(y,x) = x(1-(k-1)yx^{k-1})^{-1/(k-1)}$ is the only remaining term of $G(y,x)$ if $h=0$. These functions play an important role in the theory of reversible formal power series (see [6, Sect. 0.3]).

At the end of this introduction we mention work of D. Gronau on the solution of the formal translation equation [4,5] and a different approach to solve the translation equation in $\mathbb{C}[\![x]\!]$ by Jabłoński and Reich [7].

2. The formal translation equation in $\mathbb{C}[\![x]\!]$

We study the translation equation

$$F(s+t,x) = F\left(s, F(t,x)\right), \qquad s,t \in \mathbb{C}, \tag{T}$$

where

$$F(s,x) = x + \sum_{n\geq k} c_n(s)x^n \in \mathbb{C}[\![x]\!], \qquad k \geq 2,\ s \in \mathbb{C}, \tag{1}$$

with coefficient functions $c_n\colon \mathbb{C} \to \mathbb{C}$, $n \geq k$, and $c_k \neq 0$. Due to the form (1) we have

$$F\left(s, F(t,x)\right) = x + \sum_{n\geq k} c_n(t)x^n + \sum_{\nu\geq k} c_\nu(s)\left[x + \sum_{n\geq k} c_n(t)x^n\right]^\nu, \qquad s,t \in \mathbb{C}.$$

The family $F = (F(s,x))_{s\in\mathbb{C}}$ satisfies (T) if and only if the coefficient functions c_n satisfy a system of functional equations of the form

$$\begin{aligned}
c_n(s+t) &= c_n(t) + c_n(s), \qquad\qquad k \leq n \leq 2k-2,\\
c_{2k-1}(s+t) &= c_{2k-1}(t) + c_{2k-1}(s) + kc_k(s)c_k(t)\\
c_{2k}(s+t) &= c_{2k}(t) + c_{2k}(s) + kc_k(s)c_{k+1}(t) + (k+1)c_{k+1}(s)c_k(t)\\
c_n(s+t) &= c_n(t) + c_n(s) + kc_k(s)c_{n-(k-1)}(t)\\
&\quad + \left(n-(k-1)\right)c_{n-(k-1)}(s)c_k(t)\\
&\quad + \tilde{P}_n\left(c_k(s),\ldots,c_{n-k}(s),c_k(t),\ldots,c_{n-k}(t)\right), \qquad n > 2k,
\end{aligned} \tag{2}$$

for all $s,t \in \mathbb{C}$, where $\tilde{P}_n$ are universal polynomials which are linear in $c_j(s)$ for $k \leq j \leq n-k$. The particular form of these equations is obvious for $n \leq 2k$.

In order to prove the form of the remaining equations in (2), for $\nu \geq k$ we compute the first terms of

$$\begin{aligned} &c_\nu(s)\left[x + \sum_{n\geq k} c_n(t)x^n\right]^\nu \\ &\quad = c_\nu(s)\left(\sum_{j=0}^{\nu}\binom{\nu}{j}x^j\left[\sum_{n\geq k} c_n(t)x^n\right]^{\nu-j}\right) \\ &\quad = c_\nu(s)\left(x^\nu + \nu\sum_{n\geq k} c_n(t)x^{n+\nu-1} + \sum_{j=0}^{\nu-2}\binom{\nu}{j}x^j\left[\sum_{n\geq k} c_n(t)x^n\right]^{\nu-j}\right) \\ &\quad \equiv c_\nu(s)\left(x^\nu + \nu c_k(t)x^{\nu-1+k}\right) \bmod x^{\nu+k}, \end{aligned}$$

since for $0 \leq j \leq \nu-2$ we estimate that $\operatorname{ord}(x^j[\sum_{n\geq k} c_n(t)x^n]^{\nu-j}) = k(\nu-j)+j \geq k\nu+(\nu-2)(1-k) = \nu+2(-1+k) > \nu-1+k$. Assume that $\nu > n-k+1$, then the order of $c_\nu(s)[F(t,x)]^\nu - c_\nu(s)x^\nu$ is greater than n. Therefore, these $c_\nu(s)$ do not occur in $c_n(s+t)$ with exception of $c_n(s)$. Moreover, $c_{n-k+1}(s)$ occurs only in the summand $c_{n-k+1}(s)(n-k+1)c_k(t)$.

For $\nu \geq k$ and $j > n-k+1$ we analyze in which terms of $[F(t,x)]^\nu$ the coefficient function $c_j(t)$ occurs: Since $j > k \geq 2$, the order of these terms is at least $j+\nu-1$ which is greater than $n-k+1+k-1 = n$. Thus these terms do not occur in $c_n(s+t)$. Obviously the terms $c_n(t)$ and $kc_k(s)c_{n-k+1}(t)$ are summands in $c_n(s+t)$.

According to (2), c_k is an additive function. Each solution $(F(s,x))_{s\in\mathbb{C}}$ of (T) of the form $F(s,x) \equiv x + c_k(s)x^k \bmod x^{k+1}$ is called an iteration group of type II. In the present manuscript we restrict ourselves to iteration groups of type II.

Let F be an iteration group of type II. From $F(0+0,x) = F(0,F(0,x))$ we obtain $F(0,x) = x$, i. e. $c_n(0) = 0$ for $n \geq k$.

Lemma 1. *If F is an iteration group of type II, then there exist polynomials $P_n(y) \in \mathbb{C}[y]$, such that*

$$c_n(s) = P_n\left(c_k(s)\right), \qquad s \in \mathbb{C},\ n \geq k.$$

Proof. The assertion is trivial for $n = k$. Assume that $n > k$. Using (2) for $n = 2k$ we obtain from $c_{2k}(s+t) = c_{2k}(t+s)$ that

$$c_{k+1}(s)c_k(t) = c_{k+1}(t)c_k(s), \qquad s,t \in \mathbb{C}.$$

Since $c_k \neq 0$ there exists some $t_0 \in \mathbb{C}$ so that $c_k(t_0) \neq 0$ and we derive

$$c_{k+1}(s) = \frac{c_{k+1}(t_0)}{c_k(t_0)}\,c_k(s), \qquad s \in \mathbb{C}.$$

Hence c_{k+1} is a polynomial in c_k. Assume that $n > k+1$ and c_j are polynomials in c_k for $k \leq j < n$. Then we have $n+k-1 > 2k$ and using (2) we obtain from $c_{n+k-1}(s+t) = c_{n+k-1}(t+s)$ that

$$\begin{aligned}&kc_k(s)c_n(t) + nc_n(s)c_k(t) + \tilde{P}_{n+k-1}\left(c_k(s),\ldots,c_{n-1}(s),c_k(t),\ldots,c_{n-1}(t)\right)\\ &\quad = kc_k(t)c_n(s)+nc_n(t)c_k(s)+\tilde{P}_{n+k-1}\left(c_k(t),\ldots,c_{n-1}(t),c_k(s),\ldots,c_{n-1}(s)\right).\end{aligned}$$

Therefore $c_n(s)$ is equal to

$$\frac{c_n(t_0)}{c_k(t_0)}\,c_k(s) + \frac{\tilde{P}_{n+k-1}\left(c_k(t_0),\ldots,c_{n-1}(s)\right) - \tilde{P}_{n+k-1}\left(c_k(s),\ldots,c_{n-1}(t_0)\right)}{(n-k)c_k(t_0)}$$

for $n > k+1$. By induction we deduce that c_n is a polynomial in c_k. ☐

From (2) and Lemma 1 we deduce for $n \geq k$ that

$$\begin{aligned}&P_n\left(c_k(s)+c_k(t)\right)\\ &\quad = P_n\left(c_k(s+t)\right) = c_n(s+t)\\ &\quad = P_n(c_k(t)) + P_n(c_k(s)) + kc_k(s)P_{n-(k-1)}(c_k(t))\\ &\qquad + \left(n-(k-1)\right)P_{n-(k-1)}(c_k(s))c_k(t)\\ &\qquad + \tilde{P}_n\left(P_k(c_k(s)),\ldots,P_{n-k}(c_k(s)),P_k(c_k(t)),\ldots,P_{n-k}(c_k(t))\right).\end{aligned} \tag{3}$$

Lemma 2. *Let a be a nontrivial additive function, $a \neq 0$. Then the following assertions hold true:*

1. *a takes infinitely many values.*
2. *Consider $P(x_1,x_2) \in \mathbb{C}[x_1,x_2]$. If $P(a(s),a(t)) = 0$ for all $s,t \in \mathbb{C}$, then $P = 0$.*

Proof. The first assertion is clear. Since $a(s)$ and $a(t)$ run with s and t independently through infinitely many values, it follows from $P(a(s),a(t)) = 0$ for all $s,t \in \mathbb{C}$ by standard arguments that $P(x_1,x_2) = 0$. ☐

Since $c_k \neq 0$, from (3) we obtain the polynomial identity

$$\begin{aligned}P_n(y+z) = P_n(z) + P_n(y) + kyP_{n-(k-1)}(z) + \left(n-(k-1)\right)P_{n-(k-1)}(y)z\\ + \tilde{P}_n\left(P_k(y),\ldots,P_{n-k}(y),P_k(z),\ldots,P_{n-k}(z)\right)\end{aligned} \tag{4}$$

for all $n \geq k$. This system of identities is equivalent to the formal translation equation

$$G(y+z,x) = G\left(y,G(z,x)\right) \tag{$\mathrm{T}_{\mathrm{formal}}$}$$

in $(\mathbb{C}[y,z])[\![x]\!]$ for

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n,$$

$P_n(y) \in \mathbb{C}[y], n > k \geq 2$, and the boundary condition

$$G(0,x) = x. \tag{B}$$

Hence we obtain

Theorem 3. *$F(s,x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ is a solution of* (T) *if and only if $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a solution of* ($\mathrm{T}_{\text{formal}}$) *and* (B).

In the sequel we solve the system consisting of the formal translation equation ($\mathrm{T}_{\text{formal}}$) and the boundary condition (B). In $\mathbb{C}[y]$ we have the formal derivation with respect to y. In $(\mathbb{C}[y])[\![x]\!]$ we have the formal derivation with respect to x. Moreover the mixed chain rule is valid for formal derivations.

In the present context

$$\frac{\partial}{\partial y} G(y,x)|_{y=0} = x^k + \sum_{n>k} h_n x^n = H(x)$$

is called the infinitesimal generator of G. (We set $h_k := 1$.) Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1.

Differentiation of ($\mathrm{T}_{\text{formal}}$) with respect to y yields

$$\frac{\partial}{\partial t} G(t,x)|_{t=y+z} = \frac{\partial}{\partial y} G\left(y, G(z,x)\right).$$

For $y = 0$ we get

$$\frac{\partial}{\partial z} G(z,x) = H(G(z,x)). \tag{$\mathrm{D}_{\text{formal}}$}$$

Differentiation of ($\mathrm{T}_{\text{formal}}$) with respect to z and application of the mixed chain rule yields

$$\frac{\partial}{\partial t} G(t,x)|_{t=y+z} = \frac{\partial}{\partial t} G(y,t)|_{t=G(z,x)} \frac{\partial}{\partial z} G(z,x).$$

For $z = 0$ we get

$$\frac{\partial}{\partial y} G(y,x) = H(x)\frac{\partial}{\partial x} G(y,x). \tag{$\mathrm{PD}_{\text{formal}}$}$$

Combining ($\mathrm{D}_{\text{formal}}$) and ($\mathrm{PD}_{\text{formal}}$) yields an Aczél–Jabotinsky differential equation of the form

$$H(x)\frac{\partial}{\partial x} G(y,x) = H(G(y,x)). \tag{$\mathrm{AJ}_{\text{formal}}$}$$

3. The differential equation ($\mathrm{PD}_{\text{formal}}$)

Now we solve ($\mathrm{PD}_{\text{formal}}$) together with (B) in order to solve ($\mathrm{T}_{\text{formal}}$). The advantage of this procedure lies in the circumstance that no substitution of the unknown series $G(y,x)$ is needed and that ($\mathrm{PD}_{\text{formal}}$) is a linear equation.

Writing $G(y,x)$ as $\sum_{n\geq 0} P_n(y)x^n$ with $P_n(y)\in\mathbb{C}[y]$, then ($\mathrm{PD}_{\mathrm{formal}}$) reads as

$$\sum_{n\geq 0} P_n'(y)x^n = \left(\sum_{n\geq 0} nP_n(y)x^{n-1}\right)\left(x^k+\sum_{\ell>k} h_\ell x^\ell\right)$$
$$= \sum_{n\geq k}\left(\sum_{r=k}^{n} h_r(n+1-r)P_{n+1-r}(y)\right)x^n. \tag{5}$$

In a completely algebraic way it is possible to prove

Theorem 4. 1. *For any generator $H(x)=x^k+\sum_{n>k} h_n x^n$ the differential equation* ($\mathrm{PD}_{\mathrm{formal}}$) *together with* (B) *has exactly one solution. It is given by*

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

2. *The polynomials P_n, $n\geq k$, have a formal degree $\lfloor (n-1)/(k-1)\rfloor$ and they are of the form*

$$P_n(y) = \begin{cases} h_n y & \text{if } n<2k-1\\ h_{2k-1}y+\frac{k}{2}\,y^2 & \text{if } n=2k-1\\ h_n y + \frac{n+1}{2}\,h_{n-k+1}y^2+\Phi_n(y,h_{k+1},\dots,h_{n-k}) & \text{if } n\geq 2k,\end{cases} \tag{6}$$

where Φ_n are polynomials in y and in the coefficients $h_{k+1},\dots,h_{n-k}$. They satisfy $\Phi_n(0,h_{k+1},\dots,h_{n-k})=0$. The polynomial Φ_{2k} is of the form

$$\Phi_{2k}(y) = \begin{cases} \Phi_4(y)=y^3 & \text{if } k=2\\ 0 & \text{if } k>2.\end{cases}$$

Using the convention

$$\delta_{k,2} = \begin{cases} 1 & \text{if } k=2\\ 0 & \text{if } k\neq 2,\end{cases}$$

we write

$$\Phi_{2k}(y) = y^3\delta_{k,2}.$$

For $n>2k$ a formal degree of Φ_n as a polynomial in y is $\lfloor (n-1)/(k-1)\rfloor$. If $k=2$ this formula holds true also for $n=2k$, i.e. a formal degree of $\Phi_{2k}(y)=\Phi_4(y)$ is equal to $3=\lfloor (4-1)/(2-1)\rfloor$.

Proof. For $0\leq n<k$ we obtain $P_n'(y)=0$, whence P_n is constant. According to (B) $P_1=1$ and $P_n=0$ for $n=0$ or $2\leq n<k$. In the sequel when solving differential equations for P_n we will not always mention that due to (B) we

find a unique solution P_n. For $k \le n < 2k-1$ we have $P_n'(y) = h_n$, therefore $P_n(y) = h_n y$. For $n = 2k-1$ we have $P_{2k-1}'(y) = h_{2k-1} + ky$, whence $P_{2k-1}(y) = h_{2k-1}y + \frac{k}{2}\,y^2$. In all these situations the assertions about formal degrees of $P_n(y)$ are satisfied.

For $n = 2k$ we obtain

$$\begin{aligned} P_{2k}'(y) &= h_{2k} + \sum_{r=k}^{k+1} h_r(2k+1-r)P_{2k+1-r}(y) \\ &= h_{2k} + h_k(k+1)\left(h_{k+1}y + y^2\delta_{k,2}\right) + h_{k+1}ky \\ &= h_{2k} + h_{k+1}(2k+1)y + 3y^2\delta_{k,2}. \end{aligned}$$

The term $3y^2\delta_{k,2}$ occurs since $k+1 = 2k-1$ for $k = 2$. Thus, $P_{2k}(y) = h_{2k}y + \frac{2k+1}{2}\,h_{k+1}y^2 + y^3\delta_{k,2}$. A formal degree of $P_{2k}(y)$ is indeed equal to $\lfloor (2k-1)/(k-1)\rfloor$. For $2k < n < 3k-2$, whence $k > 2$, we derive

$$\begin{aligned} P_n'(y) &= h_n + \sum_{r=k}^{n-k+1} h_r(n+1-r)P_{n+1-r}(y) \\ &= h_n + \sum_{r=k}^{n-k+1} h_r(n+1-r)h_{n+1-r}y \\ &= h_n + (n+1)h_{n+1-k}y + \sum_{r=k+1}^{n-k} h_r(n+1-r)h_{n+1-r}y, \end{aligned}$$

since $P_{n+1-r}(y)$ is equal to $h_{n+1-r}y$ for $k \le r \le n-k+1$ (and $n-k+1 \le 2k-2$ by assumption). Therefore,

$$P_n(y) = h_n y + \frac{n+1}{2}\,h_{n+1-k}y^2 + \sum_{r=k+1}^{n-k} h_r\,\frac{n+1-r}{2}\,h_{n+1-r}y^2.$$

Moreover, $\Phi_n(y, h_{k+1}, \ldots, h_{n-k}) = \sum_{r=k+1}^{n-k} h_r\,\frac{n+1-r}{2}\,h_{n+1-r}y^2$ is a polynomial in y of formal degree 2 and $P_n(y)$ is also of formal degree $2 = \lfloor (n-1)/(k-1)\rfloor$.

Now consider $n = 3k-2$ and again $k > 2$ (since $3k-2 = 2k$ for $k = 2$). Then $n-k+1 = 2k-1$ and we have

$$\begin{aligned} P_{3k-2}'(y) &= h_{3k-2} + \sum_{r=k}^{2k-1} h_r(3k-1-r)P_{3k-1-r}(y) \\ &= h_{3k-2} + h_k(2k-1)\left(h_{2k-1}y + \frac{k}{2}\,y^2\right) + h_{2k-1}kh_ky \\ &\quad + \sum_{r=k+1}^{2k-2} h_r(3k-1-r)h_{3k-1-r}y \end{aligned}$$

and, therefore,

$$P_{3k-2}(y) = h_{3k-2}y + h_{2k-1}\frac{3k-1}{2}\,y^2 + \Phi_{3k-2}(y, h_{k+1}, \dots, h_{2k-2}),$$

where

$$\Phi_{3k-2}(y, h_{k+1}, \dots, h_{2k-2}) = \frac{(2k-1)k}{6}\,y^3 + \sum_{r=k+1}^{2k-2} h_r \frac{3k-1-r}{2}\,h_{3k-1-r}y^2.$$

A formal degree of $P_{3k-2}(y)$ and of Φ_{3k-2} as a polynomial in y is equal to $3 = \lfloor (n-1)/(k-1) \rfloor$.

Finally, for $n \geq 3k-1$ we have

$$\begin{aligned}
P'_n(y) &= \sum_{r=k}^{n} h_r(n+1-r)P_{n+1-r}(y) \\
&= h_n + \sum_{r=k}^{n-k+1} h_r(n+1-r)P_{n+1-r}(y) \\
&= h_n + h_{n-k+1}ky + \sum_{r=n-2k+3}^{n-k} h_r(n+1-r)P_{n+1-r}(y) \\
&\quad + h_{n-2k+2}(2k-1)P_{2k-1}(y) + \sum_{r=k}^{n-2k+1} h_r(n+1-r)P_{n+1-r}(y) \\
&= h_n + h_{n-k+1}ky + \sum_{r=n-2k+3}^{n-k} h_r(n+1-r)h_{n+1-r}y \\
&\quad + h_{n-2k+2}(2k-1)\left(h_{2k-1}y + \frac{k}{2}\,y^2\right) \\
&\quad + \sum_{r=k}^{n-2k+1} h_r(n+1-r)\left(h_{n+1-r}y + \frac{n+2-r}{2}\,h_{n+2-r-k}y^2\right. \\
&\quad \left. + \Phi_{n+1-r}(y, h_{k+1}, \dots, h_{n+1-r-k})\right)
\end{aligned}$$

The last sum $\sum_{r=k}^{n-2k+1} \dots$ still contains in the summand for $r = k$ the coefficient h_{n-k+1}. We obtain that

$$\begin{aligned}
P'_n(y) = h_n + h_{n-k+1}(n+1)y &+ \sum_{r=n-2k+3}^{n-k} h_r(n+1-r)h_{n+1-r}y \\
&+ h_{n-2k+2}(2k-1)\left(h_{2k-1}y + \frac{k}{2}\,y^2\right)
\end{aligned}$$

$$+\sum_{r=k+1}^{n-2k+1} h_r(n+1-r)\Bigg(h_{n+1-r}y+\frac{n+2-r}{2}\,h_{n+2-r-k}y^2$$
$$+\Phi_{n+1-r}(y,h_{k+1},\dots,h_{n+1-r-k})\big)$$
$$+(n+1-k)\Bigg(\frac{n+2-k}{2}\,h_{n+2-2k}y^2$$
$$+\Phi_{n+1-k}(y,h_{k+1},\dots,h_{n+1-2k})\Bigg).$$

Hence $P_n(y)=h_ny+\frac{n+1}{2}\,h_{n-k+1}y^2+\Phi_n(y,h_{k+1},\dots,h_{n-k})$. By the induction hypothesis, a formal degree of Φ_{n+1-r} is $\lfloor(n-r)/(k-1)\rfloor$, $k\le r<n-2k+1$. (The case $r=n-2k+1$ leads to $\Phi_{2k}(y)$ which is either zero or of degree $3=\lfloor(2k-1)/(k-1)\rfloor$ for $k=2$.) Thus a formal degree of $P_n(y)$ is equal to

$$\max\{3,\lfloor(n-k)/(k-1)\rfloor+1\}=\lfloor(n-k)/(k-1)\rfloor+1=\lfloor(n-1)/(k-1)\rfloor$$

which is also a formal degree of Φ_n as a polynomial in y. □

As in [7] it would be possible to give explicit recurrent relations for the coefficients $P_n(y)$ in Theorem 4 and in similar results of the present paper. We will not give further details here.

Theorem 5. *Each solution $G(y,x)$ of the system* ($\mathrm{PD}_{\mathrm{formal}}$) *and* (B) *is a solution of* ($\mathrm{T}_{\mathrm{formal}}$).

Proof. Let z be an indeterminate. We prove that

$$U(y,z,x):=G(y+z,x)$$
$$V(y,z,x):=G(z,G(y,x))$$

satisfy the system

$$\frac{\partial}{\partial y}\,f(y,z,x)=H(x)\frac{\partial}{\partial x}\,f(y,z,x) \tag{7}$$
$$f(0,z,x)=G(z,x). \tag{8}$$

If we further prove that the system (7) and (8) has a unique solution in $(\mathbb{C}[y,z])[\![x]\!]$, then we have shown that G satisfies ($\mathrm{T}_{\mathrm{formal}}$). First we demonstrate that U satisfies (7) and (8).

$$\frac{\partial}{\partial y}\,U(y,z,x)=\frac{\partial}{\partial w}\,G(w,x)|_{w=y+z}\overset{(\mathrm{PD}_{\mathrm{formal}})}{=}H(x)\frac{\partial}{\partial x}\,G(y+z,x)$$
$$=H(x)\frac{\partial}{\partial x}\,U(y,z,x)$$

and

$$U(0,z,x)=G(z,x).$$

Similarly we prove that V satisfies (7) and (8).

$$\begin{aligned}\frac{\partial}{\partial y} V(y,z,x) &= \frac{\partial}{\partial w} G(z,w)|_{w=G(y,x)} \frac{\partial}{\partial y} G(y,x) \\ &\overset{(\mathrm{PD_{formal}})}{=} \frac{\partial}{\partial w} G(z,w)|_{w=G(y,x)} H(x) \frac{\partial}{\partial x} G(y,x) \\ &= H(x)\frac{\partial}{\partial x} G(z,G(y,x)) = H(x)\frac{\partial}{\partial x} V(y,z,x)\end{aligned}$$

and

$$V(0,z,x) = G(z,G(0,x)) = G(z,x).$$

In order to show that there is a unique solution of (7) and (8) we write $f(y,z,x)$ as $\sum_{n\geq 0} f_n(y,z)x^n$ and we prove by induction that $f_n(y,z) = P_n(y+z)$ with exactly the same polynomials as in Theorem 4. From (7) we get

$$\sum_{n\geq 0} \frac{\partial}{\partial y} f_n(y,z)x^n = \sum_{n\geq k}\left(\sum_{r=k}^{n} h_r(n+1-r)f_{n+1-r}(y,z)\right)x^n.$$

Comparison of coefficients yields $\frac{\partial}{\partial y} f_n(y,z) = 0$ for $0 \leq n < k$. Together with (8) we have $f_1(y,z) = 1$ and $f_n(y,z) = 0$ for $n = 0$ or $2 \leq n < k$. For $n \geq k$ we obtain by the induction hypothesis that

$$\frac{\partial}{\partial y} f_n(y,z) = \sum_{r=k}^{n} h_r(n+1-r)f_{n+1-r}(y,z) = \sum_{r=k}^{n} h_r(n+1-r)P_{n+1-r}(y+z)$$

which is a polynomial $\Psi_n(y+z)$. From the proof of Theorem 4 we know that $\frac{\partial}{\partial w} P_n(w) = \Psi_n(w)$, therefore, $f_n(y,z)$ is of the form $P_n(y+z) + \bar{f}_n(z)$ with a suitable function $\bar{f}_n(z)$. On behalf of (8) we deduce that $\bar{f}_n(z) = 0$ which proves that $f_n(y,z) = P_n(y+z)$ with exactly the same polynomials P_n as in Theorem 4. Hence the solution f of (7) and (8) is uniquely determined. □

4. The differential equation ($\mathbf{D_{formal}}$)

Since in ($\mathrm{D_{formal}}$) the series $G(z,x)$ is substituted into $H(x)$ we have to assume that $G(z,x) = \sum_{n\geq 1} P_n(z)x^n$. Writing $G(z,x)$ as $\sum_{n\geq 1} P_n(z)x^n$ with $P_n(z) \in \mathbb{C}[z]$, ($\mathrm{D_{formal}}$) reads as

$$\sum_{n\geq 1} P_n'(z)x^n = \sum_{n\geq k} h_n[G(z,x)]^n,$$

which is a formal series of order at least k. Thus, $P_n'(z) = 0$ for $1 \leq n < k$, whence, P_n is constant. From (B) we deduce that $P_1 = 1$ and $P_n = 0$ for $2 \leq n < k$. Consequently

$$G(z,x) = x + \sum_{n\geq k} P_n(z)x^n.$$

The computation of $H(G(z,x))$ is described in the following lemmas.

Lemma 6. *Let ν be a positive integer, then $[G(z,x)]^\nu$, the ν-th multiplicative power of $G(z,x)$, is of the form*

$$[G(z,x)]^\nu = x^\nu + \nu P_k(z)x^{\nu+k-1} + \sum_{n>\nu+k-1}\left(\nu P_{n-\nu+1}(z) + Q_n^{(\nu)}(z)\right)x^n$$

for

$$Q_n^{(\nu)}(z) = \sum_{\substack{(j_1,j_k,\ldots,j_{n-\nu})\in\mathbb{N}_0^{n-\nu-k+2}\\ \sum j_i=\nu\\ \sum ij_i=n}} \binom{\nu}{j_1\, j_k \ldots j_{n-\nu}} \prod_{i=k}^{n-\nu} P_i(z)^{j_i}.$$

Proof. Since $[G(z,x)]^\nu$ is a formal series of order ν we write it as the formal series $\sum_{n\geq\nu} P_n^{(\nu)}(z)x^n$ with suitable polynomials $P_n^{(\nu)}(z)$. Consider some $n\geq\nu$. For the computation of $P_n^{(\nu)}(z)$ we can restrict our attention to the ν-th multiplicative power of a polynomial in x, namely to

$$\left[x + \sum_{r=k}^{n-\nu+1} P_r(z)x^r\right]^\nu.$$

The coefficient of x^n in this expression is

$$\sum_{\substack{(j_1,j_k,\ldots,j_{n-\nu+1})\in\mathbb{N}_0^{n-\nu-k+3}\\ \sum j_i=\nu\\ \sum ij_i=n}} \binom{\nu}{j_1\, j_k \ldots j_{n-\nu+1}} \prod_{i=k}^{n-\nu+1} P_i(z)^{j_i}. \tag{9}$$

Computing $\left[x + \sum_{r=k}^{n-\nu+1} P_r(z)x^r\right]^\nu$ as $\sum_f \prod_{i=1}^\nu f(i)$, where the sum is taken over all functions f from $\{1,\ldots,\nu\}$ to $\{x\}\cup\{P_i(z)x^i \mid i\geq k\}$, each summand consists of exactly ν factors. If there are exactly j_1 factors of the form x and j_i factors of the form $P_i(z)x^i$, $i\geq k$, then this summand is of degree $j_1+\sum_{i\geq k} ij_i$. Therefore, in summands of degree n no factors of the form $P_i(z)x^i$ for $i>n-\nu+1$ may occur. For simplifying the notation we always assume that $j_2=\cdots=j_{k-1}=0$. If $\sum j_i=\nu$ and $\sum ij_i=n$, then the multinomial coefficient $\binom{\nu}{j_1\, j_k\cdots j_{n-\nu+1}}$ determines the number of all functions $f\colon\{1,\ldots,\nu\}\to\{x\}\cup\{P_i(z)x^i \mid k\leq i\leq n-\nu+1\}$ such that $\prod_{i=1}^\nu f(i) = x^n\prod_{i=k}^{n-\nu+1} P_i(z)^{j_i}$.

Now we want to analyze particular cases for n and ν. If $n=\nu$, there is exactly one summand in (9) namely for $j_1=\nu$, $j_i=0$ for $i\geq k$. Therefore $P_\nu^{(\nu)}=1$. For $\nu<n\leq\nu+k-2$ there exist no sequences $(j_1,j_k,\ldots,j_{n-\nu+1})$ satisfying the two conditions of (9), whence $P_n^{(\nu)}=0$. For $n=\nu+k-1$ there exists exactly one summand in (9) with nonzero indices $j_1=\nu-1$, $j_k=1$, so that $P_{\nu+k-1}^{(\nu)}(z)=\nu P_k(z)$. Finally for $n>\nu+k-1$ there exists one nonzero summand in (9) for $j_1 = \nu - 1$ and $j_{n-\nu+1} = 1$. There may exist further summands for sequences $(j_1, j_k, \dots, j_{n-\nu+1})$ where $j_{n-\nu+1} = 0$. Thus $P_n^{(\nu)}(z) = \nu P_{n-\nu+1}(z) + Q_n^{(\nu)}(z)$ and $Q_n^{(\nu)}(z)$ is the polynomial given above. □

Lemma 7. *Consider $H(x) = \sum_{n\geq k} h_n x^n$ and $G(z,x) = x + \sum_{n\geq k} P_n(z)x^n$, where $k \geq 2$. Then*

$$\begin{aligned} H\left(G(z,x)\right) = &\sum_{n=k}^{2k-2} h_n x^n + \left(h_{2k-1} + h_k k P_k(z)\right) x^{2k-1} \\ &+ \sum_{n>2k-1} (h_n + h_{n-k+1}(n-k+1)P_k(z) + h_k k P_{n-k+1}(z) \\ &+ Q_n\left(P_k(z), \dots, P_{n-k}(z), h_k, \dots, h_{n-k}\right)) x^n \end{aligned}$$

with suitable polynomials Q_n.

Proof. By Lemma 6 we obtain

$$\begin{aligned} H\left(G(z,x)\right) = \sum_{\nu\geq k} h_\nu \Bigg(& x^\nu + \nu P_k(z) x^{\nu+k-1} \\ &+ \sum_{n>\nu+k-1} \left(\nu P_{n-\nu+1}(z) + Q_n^{(\nu)}(z)\right) x^n \Bigg) \end{aligned}$$

which is

$$\begin{aligned} &\sum_{n=k}^{2k-2} h_n x^n + \left(h_{2k-1} + h_k k P_k(z)\right) x^{2k-1} \\ &\quad + \sum_{n>2k-1} \Bigg(h_n + h_{n-k+1}(n-k+1)P_k(z) \\ &\quad + \sum_{r=k}^{n-k} h_r \left(r P_{n-r+1}(z) + Q_n^{(r)}(z)\right)\Bigg) x^n. \end{aligned} \tag{10}$$

Splitting the last sum for $r = k$ and $k+1 \leq r \leq n-k$ and collecting the terms in a suitable way yields the desired expression. For the sake of completeness we mention that

$$\begin{aligned} &Q_n(P_k(z), \dots, P_{n-k}(z), h_k, \dots, h_{n-k}) \\ &\quad = h_k Q_n^{(k)}(z) + \sum_{r=k+1}^{n-k} h_r \left(r P_{n-r+1}(z) + Q_n^{(r)}(z)\right). \end{aligned}$$

□

For $n = 2k$ we have

$$Q_{2k}(P_k(z), h_k) = h_k Q_{2k}^{(k)}(z) = h_k \sum_{\substack{(j_1, j_k) \in \mathbb{N}_0^2 \\ j_1 + j_k = k \\ j_1 + k j_k = 2k}} \binom{k}{j_k} P_k(z)^{j_k} = P_2(z)^2 \delta_{k,2}.$$

Lemma 8. *Assume that $\lfloor (i-1)/(k-1) \rfloor$ is a formal degree of the polynomial $P_i(z)$ for $k \leq i \leq n - \nu + 1$.*

1. *For $n > \nu + k - 1$ a formal degree of $Q_n^{(\nu)}(z)$ is equal to*

$$\left\lfloor \frac{n - \nu}{k - 1} \right\rfloor .$$

2. *For $n > 2k$ a formal degree of $Q_n(P_k(z), \ldots, P_{n-k}(z), h_k, \ldots, h_{n-k})$ as a polynomial in z is equal to*

$$\left\lfloor \frac{n-1}{k-1} \right\rfloor - 1.$$

For $k = 2$ this formula holds true also for $n = 2k = 4$.

Proof. By Lemma 6 a formal degree of $Q_n^{(\nu)}(z)$ is equal to

$$\begin{aligned}
\sum_{i=k}^{n-\nu} j_i \left\lfloor \frac{i-1}{k-1} \right\rfloor &\leq \left\lfloor \frac{\sum_{i=k}^{n-\nu} j_i (i-1)}{k-1} \right\rfloor \\
&= \left\lfloor \frac{\sum_{i=k}^{n-\nu} i j_i - \sum_{i=k}^{n-\nu} j_i}{k-1} \right\rfloor \\
&= \left\lfloor \frac{n - j_1 - (\nu - j_1)}{k-1} \right\rfloor \\
&= \left\lfloor \frac{n - \nu}{k-1} \right\rfloor .
\end{aligned}$$

According to the proof of Lemma 7 a formal degree of Q_n as a polynomial in z is equal to $\lfloor (n-k)/(k-1) \rfloor = \lfloor (n-1)/(k-1) \rfloor - 1$ which is a formal degree of $Q_n^{(k)}(z)$. □

By Lemma 7 the formal differential equation ($\mathrm{D}_{\mathrm{formal}}$) reads as

$$\begin{aligned}
\sum_{n \geq k} P_n'(z) x^n = \sum_{n=k}^{2k-2} h_n x^n &+ \left(h_{2k-1} + h_k k P_k(z)\right) x^{2k-1} \\
&+ \sum_{n > 2k-1} \left(h_n + h_{n-k+1}(n-k+1) P_k(z) + h_k k P_{n-k+1}(z)\right. \\
&+ \left. Q_n\left(P_k(z), \ldots, P_{n-k}(z), h_k, \ldots, h_{n-k}\right)\right) x^n. \qquad (11)
\end{aligned}$$

In a completely algebraic way it is possible to prove

Theorem 9. 1. *For any generator $H(x) = x^k + \sum_{n>k} h_n x^n$ the differential equation* ($\mathrm{D}_{\mathrm{formal}}$) *together with* (B) *has exactly one solution. It is given by*

$$G(z,x) = x + zx^k + \sum_{n>k} P_n(z)x^n \in (\mathbb{C}[z])[\![x]\!].$$

2. *The polynomials P_n, $n \geq k$, have a formal degree $\lfloor (n-1)/(k-1) \rfloor$ and their structure is similar to* (6).

Proof. Comparing coefficients in (11) we obtain for $k \leq n \leq 2k-2$ that $P_n'(z) = h_n$. Together with (B) we deduce that $P_n(z) = h_n z$. For $n = 2k-1$ we derive that $P_{2k-1}(z) = \frac{k}{2} z^2 + h_{2k-1}z$. For $n = 2k$ we have $Q_{2k} = P_2(z)^2\delta_{k,2} = z^2\delta_{k,2}$ and

$$\begin{aligned} P_{2k}'(z) &= h_{2k} + h_{k+1}(k+1)P_k(z) + kP_{k+1}(z) + z^2\delta_{k,2} \\ &= h_{2k} + h_{k+1}(k+1)z + k\left(h_{k+1}z + z^2\delta_{k,2}\right) + z^2\delta_{k,2} \\ &= h_{2k} + h_{k+1}(2k+1)z + 3z^2\delta_{k,2}. \end{aligned}$$

The term $kz^2\delta_{k,2}$ occurs since $k+1 = 2k-1$ for $k = 2$. Therefore,

$$P_{2k}(z) = h_{2k}z + h_{k+1}\frac{2k+1}{2}\, z^2 + z^3\delta_{k,2}.$$

For $n \leq 2k$ a formal degree of $P_n(z)$ is obviously $\lfloor (n-1)/(k-1) \rfloor$.

Assume that $n > 2k$. Then

$$\begin{aligned} P_n'(z) = h_n &+ h_{n-k+1}(n-k+1)P_k(z) + kP_{n-k+1}(z) \\ &+ Q_n\left(P_k(z), \dots, P_{n-k}(z), h_k, \dots, h_{n-k}\right). \end{aligned}$$

By the induction hypothesis we write $P_{n-k+1}(z)$ as $h_{n-k+1}z + \tilde{\Phi}_{n-k+1}$ $(z, h_k, \dots, h_{n-2k+2})$ with a suitable polynomial $\tilde{\Phi}_{n-k+1}$.

As a polynomial in z a formal degree of $\tilde{\Phi}_{n-k+1}$ is equal to $\lfloor (n-k)/(k-1) \rfloor$. Thus

$$\begin{aligned} P_n'(z) = h_n &+ h_{n-k+1}(n+1)z + k\tilde{\Phi}_{n-k+1}(z, h_k, \dots, h_{n-2k+2}) \\ &+ Q_n\left(P_k(z), \dots, P_{n-k}(z), h_k, \dots, h_{n-k}\right) \end{aligned}$$

and consequently

$$P_n(z) = h_n z + h_{n-k+1}\frac{n+1}{2}\, z^2 + \Phi_n(z, h_k, \dots, h_{n-k})$$

and $\Phi_n(0, h_k, \dots, h_{n-k}) = 0$. From the representation of $P_n'(z)$ above and from Lemma 8 it follows that a formal degree of $P_n(z)$ is equal to $\lfloor (n-1)/(k-1) \rfloor$ which is a formal degree of Q_n (or of $\tilde{\Phi}_{n-k+1}$) increased by 1. Obviously a formal degree of $P_n(z)$ is a formal degree of Φ_n. □

We omit here a detailed proof of the following theorem.

Theorem 10. *Each solution $G(z,x)$ of both* ($\mathrm{D}_{\mathrm{formal}}$) *and* (B) *is a solution of* ($\mathrm{T}_{\mathrm{formal}}$).

For proving this theorem we show that

$$\begin{aligned} U(y,z,x) &:= G(y+z,x) \\ V(y,z,x) &:= G\left(z, G(y,x)\right) \end{aligned}$$

both satisfy the system

$$\begin{aligned} \frac{\partial}{\partial z} f(y,z,x) &= H\left(f(y,z,x)\right) \\ f(y,0,x) &= G(y,x), \end{aligned}$$

where $\mathrm{ord}(f(y,z,x)) \geq 1$. This system has the unique solution $f(y,z,x) = x + \sum_{n\geq k} P_n(y+z)x^n$ with exactly the same polynomials P_n as in Theorem 9.

5. The differential equation ($\mathrm{AJ}_{\mathrm{formal}}$)

Now we turn our attention to the Aczél–Jabotinsky differential equation. Again we assume that $G(y,x) = \sum_{n\geq 1} P_n(y)x^n$ and $H(x) = \sum_{n\geq k} h_n x^n$ and $h_k = 1$. The left hand side of ($\mathrm{AJ}_{\mathrm{formal}}$) is the right hand side of ($\mathrm{PD}_{\mathrm{formal}}$). It was computed as the right hand side of (5) as

$$\sum_{n\geq k}\left(\sum_{r=k}^{n} h_r(n+1-r)P_{n+1-r}(y)\right)x^n.$$

For the right hand side of ($\mathrm{AJ}_{\mathrm{formal}}$) we have

$$H(G(y,x)) \equiv P_1(y)^k x^k \bmod x^{k+1}.$$

Comparing the coefficients of x^k we obtain that $P_1(y) = P_1(y)^k$. Therefore, $P_1(y)$ is a $(k-1)$-th root of 1 and by (B) $P_1(y) = 1$. Consequently

$$H\left(G(y,x)\right) \equiv x^k + \left(h_{k+1} + kP_2(y)\right)x^{k+1} \bmod x^{k+2}.$$

Comparison of coefficients yields that $2P_2(y) + h_{k+1} = h_{k+1} + kP_2(y)$. Hence for $k \neq 2$ (i.e. $k > 2$) we derive $P_2(y) = 0$. Similarly we prove that $P_n(y) = 0$ for $2 < n < k$. Therefore,

$$G(y,x) = x + \sum_{n\geq k} P_n(y)x^n.$$

Consequently, the left hand side of ($\mathrm{AJ}_{\mathrm{formal}}$) can be written as

$$\sum_{n\geq k}\left(h_n + \sum_{r=k}^{n+1-k} h_r(n+1-r)P_{n+1-r}(y)\right)x^n.$$

Using the notation introduced in Lemma 6 the right hand side of ($\mathrm{AJ}_{\mathrm{formal}}$) is given by (10) as was shown in the proof of Lemma 7, i.e.

$$\begin{aligned} H\left(G(y,x)\right) = &\sum_{n=k}^{2k-2} h_n x^n + \left(h_{2k-1} + h_k k P_k(y)\right) x^{2k-1} \\ &+ \sum_{n>2k-1} (h_n + h_{n-k+1}(n-k+1)P_k(y) \\ &+ \sum_{r=k}^{n-k} h_r \left(r P_{n-r+1}(y) + Q_n^{(r)}(y)\right)\Bigg) x^n. \end{aligned}$$

We need some more information about the polynomials $Q_n^{(r)}(y)$.

Lemma 11. *Assume that $n > 2k-1$ and $k \geq 2$.*

1. $Q_n^{(r)}(y) = 0$ *for $r \geq k$ and $n - 2k + 2 < r \leq n - k$ (thus $k > 2$).*
2. $\sum_{r=k}^{n-k} h_r Q_n^{(r)}(y) = 0$ *for $2k \leq n \leq 3(k-1)$ (thus $k > 2$).*
3. $\sum_{r=k}^{n-k} h_r Q_n^{(r)}(y) = \sum_{r=k}^{n-2k+2} h_r Q_n^{(r)}(y)$ *for all $k \geq 2$.*
4. $Q_n^{(n-2k+2)}(y) = \binom{n+2-2k}{2} P_k(y)^2$ *for $n \geq 3k-2$.*
5. *Assume that $n > 3k-2$. If $P_k(y)$ is a polynomial and $P_i(y)$, $i \geq k+1$, are given by*

$$P_i(y) = \begin{cases} h_i P_k(y) & \text{if } i < 2k-1 \\ h_{2k-1} P_k(y) + \frac{k}{2} P_k(y)^2 & \text{if } i = 2k-1 \\ h_i P_k(y) + \frac{i+1}{2} h_{i-k+1} P_k(y)^2 & \\ \qquad + \Phi_i(P_k(y), h_k, \ldots, h_{i-k}) & \text{if } i \geq 2k, \end{cases}$$

   with polynomials Φ_i, then

$$\begin{aligned} \sum_{r=k}^{n-k} h_r Q_n^{(r)}(y) = &\, h_{n+2-2k} \left(\binom{n+2-2k}{2} + k(k-1) \right) P_k(y)^2 \\ &+ \hat{Q}_n(P_k(y), h_k, \ldots, h_{n+1-2k}), \end{aligned}$$

   where $\hat{Q}_n$ is a polynomial in $P_k(y)$ and in the coefficients $h_k, \ldots, h_{n+1-2k}$.
6. *If $k > 2$, then $\hat{Q}_{3k-1}(P_k(y), h_k) = 0$.*

Proof. Assume that $n - 2k + 2 < r \leq n - k$, then $n < 2k + r - 2$. According to Lemma 6

$$Q_n^{(r)}(y) = \sum_{\substack{(j_1, j_k, \ldots, j_{n-r}) \in \mathbb{N}_0^{n-r-k+2} \\ \sum j_i = r \\ \sum i j_i = n}} \binom{r}{j_1\, j_k \ldots j_{n-r}} \prod_{i=k}^{n-r} P_i(y)^{j_i}.$$

We have to show that this is an empty sum.

If $j_1 = r$, then $j_i = 0$ for all $i > 1$ and $\sum i j_i = r \leq n - k < n$. Thus there are no sequences of this form.

If $j_1 = r-1$, then there exists exactly one $i_0 > 0$ such that $j_{i_0} \neq 0$, and then $j_{i_0} = 1$. From $n = \sum ij_i = r-1+i_0$ we derive that $i_0 = n-r+1$ which is impossible since by assumption $i_0 \leq n-r$.

If $j_1 = r-2$, then $n = \sum ij_i \geq r-2+2k$ which is a contradiction to $n < 2k+r-2$.

If $j_1 = r-\nu$ for $2 < \nu \leq r$, then $n = \sum ij_i \geq r-\nu+\nu k > r-\nu+2k+(\nu-2) = r+2k-2$ which is a contradiction $n < 2k+r-2$. This completes the proof of the first assertion.

The second and third assertion follow immediately from the first one.

In order to prove the fourth assertion, we assume that $r = n+2-2k$ and $n \geq 3k-2$. We prove that there exists only one summand in $Q_n^{(r)}(y)$, namely for the sequence $(n-2k, 2, 0, \ldots, 0)$. This sequence yields the summand $\binom{n+2-2k}{2} P_k(y)^2$. This is the only possible sequence starting with $j_1 = n-2k = r-2$, since $\sum ij_i = n$. Obviously the case $j_1 = r = n+2-2k$ is impossible. The case $j_1 = r-1$ is also impossible since it would imply the existence of exactly one $i_0 \leq n-r$ such that $j_{i_0} = 1$, but then $i_0 = n+1-r$ which is not possible. If $j_1 \leq r-3$, then we obtain as in the proof of the first assertion that $n = \sum ij_i > r-2+2k = n$ which is a contradiction.

For proving the fifth assertion, we already know that

$$\sum_{r=k}^{n-k} h_r Q_n^{(r)}(y) = h_{n-2k+2} \binom{n+2-2k}{2} P_k(y)^2 + \sum_{r=k}^{n-2k+1} h_r Q_n^{(r)}(y).$$

Due to the construction of P_i, the coefficient h_{n-2k+2} occurs in $P_{n-2k+2}(y)$, in $P_{n-k+1}(y)$ and it can occur in $P_i(y)$ for $i > n-k+1$. Because of Lemma 6 only $P_{n-2k+2}(y)$ can occur as a factor of $Q_n^{(r)}(y)$ from our sum. It can occur in those $Q_n^{(r)}(y)$ which satisfy $n-r \geq n+2-2k$, thus, for $r \leq 2k-2$ and of course $k \leq r \leq n-2k+1$. The polynomials $P_i(y)$ for $i \geq n-k+1$ do not occur in these $Q_n^{(r)}(y)$, since we have $k \leq r \leq n-k$. Hence, h_{n-2k+2} occurs in $Q_n^{(r)}(y)$ just as a coefficient of $P_{n-2k+2}(y)$.

Now we study all the summands for $(j_1, j_k, \ldots, j_{n-r})$ of $Q_n^{(r)}(y)$ for $k \leq r \leq 2k-2$ with $j_{n-2k+2} \neq 0$.

Consider the case $j_{n-2k+2} = 1$. If $\sum_{i>1} j_i > 2$, then $\sum ij_i \geq n+2-2k+2k > n$, whence this situation does not occur.

If $\sum_{i>1} j_i = 1$, then $j_1 = r-1$ and $\sum ij_i = r-1+n-2k+2 \leq 2k-2-1+n-2k+2 = n-1$, whence this situation is also impossible.

If $\sum_{i>1} j_i = 2$, then $j_1 = r-2$, there exists exactly one i_0 with $k \leq i_0 \leq n-r$, $i_0 \neq n-2k+2$ and $j_{i_0} = 1$. Then from $n = \sum ij_i = r-2+i_0+n-2k+2$ we obtain $i_0 = 2k-r$. From $i_0 \geq k$ we get $r \leq k$ and consequently $r = k$. Moreover $n-2k+2 > k$ since $n > 3k-2$. So in this situation there exists exactly one summand of $Q_n^{(k)}(y)$ which contains $P_{n+2-2k}(y)$ as a factor, namely the summand for $j_1 = k-2$, $j_k = 1$ and $j_{n+2-2k} = 1$ which is

$$k(k-1)P_k(y)P_{n-2k+2}(y).$$

The situation $j_{n+2-2k} \geq 2$ is impossible since then $\sum ij_i \geq 2(n-2k+2) + r - 2 \geq 2n - 4k + 4 + k - 2 = n + (n-(3k-2)) > n$. In conclusion, the coefficient h_{n-2k+2} occurs only in the form

$$h_{n-2k+2}k(k-1)P_k(y)^2$$

in $\sum_{r=k}^{n-2k+1} h_r Q_n^{(r)}(y)$. This finishes the proof of the fifth assertion.

By definition

$$\hat{Q}_{3k-1}(P_k(y), h_k) = \sum_{r=k}^{k+1} h_r Q_{3k-1}^{(r)}(y) - h_{k+1}\left(\binom{k+1}{2} + k(k-1)\right) P_k(y)^2.$$

The polynomial $Q_{3k-1}^{(k+1)}(y)$ was computed in 4. The polynomial $Q_{3k-1}^{(k)}(y)$ contains the summand $k(k-1)P_k(y)P_{k+1}(y)$ as described in 5. If $k > 2$ this summand is equal to $h_{k+1}k(k-1)P_k(y)^2$. There are no other summands in $Q_{3k-1}^{(k)}(y)$, since if $(j_1, j_k, \ldots, j_{2k-1})$ satisfies $\sum j_i = k$ and $\sum ij_i = 3k-1$, then $2k - 1 = \sum ij_i - \sum j_i = \sum_{i=k}^{2k-1}(i-1)j_i$ which has the unique solution $j_k = j_{k+1} = 1$ and $j_i = 0$ for $i > k+1$. □

Lemma 12. *Assume that $n \geq 2k+1$. For any coefficients $h_{k+1}, \ldots, h_{n-k}$ we have*

$$\sum_{r=k+1}^{n-k} (2r - n - 1)h_r h_{n+1-r} = 0.$$

We present only a sketch of the proof. If $n \equiv 0 \bmod 2$, then the above sum consists of an even number of summands. In pairs the first and the last summand, the second and the last but one etc. cancel. If $n \equiv 1 \bmod 2$, then the above sum consists of an odd number of summands. Again in pairs the first and the last summand, the second and the last but one etc. cancel. Finally, there remains one summand for $r = \lfloor n/2 \rfloor + 1$ which is also zero, since in this particular situation $2r - n - 1 = 0$.

Theorem 13. 1. *For any generator $H(x) = x^k + \sum_{n>k} h_n x^n$ and for any polynomial $P_k(y) \in \mathbb{C}[y]$ with $P_k(0) = 0$ the differential equation ($\mathrm{AJ}_{\mathrm{formal}}$) together with (B) has exactly one solution of the form*

$$G(y,x) = x + P_k(y)x^k + \sum_{n>k} P_n(y)x^n \in (\mathbb{C}[y])[\![x]\!].$$

The polynomials $P_n(y)$ for $n > k$ are given by

$$P_n(y) = \begin{cases} h_n P_k(y) & \text{if } n < 2k-1 \\ h_{2k-1}P_k(y) + \frac{k}{2}\, P_k(y)^2 & \text{if } n = 2k-1 \\ h_n P_k(y) + \frac{n+1}{2}\, h_{n-k+1}P_k(y)^2 & \\ \qquad\qquad + \Phi_n(P_k(y), h_{k+1}, \dots, h_{n-k}) & \text{if } n \geq 2k, \end{cases}$$

with polynomials Φ_n, $n \geq 2k$, in $P_k(y)$ and $h_{k+1}, \dots, h_{n-k}$.

2. *Assume that $P_k(y) = y$. The polynomials P_n, $n \geq k$, have a formal degree $\lfloor (n-1)/(k-1) \rfloor$ and their structure is similar to (6).*

Proof. Comparing the coefficients of x^{2k-1} in ($\mathrm{AJ}_{\mathrm{formal}}$) we obtain that $h_{2k-1} + kP_k(y) = h_{2k-1} + kP_k(y)$. Therefore, this equation does not determine $P_k(y)$. According to (B) the polynomial $P_k(y)$ must vanish at $y = 0$. Therefore at the moment $P_k(y)$ can be any polynomial with $P_k(0) = 0$.

Now consider $2k \leq n \leq 3(k-1)$, whence $k > 2$. Comparison of coefficients yields

$$h_n + \sum_{r=k}^{n+1-k} h_r(n+1-r)P_{n+1-r}(y)$$
$$= h_n + h_{n-k+1}(n-k+1)P_k(y) + \sum_{r=k}^{n-k} h_r \left(rP_{n-r+1}(y) + Q_n^{(r)}(y) \right).$$

Using Lemma 11.2 (i.e. point 2 of Lemma 11) we obtain

$$(n+1-k)P_{n+1-k}(y) + \sum_{r=k+1}^{n-k} h_r(n+1-r)P_{n+1-r}(y) + h_{n+1-k}kP_k(y)$$
$$= h_{n-k+1}(n-k+1)P_k(y) + kP_{n-k+1}(y) + \sum_{r=k+1}^{n-k} h_r r P_{n-r+1}(y).$$

Therefore

$$(n+1-2k)P_{n+1-k}(y) = (n+1-2k)h_{n+1-k}P_k(y)$$
$$+ \sum_{r=k+1}^{n-k} h_r(2r-n-1)P_{n+1-r}(y).$$

For $n = 2k$ we get $P_{k+1}(y) = h_{k+1}P_k(y)$. Consider some n with $2k < n \leq 3(k-1)$. By induction we have already shown that $P_r(y) = h_r P_k(y)$ for $k+1 \leq r < n+1-k$. Then by an application of Lemma 12 we obtain that $P_{n+1-k}(y) = h_{n-k+1}P_k(y)$. Therefore

$$P_r(y) = h_r P_k(y), \qquad k+1 \leq r \leq 2k-2.$$

For $n = 3k - 2$ we get

$$h_{3k-2} + \sum_{r=k}^{2k-1} h_r(3k-1-r)P_{3k-1-r}(y)$$
$$= h_{3k-2} + h_{2k-1}(2k-1)P_k(y) + \sum_{r=k}^{2k-2} h_r \left(rP_{3k-1-r}(y) + Q^{(r)}_{3k-2}(y)\right).$$

Either by direct calculations (for $k = 2$) or by Lemma 11.2 (for $k > 2$) we derive that

$$(2k-1)P_{2k-1}(y) + \sum_{r=k+1}^{2k-2} h_r(3k-1-r)P_{3k-1-r}(y) + h_{2k-1}kP_k(y)$$
$$= h_{2k-1}(2k-1)P_k(y) + kP_{2k-1}(y) + \left(\sum_{r=k+1}^{2k-2} h_r rP_{3k-1-r}(y)\right) + Q^{(k)}_{3k-2}(y).$$

According to Lemma 11.4 we obtain

$$(k-1)P_{2k-1}(y)$$
$$= (k-1)h_{2k-1}P_k(y) + \sum_{r=k+1}^{2k-2} h_r(2r-3k+1)P_{3k-1-r}(y) + \binom{k}{2} P_k(y)^2$$

and therefore, due to Lemma 12,

$$P_{2k-1}(y) = h_{2k-1}P_k(y) + \frac{k}{2}\, P_k(y)^2.$$

Finally assume that $n > 3k - 2$. Then

$$h_n + (n+1-k)P_{n+1-k}(y) + \sum_{r=k+1}^{n-k} h_r(n+1-r)P_{n+1-r}(y) + h_{n-k+1}kP_k(y)$$
$$= h_n + h_{n-k+1}(n-k+1)P_k(y) + kP_{n-k+1}(y)$$
$$+ \sum_{r=k+1}^{n-k} h_r rP_{n-r+1}(y) + \sum_{r=k}^{n-k} h_r Q^{(r)}_n(y).$$

Then

$$(n+1-2k)\left(P_{n+1-k}(y) - h_{n-k+1}P_k(y)\right)$$
$$= \sum_{r=k+1}^{n-k} h_r(2r-n-1)P_{n+1-r}(y) + \sum_{r=k}^{n-k} h_r Q^{(r)}_n(y). \tag{12}$$

For $n = 3k - 1$ we deduce

$$\begin{aligned}
&k\left(P_{2k}(y) - h_{2k}P_k(y)\right)\\
&\quad = \sum_{r=k+1}^{2k-1} h_r(2r-3k)P_{3k-r}(y) + \sum_{r=k}^{2k-1} h_r Q_{3k-1}^{(r)}(y)\\
&\quad = h_{k+1}(2-k)P_{2k-1}(y) + \sum_{r=k+2}^{2k-1} h_r(2r-3k)P_{3k-r}(y) + \sum_{r=k}^{2k-1} h_r Q_{3k-1}^{(r)}(y)\\
&\quad = h_{k+1}(2-k)\left(h_{2k-1}P_k(y) + \frac{k}{2}\,P_k(y)^2\right) + \sum_{r=k+2}^{2k-1} h_r(2r-3k)h_{3k-r}P_k(y)\\
&\qquad + \sum_{r=k}^{k+1} h_r Q_{3k-1}^{(r)}(y)\\
&\quad = h_{k+1}(2-k)\frac{k}{2}\,P_k(y)^2 + \sum_{r=k+1}^{2k-1} h_r(2r-3k)h_{3k-r}P_k(y) + \sum_{r=k}^{k+1} h_r Q_{3k-1}^{(r)}(y)\\
&\quad = h_{k+1}(2-k)\frac{k}{2}\,P_k(y)^2 + \sum_{r=k}^{k+1} h_r Q_{3k-1}^{(r)}(y).
\end{aligned}$$

(By Lemma 12 the sum $\sum_{r=k+1}^{2k-1} h_r(2r-3k)h_{3k-r}P_k(y)$ disappears. According to Lemma 11.3, both for $k = 2$ and $k > 2$ the sum $\sum_{r=k}^{2k-1} h_r Q_{3k-1}^{(r)}(y)$ consists of two summands only.) For $k = 2$ we compute explicitly

$$\begin{aligned}
P_4(y) &= h_4P_2(y) + \frac{1}{2}\,\left(Q_5^{(2)} + h_3Q_5^{(3)}\right)\\
&= h_4P_2(y) + \frac{1}{2}\,\left(2P_2(y)P_3(y) + 3h_3P_2(y)^2\right)\\
&= h_4P_2(y) + \frac{5}{2}\,h_3P_2(y)^2 + P_2(y)^3.
\end{aligned}$$

According to Lemma 11.5 and Lemma 11.6, for $k > 2$ we obtain

$$\begin{aligned}
P_{2k}(y) &= h_{2k}P_k(y) + \frac{h_{k+1}}{k}\,\left(\frac{(2-k)k}{2} + \binom{k+1}{2} + k(k-1)\right)P_k(y)^2\\
&\quad + \hat{Q}_{3k-1}(P_k(y), 1)\\
&= h_{2k}P_k(y) + \frac{2k+1}{2}\,h_{k+1}P_k(y)^2.
\end{aligned}$$

Therefore $P_{2k}(y)$ satisfies the assertion. Consider $n > 3k - 1$ and by induction we assume that the polynomials $P_r(y)$ for $r < n + 1 - k$ have the desired representation. The right hand side of (12) is then

$$\begin{aligned}
&\sum_{r=k+1}^{n-2k+1} h_r(2r-n-1)\Big(h_{n+1-r}P_k(y)+h_{n-r-k+2}\frac{n-r+2}{2}\,P_k(y)^2\\
&\qquad+\Phi_{n-r+1}\left(P_k(y),h_{k+1},\dots,h_{n-r-k+1}\right)\Big)\\
&\qquad+h_{n-2k+2}(n-4k+3)\left(h_{2k-1}P_k(y)+\frac{k}{2}\,P_k(y)^2\right)\\
&\qquad+\sum_{r=n-2k+3}^{n-k} h_r(2r-n-1)h_{n-r+1}P_k(y)+h_{n+2-2k}\\
&\qquad\times\left(\left(\binom{n+2-2k}{2}+k(k-1)\right)P_k(y)^2+\hat{Q}_n(P_k(y),h_k,\dots,h_{n+1-2k})\right)\\
&\quad=\sum_{r=k+1}^{n-k} h_r(2r-n-1)h_{n-r+1}P_k(y)\\
&\qquad+h_{n+2-2k}\left((n-4k+3)\frac{k}{2}+\binom{n+2-2k}{2}+k(k-1)\right)P_k(y)^2\\
&\qquad+\sum_{r=k+1}^{n-2k+1} h_r(2r-n-1)\Big(h_{n-r-k+2}\frac{n-r+2}{2}\,P_k(y)^2\\
&\qquad+\Phi_{n-r+1}\left(P_k(y),h_{k+1},\dots,h_{n-r-k+1}\right)\Big)\\
&\qquad+\hat{Q}_n(P_k(y),h_k,\dots,h_{n+1-2k})\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P_{n+1-k}(y)\\
&\quad=h_{n+1-k}P_k(y)\\
&\qquad+\frac{h_{n+2-2k}}{n+1-2k}\,\frac{(n-4k+3)k+(n+2-2k)(n+1-2k)+2k(k-1)}{2}\,P_k(y)^2\\
&\qquad+\Phi_{n+1-k}(P_k(y),h_{k+1},\dots,h_{n+1-2k})\\
&\quad=h_{n+1-k}P_k(y)+h_{n+2-2k}\frac{(n+1-2k)(n+2-k)}{2(n+1-2k)}\,P_k(y)^2\\
&\qquad+\Phi_{n+1-k}(P_k(y),h_{k+1},\dots,h_{n+1-2k})
\end{aligned}$$

as it was claimed. Moreover we saw, that it was not necessary to impose any further condition on $P_k(y)$. So $P_k(y)$ can be any polynomial with $P_k(0)=0$.

If $P_k(y)=y$, then for $k\le n\le 2k$ we immediately see that $P_n(y)$ has a formal degree $\lfloor (n-1)/(k-1)\rfloor$. By induction, for $n>3k-1$ we derive from Lemma 8.1 and (12) that a formal degree of $P_{n+1-k}(y)$ is

$$\max\left\{\lfloor (n-k-1)/(k-1)\rfloor, \lfloor (n-k)/(k-1)\rfloor\right\} = \lfloor (n-k)/(k-1)\rfloor$$

which finishes the proof. □

Theorem 14. *Each solution $G(y,x)$ of* ($\mathrm{AJ}_{\mathrm{formal}}$) *with* $G(y,x) \equiv x + yx^k \bmod x^{k+1}$ *is a solution of* ($\mathrm{T}_{\mathrm{formal}}$).

For proving this theorem we show that

$$\begin{aligned} U(y,z,x) &:= G(y+z,x) \\ V(y,z,x) &:= G\left(z, G(y,x)\right) \end{aligned}$$

both satisfy the system

$$\begin{aligned} H(x)\frac{\partial}{\partial x} f(y,z,x) &= H\left(f(y,z,x)\right) \\ f(y,z,x) &= x + (y+z)x^k + \sum_{n>k} f_n(y,z)x^n. \end{aligned}$$

Proceeding as in the proof of Theorem 13, it is easy to prove that this system has the unique solution $f(y,z,x) = x + (y+z)x^k + \sum_{n>k} P_n(y+z)x^n$ with exactly the same polynomials P_n as in Theorem 13.2.

6. Reordering of the summands

From the particular representation of $G(y,x)$ as

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$$

with polynomials P_n which have a formal degree $\lfloor (n-1)/(k-1)\rfloor$ and which satisfy $P_n(0) = 0$, we also get a representation of $G(y,x)$ as

$$G(y,x) = \sum_{n\geq 0} \phi_n(x)y^n \tag{13}$$

as an element of $(\mathbb{C}[\![x]\!])[\![y]\!]$. Introducing coefficients of the polynomials $P_r(y)$ so that $P_r(y) = \sum_{j=1}^{d_r} P_{r,j}y^j$, where $d_r = \lfloor (r-1)/(k-1)\rfloor$ is a formal degree of $P_r(y)$, $r \geq k$, for $n \geq 1$ we have

$$\phi_n(x) = \sum_{r\geq k} P_{r,n}x^r.$$

Since the degrees d_r are monotonically increasing the sum $\sum_{n\geq 0}\phi_n(x)$ belongs to $\mathbb{C}[\![x]\!]$ and $(\phi_n(x)y^n)_{n\geq 0}$ is a summable family in $(\mathbb{C}[\![x]\!])[\![y]\!]$. This allows us to rewrite ($\mathrm{PD}_{\mathrm{formal}}$) and (B) as

$$\sum_{n\geq 1} n\phi_n(x)y^{n-1} = H(x)\sum_{n\geq 0}\phi_n'(x)y^n, \tag{14}$$

$$\phi_0(x) = x, \tag{15}$$

where $(\phi_n'(x)y^n)_{n\geq 0}$ is also a summable family. We note that (14) is satisfied if and only if

$$\phi_{n+1}(x) = \frac{1}{n+1}\, H(x)\phi_n'(x) \tag{14_n}$$

holds true for all $n \geq 0$. Therefore

$$\begin{aligned}
\phi_1(x) &= H(x),\\
\phi_2(x) &= H(x)H'(x)/2,\\
\phi_3(x) &= H(x)\left(H(x)H'(x)\right)'/6 = \left(H(x)H'(x)^2 + H(x)^2H''(x)\right)/6.
\end{aligned}$$

Now, given a formal series $H(x) = \sum_{n\geq k} h_n x^n$, we want to solve the system ((14),(15)) and obtain some further properties of its solutions.

Theorem 15. *For any generator $H(x) = \sum_{n\geq k} h_n x^n$, $k \geq 2$, $h_k \neq 0$, the system ((14),(15)) has a unique solution. For $n \geq 0$ the order of $\phi_n(x)$ is equal to $n(k-1)+1$ and $\phi_n(0) = 0$.*

Proof. From (15) we deduce that $\phi_0(x) = x$ which is of order $1 = 0(k-1)+1$. Assume that $n \geq 0$ and the assertions are true for n. Then $\phi_{n+1}(x)$ is uniquely given by $\frac{1}{n+1}\, H(x)\phi_n'(x)$ and $\operatorname{ord}(\phi_{n+1}(x)) = \operatorname{ord}(H(x)) + \operatorname{ord}(\phi_n'(x)) = k + n(k-1)+1-1 = (n+1)(k-1)+1$. Moreover $\phi_0(0) = 0$ and $\phi_n(0) = 0$, $n \geq 1$, since $H(0) = 0$, which finishes the proof. □

Corollary 16. *We assume that $\sum_{n\geq 0} \phi_n(x)y^n = \sum_{r\geq 1} P_r(y)x^r$ is the solution of ((14),(15)) for a given generator $H(x)$. Writing*

$$P_r(y) = \sum_{j\geq 0} P_{r,j}y^j, \quad r \geq 1, \text{ and } \phi_n(x) = \sum_{r\geq 1} P_{r,n}x^r, \quad n \geq 0,$$

we deduce that $P_r = 0$ for $2 \leq r < k$. Moreover for $r \geq k$ the series $P_r(y)$ is a polynomial which has a formal degree $\lfloor (r-1)/(k-1)\rfloor$ and which satisfies $P_r(0) = 0$. Consequently

$$\sum_{n\geq 0} \phi_n(x)y^n = x + \sum_{r\geq k} P_r(y)x^r \in (\mathbb{C}\,[y])[\![x]\!].$$

Proof. From $\phi_0(x) = x$ we obtain that $P_{1,0} = 1$ and $P_{r,0} = 0$ for all $r \geq 2$. Since $\operatorname{ord}(\phi_n(x)) = n(k-1)+1$ for $n \geq 0$, we see that $P_{r,n} = 0$ for $1 \leq r \leq n(k-1)$. Especially for $1 \leq r \leq k-1$ and $n \geq 1$ we have $r \leq n(k-1)$ and, therefore, $P_{n,r} = 0$ for $n \geq 1$ and $1 \leq r \leq k-1$. Thus $P_1 = 1$ and $P_r = 0$ for $2 \leq r < k$. Assume that $r \geq k$. Then there exists some integer $s \geq 1$ so that $(s-1)(k-1)+1 \leq r < s(k-1)+1$. Then $P_{r,n} = 0$ for all $n \geq s$. In the case $r = (s-1)(k-1)+1$ we have $P_{r,s-1} \neq 0$ since $\operatorname{ord}(\phi_{s-1}(x)) = r$. Therefore, $s-1 = \lfloor (r-1)/(k-1)\rfloor$ is a formal degree for P_r. Moreover $P_r(0) = P_{r,0} = 0$. □

Since $\phi_1(x) = H(x)$ and $P_r(0) = 0$ for $r \geq k$ we have

$$P_r(y) = h_r y + \sum_{j=2}^{\lfloor (r-1)/(k-1) \rfloor} P_{r,j} y^j, \qquad r \geq k.$$

Now we want to describe the coefficients $P_{r,j}$ for $j \geq 2$, $r \geq 2k-1$.

Consider the situation for $j = 2$. Then

$$\begin{aligned}
\phi_2(x) &= \frac{1}{2} H(x) H'(x) \\
&= \frac{1}{2} \left(\sum_{\nu \geq k} h_\nu x^\nu \right) \left(\sum_{\mu \geq k} \mu h_\mu x^{\mu - 1} \right) \\
&= \frac{1}{2} \sum_{r \geq 2k-1} \left(\sum_{\substack{\nu+\mu=r+1 \\ \nu \geq k,\ \mu \geq k}} \mu h_\nu h_\mu \right) x^r \\
&= \frac{1}{2} \sum_{r \geq 2k-1} \left(\sum_{\mu=k}^{r+1-k} \mu h_\mu h_{r+1-\mu} \right) x^r.
\end{aligned}$$

Corollary 17. *The coefficients $P_{r,2}$ for $r \geq 2k-1$ are of the form*

$$\begin{aligned}
P_{2k-1,2} &= \frac{k}{2} h_k^2 \\
P_{2k,2} &= \frac{2k+1}{2} h_k h_{k+1} \\
P_{r,2} &= \begin{cases} \frac{r+1}{2} \left(h_k h_{r+1-k} + \sum_{\nu=k+1}^{r/2} h_\nu h_{r+1-\nu} \right) & r \equiv 0 \bmod 2 \\ \frac{r+1}{2} \left(h_k h_{r+1-k} + \sum_{\nu=k+1}^{(r-1)/2} h_\nu h_{r+1-\nu} + \frac{1}{2} h_{(r+1)/2}^2 \right) & r \equiv 1 \bmod 2. \end{cases}
\end{aligned}$$

This proves the second summand of $P_n(y)$, $n \geq 2k-1$, in (6).

Using (14_n), it is possible to prove the following

Theorem 18. *If $\phi_n(x) = \sum_{r \geq n(k-1)+1} P_{r,n} x^r$ and $H(x) = \sum_{r \geq k} h_r x^r$, then*

$$\phi_{n+1}(x) = \frac{1}{n+1} \sum_{r \geq (n+1)(k-1)+1} \left(\sum_{\nu=n(k-1)+1}^{r+1-k} \nu h_{r+1-\nu} P_{\nu,n} \right) x^r, \qquad n \geq 0.$$

By induction this formula allows the computation of the coefficients $P_{r,n+1}$ of $\phi_{n+1}(x)$ for $r = (n+1)(k-1)+1$ and $r = (n+1)(k-1)+2$. For $n \geq 0$ we obtain

$$P_{(n+1)(k-1)+1,n+1} = \frac{h_k^{n+1}}{(n+1)!} \prod_{j=1}^{n} (j(k-1)+1),$$

and

$$P_{(n+1)(k-1)+2,n+1} = \frac{h_k^n h_{k+1}}{(n+1)!} \sum_{r=1}^{n+1} \prod_{s=r+1}^{n+1} ((s-1)(k-1)+2) \prod_{j=1}^{r-1} (j(k-1)+1)\,.$$

Theorem 19. *Let $H(x) = \sum_{n\geq k} h_n x^n, k \geq 2, h_k \neq 0$, be a generator and assume that $\sum_{n\geq 0} \phi_n(x)y^n$ is a solution of ((14),(15)). Then $\phi_0(x) = x$ and*

$$\phi_n(x) = \frac{1}{n!} \sum_{r\geq n(k-1)+1} \left(\sum_{(\nu_1,\ldots,\nu_{n-1})}^{*r} \prod_{s=1}^{n-1} h_{\nu_s} \left(r+s-\sum_{t=1}^{s} \nu_t \right) h_{r+(n-1)-\sum_{t=1}^{n-1}\nu_t} \right) x^r$$

for $n \geq 1$. In $\sum_{(\nu_1,\ldots,\nu_{n-1})}^{*r}$ we are taking the sum over all $(n-1)$-tuples $(\nu_1,\ldots,\nu_{n-1})$ of integers, such that $k \leq \nu_s \leq r-(n-s)k+(n-1)-\sum_{t=1}^{s-1}\nu_t$.

This theorem shows that the coefficient $P_{r,n}$ of x^r in $\phi_n(x)$ depends only on $h_k,\ldots,h_{r-(n-1)(k-1)}$.

Proof. We prove this theorem by induction on n. For $n=0$ see (15). For $n=1$ the sum $\sum_{(\nu_1,\ldots,\nu_{n-1})}^{*r}$ consists of one summand (for the empty tuple) and the formula above specializes to $\phi_1(x) = \sum_{r\geq k} h_r x^r$ which is $H(x)$ as we had already seen above. Assume that $n \geq 1$ and that ϕ_n has the representation above. By $P_{r,n}$ we indicate the coefficient

$$\sum_{(\nu_1,\ldots,\nu_{n-1})}^{*r} \prod_{s=1}^{n-1} h_{\nu_s} \left(r+s-\sum_{t=1}^{s} \nu_t \right) h_{r+(n-1)-\sum_{t=1}^{n-1}\nu_t}$$

of x^r in ϕ_n for $r \geq n(k-1)+1$. From (14_n) we derive that

$$\begin{aligned}\phi_{n+1}(x) &= \frac{1}{n+1} H(x) \frac{1}{n!} \sum_{r\geq n(k-1)+1} r P_{r,n} x^{r-1} \\ &= \frac{1}{(n+1)!} \sum_{r\geq(n+1)(k-1)+1} \left(\sum_{\ell=k}^{r-n(k-1)} h_\ell (r+1-\ell) P_{r+1-\ell,n} \right) x^r.\end{aligned}$$

Now we expand $\sum_{\ell=k}^{r-n(k-1)} h_\ell(r+1-\ell)P_{r+1-\ell,n}$ and obtain

$$\begin{aligned}&\sum_{\ell=k}^{r-n(k-1)} h_\ell(r+1-\ell) \\ &\quad\times \sum_{(\nu_1,\ldots,\nu_{n-1})}^{*(r+1-\ell)} \prod_{s=1}^{n-1} h_{\nu_s} \left(r+1-\ell+s-\sum_{t=1}^{s}\nu_t \right) h_{r+1-\ell+(n-1)-\sum_{t=1}^{n-1}\nu_t}. \qquad (*)\end{aligned}$$

Let $(\nu_1,\ldots,\nu_{n-1})$ be an $(n-1)$-tuple from the sum $\sum_{(\nu_1,\ldots,\nu_{n-1})}^{*(r+1-\ell)}$. Consider the n-tuples $(\mu_1,\ldots,\mu_n)$ given by $\mu_1 := \ell$ and $\mu_{s+1} := \nu_s$ for $1 \leq s \leq n-1$.

Then

$$k \leq \mu_1 \leq r - nk + n = r - (n+1-1)k + (n+1) - 1.$$

Moreover, for $1 \leq s \leq n-1$ we have

$$k \leq \nu_s \leq r + 1 - \ell - (n+1-(s+1))k + (n-1) - \sum_{t=1}^{s-1} \nu_t.$$

Therefore we also have

$$k \leq \mu_{s+1} \leq r - (n+1-(s+1))k + (n+1) - 1 - \mu_1 - \sum_{t=1}^{s-1} \mu_{t+1}.$$

Thus

$$k \leq \mu_s \leq r - (n+1-s)k + (n+1) - 1 - \sum_{t=1}^{s-1} \mu_t, \qquad 2 \leq s \leq n,$$

and each summand in (*) yields a summand of the form

$$\sum_{(\mu_1,\dots,\mu_n)}^{*r} \prod_{s=1}^{n} h_{\mu_s} \left(r + s - \sum_{t=1}^{s} \mu_t \right) h_{r+n-\sum_{t=1}^{n} \mu_t}. \tag{**}$$

Conversely each n-tuple $(\mu_1, \dots, \mu_n)$ belonging to this sum yields a summand of (*) namely the summand for $\ell = \mu_1$ and $(\nu_1, \dots, \nu_{n-1}) = (\mu_2, \dots, \mu_n)$. Thus (*) equals (**) and ϕ_{n+1} has the desired representation. $\square$

Now we describe the solutions of (14) and (15) still in another way. We need some preparatory remarks and definitions:

For $n \geq 1$ let I_n be the set of all nonnegative integer sequences $(i_j)_{j\geq 0}$ with $i_0 \geq 1$, $\sum_{j\geq 0} i_j = n$, and $\sum_{j\geq 0} j i_j = n-1$. Therefore, only finitely many components i_j can be different from zero. To be more precise,

$$I_n = \left\{ (i_j)_{j\geq 0} \;\middle|\; i_j \in \mathbb{Z},\; i_j \geq 0,\; i_0 \geq 1,\; \sum_{j=0}^{n-1} i_j = n,\; \sum_{j=1}^{n-1} j i_j = n-1 \right\}.$$

If $(i_j)_{j\geq 0}$ belongs to I_n, $n \geq 1$, then $i_j = 0$ for $j \geq n$. The sets I_n are all finite. For instance I_1 contains only one sequence, namely $\iota := (1, 0, 0, \dots)$.

Consider two integer sequences $u = (u_j)_{j\geq 0}$ and $v = (v_j)_{j\geq 0}$. We define

$$u \prec v$$

if either

$$u_0 = v_0,\; u_1 = v_1 - 1,\; u_j = v_j \quad \text{for } j > 1 \tag{$\prec_1$}$$

or

$$u_0 = v_0 - 1,\; \exists s > 1 : u_{s-1} = v_{s-1} + 1,\; u_s = v_s - 1,\; u_j = v_j \quad \text{for } j \notin \{0, s-1, s\}. \tag{$\prec_2$}$$

If we put $r := s - 1$ then ($\prec_2$) reads as

$$v_0 = u_0 + 1, \exists r \geq 1 : v_r = u_r - 1,\ v_{r+1} = u_{r+1} + 1,\ v_j = u_j \text{ for } j \notin \{0, r, r+1\}\,.$$

Assume that $v \in I_n$, $n > 1$, and that $u \prec v$. If u is also supposed to be a sequence of nonnegative integers with $u_0 \geq 1$, then v must satisfy certain properties: For applying ($\prec_1$) it is necessary to have $v_1 \geq 1$, for ($\prec_2$) $v_0 > 1$ and $v_s \geq 1$ for some $s > 1$.

Lemma 20. *Let $n \geq 1$ and assume that $i = (i_j)_{j \geq 0}$ belongs to I_n.*

1. *If $i_0 = 1$, then $i = (1, n-1, 0, 0 \ldots)$.*
2. *If $i_{n-1} \neq 0$, then $i_{n-1} = 1$ and $i_j = 0$ for all $1 \leq j < n-1$. If $n > 1$, then $i_0 = n - 1$.*
3. *Let $\ell = (\ell_j)_{j \geq 0}$ be a nonnegative integer sequence. If $n > 1$ and $\ell \prec i$, then $\ell \in I_{n-1}$.*
4. *If $i \prec \ell$, then $\ell \in I_{n+1}$.*
5. *For each $\ell \in I_{n+1}$ there exists some $k = (k_j)_{j \geq 0} \in I_n$ such that $k \prec \ell$.*
6. $$I_{n+1} = \bigcup_{k \in I_n} \{\ell \mid k \prec \ell\}\,.$$
7. *The set of all $\ell \in I_{n+1}$ with $i \prec \ell$ is the union of $\{(i_0, i_1 + 1, i_2, i_3, \ldots)\}$ and*
$$\big\{(i_0+1, \ldots, i_{r-1}, i_r - 1, i_{r+1}+1, i_{r+2}, i_{r+3}, \ldots) \,\big|\, 1 \leq r \leq n-1,\ i_r > 0\big\}.$$

Proof. The proof of assertions 1., 2., 4., and 6. is left to the reader.

In order to prove 3., assume that $n > 1$, $i \in I_n$ and $\ell \prec i$ is a nonnegative integer sequence. If ($\prec_1$) is applied, thus $i_1 \geq 1$, then

$$\sum_{j \geq 0} \ell_j = i_0 + (i_1 - 1) + \sum_{j \geq 2} i_j = \sum_{j \geq 0} i_j - 1 = n - 1$$

and

$$\sum_{j \geq 0} j\ell_j = (i_1 - 1) + \sum_{j \geq 2} j i_j = \sum_{j \geq 0} j i_j - 1 = n - 2.$$

Moreover, $\ell_0 = i_0 > 0$ and $\ell_j \geq 0$ for $j \geq 1$. If ($\prec_2$) is applied, thus there exists some $s > 1$ so that $i_s \geq 1$, then

$$\sum_{j \geq 0} \ell_j = (i_0 - 1) + \sum_{j=1}^{s-2} i_j + (i_{s-1} + 1) + (i_s - 1) + \sum_{j > s} i_j = \sum_{j \geq 0} i_j - 1 = n - 1$$

and

$$\sum_{j\geq 0} j\ell_j = \sum_{j=0}^{s-2} ji_j + (i_{s-1}+1)(s-1) + (i_s-1)s$$
$$+\sum_{j>s} ji_j = \sum_{j\geq 0} ji_j + s - 1 - s = n-2.$$

We have to show that $\ell_0 = i_0 - 1$ is positive. Assuming that $i_0 = 1$, we obtain from 1. that $i = (1, n-1, 0, \dots)$ and, therefore, there does not exist some $s > 1$ such that $i_s \geq 1$. So for $i_0 = 1$ ($\prec_2$) cannot be applied and in our situation $i_0 \geq 2$, whence $\ell_0 \geq 1$.

In order to prove 5., let ℓ belong to I_{n+1}. Since $\sum_{j\geq 0} j\ell_j = n \geq 1$, there exists at least one $j \geq 1$ so that $\ell_j \geq 1$. Let s be the largest index with this property. If $s = 1$, then $\ell = (1, n, 0, 0, \dots)$ and by assumption $n \geq 1$, so that $k = (1, n-1, 0, 0, \dots)$ belongs to I_n and $k \prec \ell$. If $s > 1$, then $\ell_0 > 1$ and $k = (\ell_0 - 1, \ell_1, \dots, \ell_{s-1} + 1, \ell_s - 1, 0, 0, \dots)$ belongs to I_n and $k \prec \ell$.

Finally for 7., let i belong to I_n. If $i \prec \ell$, then either $\ell_1 = i_1 + 1$ and $\ell_j = i_j$ for $j \neq 1$, or $\ell_0 = i_0 + 1$ and there exists some $r \geq 1$ so that $\ell_r = i_r - 1$, $\ell_{r+1} = i_{r+1} + 1$ and $\ell_j = i_j$ for $j \notin \{0, r, r+1\}$. From $\ell_r \geq 0$ we deduce that $i_r > 0$. □

For $n \geq 1$, $u = (u_j)_{j\geq 0} \in I_n$, $v = (v_j)_{j\geq 0}$, and $u \prec v$ we define

$$\tilde{K}(u,v) := \begin{cases} u_0 & \text{if } (\prec_1) \text{ is applied} \\ u_{s-1} & \text{if } (\prec_2) \text{ is applied.} \end{cases}$$

We note that in the first case $u_0 = v_0$ and in the second case $u_{s-1} = v_{s-1} + 1 = u_r = v_r + 1$ for $r = s-1$.

For $v \in I_n$, $n > 1$, we set

$$K(v) := \sum_{\substack{u \in I_{n-1} \\ u \prec v}} \tilde{K}(u,v) K(u)$$

and $K(\iota) := 1$ for $\iota := (1, 0, 0, \dots)$. This definition determines the value $K(v)$ recursively.

The proof of the next lemma is trivial, thus it is omitted.

Lemma 21. *Consider $n \geq 2$, then*

$$K(1, n-1, 0, 0 \dots) = 1.$$

If $i_1 = \dots = i_{n-2} = 0$, then

$$K(n-1, i_1, \dots, i_{n-2}, 1, 0, 0, \dots) = 1.$$

Theorem 22. *Let $H(x)$ be a generator. Then the sequence $(\phi_n)_{n\geq 0}$ given by $\phi_0(x) = x$ and*

$$\phi_n(x) = \frac{1}{n!}\sum_{i\in I_n} K(i)\prod_{j=0}^{n-1}\left[H^{(j)}(x)\right]^{i_j}, \qquad n \geq 1,$$

satisfies the system ((14),(15)) *where $H^{(j)}(x) = \frac{d^j}{dx^j}\,H(x), j \geq 0$.*

Proof. By induction we proof that (14_n) is satisfied for all $n \geq 0$. $\phi_1(x) = H(x)$ satisfies this equation for $n = 0$. Now let $n > 0$. We have

$$\begin{aligned}
&\frac{1}{n+1}\,H(x)\phi_n'(x)\\
&\quad= \frac{1}{(n+1)!}\,H(x)\frac{\partial}{\partial x}\left(\sum_{i\in I_n} K(i)\prod_{j=0}^{n-1}\left[H^{(j)}(x)\right]^{i_j}\right)\\
&\quad= \frac{1}{(n+1)!}\sum_{i\in I_n}\sum_{r=0}^{n-1}\Bigg(H(x)K(i)i_r\prod_{j=0}^{r-1}\left[H^{(j)}(x)\right]^{i_j}\left[H^{(r)}(x)\right]^{i_r-1}\\
&\qquad\times\left[H^{(r+1)}(x)\right]^{i_{r+1}+1}\prod_{j=r+2}^{n-1}\left[H^{(j)}(x)\right]^{i_j}\Bigg).
\end{aligned} \tag{16}$$

Consider the summand for $r = 0$. The sequence of exponents of $H^{(j)}(x)$ for $j \geq 0$ is $\ell = (i_0, i_1 + 1, i_2, i_3, \ldots) \in I_{n+1}$. From Lemma 20.7 we deduce that $i \prec \ell$ and $\tilde{K}(i,\ell) = i_0$. For $r > 0$ this sequence is $\ell = (i_0+1, i_1, \ldots, i_{r-1}, i_r-1, i_{r+1}+1, i_{r+2}\ldots) \in I_{n+1}$. Again we deduce that $i \prec \ell$ and $\tilde{K}(i,\ell) = i_r$. Moreover, from Lemma 20.7 and (16) we derive that

$$\begin{aligned}
\frac{1}{n+1}\,H(x)\phi_n'(x) &= \frac{1}{(n+1)!}\sum_{i\in I_n}\sum_{\substack{\ell\in I_{n+1}\\ i\prec\ell}}\tilde{K}(i,\ell)K(i)\prod_{j=0}^{n}\left[H^{(j)}(x)\right]^{\ell_j}\\
&= \frac{1}{(n+1)!}\sum_{\ell\in I_{n+1}}\sum_{\substack{i\in I_n\\ i\prec\ell}}\tilde{K}(i,\ell)K(i)\prod_{j=0}^{n}\left[H^{(j)}(x)\right]^{\ell_j}\\
&= \frac{1}{(n+1)!}\sum_{\ell\in I_{n+1}} K(\ell)\prod_{j=0}^{n}\left[H^{(j)}(x)\right]^{\ell_j}\\
&= \phi_{n+1}(x).
\end{aligned}$$

□

There exists also an approach with Lie–Gröbner-series (cf. [2] or [3, chapter 1]) to solve ((14),(15)). Define an operator

$$D\colon \mathbb{C}[\![x]\!] \to \mathbb{C}[\![x]\!], \qquad D(f(x)) := H(x)f'(x).$$

Lemma 23. *Let H be a generator of order $k \geq 2$. If $(\phi_n)_{n\geq 0}$ satisfies the system ((14),(15)), then*

$$\phi_n(x) = \frac{1}{n!}\, D^n(x), \qquad n \geq 0.$$

Proof. For $n = 0$ we obtain from (15) that $x = \phi_0(x) = D^0(x)$. Assume that $n > 0$ and that the assertion is true for $n-1$, then from (14_{n-1}) we derive

$$\begin{aligned}\phi_n(x) &= \frac{1}{n}\, H(x)\phi_{n-1}(x)\\ &= \frac{1}{n}\, H(x)\frac{\partial}{\partial x}\left(\frac{1}{(n-1)!}\, D^{n-1}(x)\right)\\ &= \frac{1}{n!}\, D^n(x)\end{aligned}$$

and the proof is finished. □

Theorem 24. *The series*

$$G(y,x) := \sum_{n\geq 0} \frac{1}{n!}\, D^n(x) y^n,$$

which is a Lie–Gröbner-series, satisfies ($\mathrm{T}_{\mathrm{formal}}$) and (B).

Proof. Let x, y, z be distinct indeterminates. The family $(D^n(x))_{n\geq 0}$ is a summable family, since $\operatorname{ord} D^0(x) = \operatorname{ord} x = 1 < k = \operatorname{ord} H(x) = \operatorname{ord} D(x)$, $\operatorname{ord} D^2(x) = \operatorname{ord}(H(x)H'(x)) = \operatorname{ord}(D(x)) + k - 1 > \operatorname{ord} D(x)$, and $\operatorname{ord} D^n(x) = \operatorname{ord}\big(D^{n-1}(x)\big) + k - 1 > \operatorname{ord} D^{n-1}(x)$. Consequently $(z^n D^n(x))_{n\geq 0}$ is also a summable family.

Similarly we see that $(D^n(G(z,x)))_{n\geq 0}$ and $(z^n D^n(G(z,x)))_{n\geq 0}$ are summable families and we obtain that

$$\begin{aligned}G(y, G(z,x)) &= \sum_{n\geq 0} \frac{1}{n!}\, y^n D^n \left(\sum_{\nu\geq 0} \frac{1}{\nu!}\, z^\nu D^\nu(x)\right)\\ &= \sum_{n\geq 0}\sum_{\nu\geq 0} \frac{1}{n!}\,\frac{1}{\nu!}\, y^n z^\nu D^{n+\nu}(x)\\ &= \sum_{N\geq 0}\sum_{n=0}^{N} \frac{1}{N!}\,\frac{N!}{n!(N-n)!}\, y^n z^{N-n} D^N(x)\\ &= \sum_{N\geq 0} \frac{1}{N!}\left(\sum_{n=0}^{N}\binom{N}{n} y^n z^{N-n}\right) D^N(x)\end{aligned}$$

$$= \sum_{N\geq 0} \frac{1}{N!} (y+z)^N D^N(x)$$
$$= G(y+z,x).$$

Thus $G(y,x)$ satisfies ($\mathrm{T}_{\mathrm{formal}}$). Moreover, (B) is satisfied, since $G(0,x) = D^0(x) = x$. □

We note that Lie–Gröbner-series in the context of iteration groups have already been used by St. Scheinberg [11] and also in [9].

7. Normal forms

Let Γ_1 be the set of all formal power series $f(x) \in \mathbb{C}[\![x]\!]$ with $f(x) \equiv x \bmod x^2$.

Theorem 25. *Assume that $G(y,x)$ is a solution of* ($\mathrm{T}_{\mathrm{formal}}$)*. For all $S \in \Gamma_1$ the series $\tilde{G}(y,x) := S^{-1}(G(y,S(x)))$ is a solution of* ($\mathrm{T}_{\mathrm{formal}}$)*. If $G(y,x)$ satisfies* (B) *then also $\tilde{G}(y,x)$ satisfies* (B)*.*

Proof. It is straight forward that

$$\begin{aligned}\tilde{G}(y+z,x) &= S^{-1}(G(y+z,S(x))) \\ &= S^{-1}(G(y,G(z,S(x)))) \\ &= S^{-1}\left(G(y,S(S^{-1}(G(z,S(x)))))\right) \\ &= S^{-1}\left(G(y,S(\tilde{G}(z,x)))\right) \\ &= \tilde{G}\left(y,\tilde{G}(z,x)\right)\end{aligned}$$

and

$$\tilde{G}(0,x) = S^{-1}(G(0,S(x))) = S^{-1}(S(x)) = x.$$

□

Let Σ be the set of all solutions of ($\mathrm{T}_{\mathrm{formal}}$) and (B). This is the set of all formal iteration groups of type II. Then the relation $\sim$ defined by

$$G_1(y,x) \sim G_2(y,x) \text{ if and only if } \exists S \in \Gamma_1 \colon G_2(y,x) = S^{-1}(G_1(y,S(x)))$$

for $G_1, G_2 \in \Sigma$, is an equivalence relation on Σ, called *conjugation* on Σ.

Theorem 26. *Assume that $H(x)$ is the infinitesimal generator of $G(y,x) \in \Sigma$ and let $S(x) \in \Gamma_1$. The infinitesimal generator of $\tilde{G}(y,x) := S^{-1}(G(y,S(x)))$ is*

$$\tilde{H}(x) = [S'(x)]^{-1} H(S(x)).$$

Proof. We have

$$\begin{aligned}\tilde{H}(x) &= \frac{\partial}{\partial y}\,\tilde{G}(y,x)|_{y=0}\\ &= \frac{\partial}{\partial z}\,S^{-1}(z)|_{z=G(0,S(x))}\frac{\partial}{\partial y}\,G\left(y,S(x)\right)|_{y=0}\\ &= \frac{\partial}{\partial z}\,S^{-1}(z)|_{z=S(x)}H\left(S(x)\right).\end{aligned}$$

Since $S^{-1}(S(x)) = x$ we deduce $\frac{\partial}{\partial x}\left(S^{-1}(S(x))\right) = 1$. Therefore

$$1 = \frac{\partial}{\partial z}\,S^{-1}(z)|_{z=S(x)}\frac{\partial}{\partial x}\,S(x),$$

and

$$\frac{\partial}{\partial z}\,S^{-1}(z)|_{z=S(x)} = [S'(x)]^{-1},$$

which proves the theorem. □

Theorem 27. *Consider $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n \in \Sigma$ with infinitesimal generator $H(x), S(x) \in \Gamma_1$, and $\tilde{G}(y,x) := S^{-1}\left(G(y,S(x))\right)$ with generator $\tilde{H}(x) = [S'(x)]^{-1}H(S(x))$. Then the following assertions hold true.*

1. *$G(z,x)$ is a solution of ($\mathrm{D}_{\mathrm{formal}}$) if and only if*

$$\frac{\partial}{\partial z}\,\tilde{G}(z,x) = \tilde{H}\left(\tilde{G}(z,x)\right).$$

2. *$G(y,x)$ is a solution of ($\mathrm{PD}_{\mathrm{formal}}$) if and only if*

$$\frac{\partial}{\partial y}\,\tilde{G}(y,x) = \tilde{H}(x)\frac{\partial}{\partial x}\,\tilde{G}(y,x).$$

3. *$G(y,x)$ is a solution of ($\mathrm{AJ}_{\mathrm{formal}}$) if and only if*

$$\tilde{H}(x)\frac{\partial}{\partial x}\,\tilde{G}(y,x) = \tilde{H}\left(\tilde{G}(y,x)\right).$$

Theorem 28. *For each $H(x) = \sum_{n\geq k} h_n x^n$, $k \geq 2$, $h_k = 1$, there exist some $S(x) \in \Gamma_1$ and exactly one $h \in \mathbb{C}$, so that*

$$(x^k + hx^{2k-1})S'(x) = H\left(S(x)\right).$$

Proof. Assume that $S(x) = x + \sum_{n\geq 2} s_n x^n$, then $(x^k + hx^{2k-1})S'(x)$ expands to

$$\begin{aligned}x^k &+ \sum_{n=k+1}^{2k-2}(n-k+1)s_{n-k+1}x^n + (ks_k + h)x^{2k-1}\\ &+ \sum_{n\geq 2k}\left((n-k+1)s_{n-k+1} + h(n-2k+2)s_{n-2k+2}\right)x^n.\end{aligned}$$

For the computation of $H(S(x))$ we note that for $\nu \geq 2$ there exist polynomials $R_{n,\nu}(s_2,\dots,s_{n-1})$ so that

$$\left[x+\sum_{n\geq 2} s_n x^n\right]^{\nu} = x^{\nu} + \nu s_2 x^{\nu+1} + \sum_{n\geq 3}\left(\nu s_n + R_{n,\nu}(s_2,\dots,s_{n-1})\right) x^{\nu+n-1}.$$

Therefore,

$$\begin{aligned} H\left(S(x)\right) &= \sum_{\nu\geq k} h_\nu \left(x^{\nu} + \nu s_2 x^{\nu+1} + \sum_{n\geq 3}\left(\nu s_n + R_{n,\nu}(s_2,\dots,s_{n-1})\right) x^{\nu+n-1}\right)\\ &= \sum_{n\geq k}\left(h_n + \sum_{r=k}^{n-1} h_r r s_{n-r+1} + \sum_{r=k}^{n-2} h_r R_{n-r+1,r}(s_2,\dots,s_{n-r})\right) x^n. \end{aligned}$$

Next we determine the coefficients of S so that $(x^k + hx^{2k-1})S'(x) = H\left(S(x)\right)$ is satisfied. For $k+1 \leq n \leq 2k-2$ comparison of coefficients yields:

$$(n-k+1)s_{n-k+1} = h_n + \sum_{r=k}^{n-1} h_r r s_{n-r+1} + \sum_{r=k}^{n-2} h_r R_{n-r+1,r}(s_2,\dots,s_{n-r}). \tag{17}$$

Let $n = k+1$ (for $k \neq 2$), then $s_2 = h_{k+1}/(2-k)$, thus it is uniquely determined. If $2 < \nu \leq k-1$ and $s_2,\dots,s_{\nu-1}$ are uniquely determined by this equation, then (17) for $n := \nu + k - 1 \leq 2k-2$ shows that $s_\nu = s_{n-k+1}$ is uniquely determined as

$$\frac{1}{n-2k+1}\left(h_n + \sum_{r=k+1}^{n-1} h_r r s_{n-r+1} + \sum_{r=k}^{n-2} h_r R_{n-r+1,r}(s_2,\dots,s_{n-r})\right).$$

Comparing the coefficients of x^{2k-1} yields

$$k s_k + h = h_{2k-1} + \sum_{r=k}^{2k-2} h_r r s_{2k-r} + \sum_{r=k}^{2k-3} h_r R_{2k-r,r}(s_2,\dots,s_{2k-r-1})$$

which shows that h is uniquely determined by

$$h = h_{2k-1} + \sum_{r=k+1}^{2k-2} h_r r s_{2k-r} + \sum_{r=k}^{2k-3} h_r R_{2k-r,r}(s_2,\dots,s_{2k-r-1}),$$

but s_k is not determined by this equation. For that reason we choose $s_k \in \mathbb{C}$ arbitrarily but fixed.

Finally for $n \geq 2k$ we obtain

$$\begin{aligned} &(n-k+1)s_{n-k+1} + h(n-2k+2)s_{n-2k+2}\\ &\quad= h_n + \sum_{r=k}^{n-1} h_r r s_{n-r+1} + \sum_{r=k}^{n-2} h_r R_{n-r+1,r}(s_2,\dots,s_{n-r}). \end{aligned}$$

Now s_{n-k+1} is uniquely determined by $s_2, \ldots, s_{n-k}$ and h as

$$\frac{1}{n-2k+1}\left(h_n + \sum_{r=k+1}^{n-1} h_r r s_{n-r+1} \right.$$
$$\left. + \sum_{r=k}^{n-2} h_r R_{n-r+1,r}(s_2, \ldots, s_{n-r}) - h(n-2k+2)s_{n-2k+2}\right).$$

Since s_k could be chosen arbitrarily, S is not uniquely determined. □

We have just shown that for each formal iteration group of type II, there exists a unique *normal form* with respect to conjugation, so that the infinitesimal generator of this normal form is $x^k + hx^{2k-1}$ for some $h \in \mathbb{C}$.

8. Normal forms and ($\mathrm{PD}_{\mathrm{formal}}$)

In this section we consider formal iteration groups $G(y, x)$ of type II in normal form. Thus the infinitesimal generator is given by

$$H(x) = x^k + hx^{2k-1}, \qquad k \geq 2,$$

with some $h \in \mathbb{C}$.

Similar computations as in the proof of Theorem 4 yield

Theorem 29. *The solution of* ($\mathrm{PD}_{\mathrm{formal}}$) *and* (B) *for* $H(x) = x^k + hx^{2k-1}$ *is given by*

$$G(y, x) = \sum_{n\geq 0} P_{n(k-1)+1}(y)x^{n(k-1)+1}$$

where

$$P_{n(k-1)+1}(y) = \begin{cases} 1 & \text{if } n = 0 \\ y & \text{if } n = 1 \\ \prod_{i=1}^{n-1}(i(k-1)+1)\dfrac{y^n}{n!} + hQ_n(y, h) & \text{if } n \geq 2, \end{cases}$$

and where $Q_n(y, h), n \geq 2$, *is a polynomial in* y *of degree* $n-1$ *and a polynomial in* h *of degree* $\lfloor n/2 \rfloor - 1$.

Moreover, in the sequel we assume that h is an indeterminate over $(\mathbb{C}[y])[\![x]\!]$. Since for $n \geq 2$ the degree of $P_{n(k-1)+1}(y)$ as a polynomial in h is $\lfloor n/2 \rfloor$, we can write $G(y, x)$ as

$$\sum_{r\geq 0} G_r(y, x)h^r \in (\mathbb{C}[\![x, y]\!])[\![h]\!].$$

From (B) we deduce that $G_0(0,x) = x$ and $G_r(0,x) = 0$ for $r \geq 1$. Instead of ($\mathrm{PD}_{\mathrm{formal}}$) we obtain

$$\begin{aligned}\sum_{r\geq 0} \frac{\partial}{\partial y} G_r(y,x)h^r &= \left(x^k + hx^{2k-1}\right)\left(\sum_{r\geq 0} \frac{\partial}{\partial x} G_r(y,x)h^r\right) \\ &= \sum_{r\geq 0} x^k \frac{\partial}{\partial x} G_r(y,x)h^r + \sum_{r\geq 0} x^{2k-1} \frac{\partial}{\partial x} G_r(y,x)h^{r+1}\end{aligned}$$

This is a system of equations for $G_r(y,x)$, $r \geq 0$, given by

$$\frac{\partial}{\partial y} G_0(y,x) = x^k \frac{\partial}{\partial x} G_0(y,x) \tag{18}$$

and

$$\frac{\partial}{\partial y} G_r(y,x) = x^k \frac{\partial}{\partial x} G_r(y,x) + x^{2k-1} \frac{\partial}{\partial x} G_{r-1}(y,x), \qquad r \geq 1. \tag{19}$$

In order to simplify notation let $[r]$ stand for $r(k-1)+1$ for any nonnegative integer r.

Theorem 30. *Consider $H(x) = x^k + hx^{2k-1}$ where h is an indeterminate over $\mathbb{C}[\![x,y]\!]$. The solution of ((18),(19)) and (B) is given by*

$$\sum_{r\geq 0} G_r(y,x)h^r$$

where

$$G_r(y,x) = \sum_{n\geq r} \sum_{\substack{(j_1,\dots,j_r)\\ 1\leq j_1\\ j_s\geq j_{s-1}+2,\ s\geq 2\\ j_r\leq n+r-1}} \frac{\prod_{i=1}^{n+r-1}[i]}{\prod_{s=1}^{r}[j_s]}\, \frac{x^{[n+r]}}{n!}\, y^n, \qquad r \geq 0.$$

Proof. In order to solve (18) we assume that

$$G_0(y,x) = \sum_{n\geq 0} \phi_n^{(0)}(x)y^n.$$

$G_0(y,x)$ satisfies (18) if and only if

$$\sum_{n\geq 1} n\phi_n^{(0)}(x)y^{n-1} = \sum_{n\geq 0} x^k \phi_n^{(0)\prime}(x)y^n$$

which is equivalent to

$$(n+1)\phi_{n+1}^{(0)}(x) = x^k \phi_n^{(0)\prime}(x), \qquad n \geq 0.$$

From (B) we deduce that $x = G_0(0,x) = \phi_0^{(0)}(x)$ and therefore $\phi_1^{(0)}(x) = x^k = x^{[1]}$, $2\phi_2^{(0)}(x) = kx^{2k-1}$ whence $\phi_2^{(0)}(x) = [1]\frac{x^{[2]}}{2!}$ and by induction $\phi_n^{(0)}(x) = \prod_{i=1}^{n-1}[i]\frac{x^{[n]}}{n!}$.

Consider $r = 1$ and assume that

$$G_1(y,x) = \sum_{n\geq 0} \phi_n^{(1)}(x) y^n.$$

$G_1(y,x)$ satisfies (19) if and only if

$$\sum_{n\geq 0}(n+1)\phi_{n+1}^{(1)}(x)y^n = \sum_{n\geq 0} x^k \phi_n^{(1)\prime}(x)y^n + x^{2k-1}\sum_{n\geq 0}\prod_{i=1}^{n}[i]\frac{x^{n(k-1)}}{n!}\ y^n$$

which is equivalent to

$$(n+1)\phi_{n+1}^{(1)}(x) = x^k\phi_n^{(1)\prime}(x) + \prod_{i=1}^{n}[i]\frac{x^{[n+2]}}{n!}\,, \qquad n\geq 0.$$

From (B) we deduce that $\phi_0^{(1)} = 0$, $\phi_1^{(1)}(x) = x^{[2]} = \frac{[1]}{[1]}\ \frac{x^{[2]}}{1!}$,

$$\phi_2^{(1)}(x) = \sum_{j_1=1}^{2}\frac{[1][2]}{[j_1]}\ \frac{x^{[3]}}{2!}$$

and by induction

$$\phi_n^{(1)}(x) = \sum_{j_1=1}^{n}\frac{\prod_{i=1}^{n}[i]}{[j_1]}\ \frac{x^{[n+1]}}{n!}\,, \qquad n\geq 2.$$

Assume that $r\geq 2$ and that G_{r-1} has the representation as was claimed. If

$$G_r(y,x) = \sum_{n\geq 0}\phi_n^{(r)}(x)y^n,$$

then $G_r(y,x)$ satisfies (19) if and only if

$$\begin{aligned}&\sum_{n\geq 0}(n+1)\phi_{n+1}^{(r)}(x)y^n\\ &\quad=\sum_{n\geq 0}x^k\phi_n^{(r)\prime}(x)y^n + x^{2k-1}\sum_{n\geq r-1}\ \sum_{\substack{(j_1,\dots,j_{r-1})\\ 1\leq j_1\\ j_s\geq j_{s-1}+2,\ s\geq 2\\ j_{r-1}\leq n+r-2}}\frac{\prod_{i=1}^{n+r-1}[i]}{\prod_{s=1}^{r-1}[j_s]}\ \frac{x^{(n+r-1)(k-1)}}{n!}\ y^n\end{aligned}$$

which is equivalent to

$$(n+1)\phi_{n+1}^{(r)}(x) = x^k\phi_n^{(r)\prime}(x), \qquad n<r-1,$$

and

$$(n+1)\phi_{n+1}^{(r)}(x) = x^k\phi_n^{(r)\prime}(x) + \sum_{\substack{(j_1,\dots,j_{r-1})\\ 1\leq j_1\\ j_s\geq j_{s-1}+2,\ s\geq 2\\ j_{r-1}\leq n+r-2}}\frac{\prod_{i=1}^{n+r-1}[i]}{\prod_{s=1}^{r-1}[j_s]}\ \frac{x^{[n+r+1]}}{n!}\,, \quad n\geq r-1.$$

From (B) we deduce that $\phi_0^{(r)} = 0$. Therefore also $\phi_1^{(r)} = \cdots = \phi_{r-1}^{(r)} = 0$.

Comparing the coefficients of y^{r-1} we obtain that

$$r\phi_r^{(r)}(x) = \sum_{\substack{(j_1,\ldots,j_{r-1})\\ 1\le j_1\\ j_s\ge j_{s-1}+2,\ s\ge 2\\ j_{r-1}\le 2r-3}} \frac{\prod_{i=1}^{2r-2}[i]}{\prod_{s=1}^{r-1}[j_s]}\ \frac{x^{[2r]}}{(r-1)!}\,,$$

whence

$$\phi_r^{(r)}(x) = \sum_{\substack{(j_1,\ldots,j_{r-1})\\ 1\le j_1\\ j_s\ge j_{s-1}+2,\ s\ge 2\\ j_{r-1}\le 2r-3}} \frac{\prod_{i=1}^{2r-2}[i]}{\prod_{s=1}^{r-1}[j_s]}\ \frac{[2r-1]}{[2r-1]}\ \frac{x^{[2r]}}{r!}\,.$$

Let $j_r := 2r-1$, then for any sequence $(j_1,\ldots,j_{r-1})$ from this sum (actually it consists of one summand only) we have $j_r - j_{r-1} \ge (2r-1)-(2r-3) = 2$. Moreover the sequence $(j_1,\ldots,j_r) = (1,3,\ldots,2r-1)$ is the only sequence of length r with the properties $j_1 \ge 1$, $j_s \ge j_{s-1}+2$ for $s\ge 2$, and $j_r \le 2r-1$. Consequently

$$\phi_r^{(r)}(x) = \sum_{\substack{(j_1,\ldots,j_r)\\ 1\le j_1\\ j_s\ge j_{s-1}+2,\ s\ge 2\\ j_r\le 2r-1}} \frac{\prod_{i=1}^{2r-1}[i]}{\prod_{s=1}^{r}[j_s]}\ \frac{x^{[2r]}}{r!}\,.$$

Assume that $n \ge r$ and

$$\phi_n^{(r)} = \sum_{\substack{(j_1,\ldots,j_r)\\ 1\le j_1\\ j_s\ge j_{s-1}+2,\ s\ge 2\\ j_r\le n+r-1}} \frac{\prod_{i=1}^{n+r-1}[i]}{\prod_{s=1}^{r}[j_s]}\ \frac{x^{[n+r]}}{n!}\,.$$

Then we obtain from

$$(n+1)\phi_{n+1}^{(r)}(x) = x^k\phi_n^{(r)\prime}(x) + \sum_{\substack{(j_1,\ldots,j_{r-1})\\ 1\le j_1\\ j_s\ge j_{s-1}+2,\ s\ge 2\\ j_{r-1}\le n+r-2}} \frac{\prod_{i=1}^{n+r-1}[i]}{\prod_{s=1}^{r-1}[j_s]}\ \frac{x^{[n+r+1]}}{n!}$$

that

$$\phi_{n+1}^{(r)}(x) = \sum_{\substack{(j_1,\ldots,j_r)\\ 1\le j_1\\ j_s\ge j_{s-1}+2,\ s\ge 2\\ j_r\le n+1+r-2}} \frac{\prod_{i=1}^{n+1+r-1}[i]}{\prod_{s=1}^{r}[j_s]}\ \frac{x^{[n+1+r]}}{(n+1)!}$$
$$+ \sum_{\substack{(j_1,\ldots,j_{r-1})\\ 1\le j_1\\ j_s\ge j_{s-1}+2,\ s\ge 2\\ j_{r-1}\le n+1+r-3}} \frac{\prod_{i=1}^{n+1+r-2}[i]}{\prod_{s=1}^{r-1}[j_s]}\ \frac{[n+1+r-1]}{[n+1+r-1]}\ \frac{x^{[n+1+r]}}{(n+1)!}\,.$$

In the right sum let $j_r := n+1+r-1$, then $j_r - j_{r-1} \geq 2$. Since, moreover, the set of all sequences $(j_1, \ldots, j_r)$ with the properties $1 \leq j_1$, $j_s \geq j_{s-1}+2$ for $s \geq 2$, and $j_r \leq n+1+r-1$ partitions into the two sets of sequences with $j_r \leq n+1+r-2$ and $j_r = n+1+r-1$ we see that

$$\phi_{n+1}^{(r)}(x) = \sum_{\substack{(j_1,\ldots,j_r)\\ 1\leq j_1\\ j_s \geq j_{s-1}+2,\ s\geq 2\\ j_r \leq n+1+r-1}} \frac{\prod_{i=1}^{n+1+r-1}[i]}{\prod_{s=1}^{r}[j_s]} \frac{x^{[n+1+r]}}{(n+1)!},$$

which finishes the proof. □

Remark 31. In the previous theorem the indeterminate h can be replaced by an arbitrary complex number h, since for each $r \geq 0$ the coefficient $G_r(y,x)$ belongs to $(\mathbb{C}[x])[\![y]\!]$ and $\operatorname{ord} G_r(y,x)$ is equal to r. Consequently, the family $(G_r(y,x))_{r\geq 0}$ is summable and, therefore, for any $h \in \mathbb{C}$ also $(h^r G_r(y,x))_{r\geq 0}$ is summable.

Hence, replacing the indeterminate by an arbitrary complex number makes sense and Theorem 30 describes the solution of ($\mathrm{PD}_{\mathrm{formal}}$) for any generator $H(x)$ of an iteration group of type II in normal form.

9. Normal forms and ($\mathbf{D}_{\mathbf{formal}}$)

In this section we consider formal iteration groups $G(y,x)$ of type II in normal form. Thus the infinitesimal generator is given by

$$H(x) = x^k + hx^{2k-1}, \qquad k \geq 2,\ h \in \mathbb{C}.$$

Again we assume that h is an indeterminate over $\mathbb{C}[\![x,y]\!]$ and we write $G(y,x)$ as

$$\sum_{r\geq 0} G_r(y,x)h^r \in (\mathbb{C}[\![x,y]\!])[\![h]\!].$$

For the proof of the main theorem we need the following lemma which can easily be proved by differentiating the primitive function, or by suitable partial differentiation.

Lemma 32. 1. *Let $s > 1$ be an integer. Then*

$$\int \frac{dy}{(1-(k-1)yx^{k-1})^s} = \frac{1}{(s-1)(k-1)x^{k-1}(1-(k-1)yx^{k-1})^{s-1}} + c(x)$$

with $c(x) \in \mathbb{C}[\![x]\!]$.

2. *If, moreover, r is a positive integer, then there exists a polynomial Q_r of degree r such that*

$$\int \frac{(\ln(1-(k-1)yx^{k-1}))^r}{(1-(k-1)yx^{k-1})^s}\,dy = \frac{Q_r(\ln(1-(k-1)yx^{k-1})}{x^{k-1}(1-(k-1)yx^{k-1})^{s-1}} + c(x)$$

with $c(x) \in \mathbb{C}[\![x]\!]$.

3. *If* $\hat{P}_n$ *is a polynomial of degree* $n \geq 0$, *then there exists a polynomial* P_n *of degree* n *such that*

$$\int \frac{\hat{P}_n(\ln(1-(k-1)yx^{k-1}))}{(1-(k-1)yx^{k-1})^s}\,dy = \frac{P_n(\ln(1-(k-1)yx^{k-1})}{x^{k-1}(1-(k-1)yx^{k-1})^{s-1}} + c(x)$$

with $c(x) \in \mathbb{C}[\![x]\!]$.

The left hand side denotes the primitive functions of the integrand in $(\mathbb{C}[\![x]\!])[\![y]\!]$.

Theorem 33. *The solution of* ($\mathrm{D}_{\mathrm{formal}}$) *and* (B) *for* $H(x) = x^k + hx^{2k-1}$ *is given by*

$$G(y,x) = \sum_{r\geq 0} G_r(y,x)h^r$$

where

$$G_r(y,x) = x^{[r]}(1-(k-1)yx^{k-1})^{-[r]/(k-1)}P_r(\ln(1-(k-1)yx^{k-1})), \qquad r \geq 0,$$

where $[r]$ *stands for* $r(k-1)+1$ *and* P_r *are polynomials of degree* r. *Moreover* $P_0 = 1$ *and* $P_1(z) = -z/(k-1)$.

Proof. In the present setting, from ($\mathrm{D}_{\mathrm{formal}}$) we derive

$$\sum_{r\geq 0} \frac{\partial}{\partial y} G_r(y,x)h^r = \left(\sum_{r\geq 0} G_r(y,x)h^r\right)^k + h\left(\sum_{r\geq 0} G_r(y,x)h^r\right)^{2k-1}. \qquad (20)$$

Comparing the coefficients of h^0 we obtain the differential equation

$$\frac{\partial}{\partial y} G_0(y,x) = G_0(y,x)^k.$$

From (B) we deduce that $G_0(0,x) = x$ and $G_r(0,x) = 0$ for $r \geq 1$. It is easy to prove that $G_0(y,x) = x(1-(k-1)yx^{k-1})^{-1/(k-1)}$ satisfies this differential equation and boundary condition.

Comparison of coefficients of h^1 yields

$$\frac{\partial}{\partial y} G_1(y,x) = kG_0(y,x)^{k-1}G_1(y,x) + G_0(y,x)^{2k-1}.$$

First we solve the homogeneous equation

$$\frac{\partial}{\partial y} \tilde{G}_1(y,x) = kG_0(y,x)^{k-1}\tilde{G}_1(y,x).$$

Straight forward computations show that $\tilde{G}_1(y,x) = c(1-(k-1)yx^{k-1})^{-k/(k-1)}$ for some $c \in \mathbb{C}$. Now by variation of the integration constant we solve the inhomogeneous equation together with the boundary equation $G_1(0,x) = 0$. We set $c = 1$ in $\tilde{G}_1(y,x)$ and we set $G_1(y,x) = C_1(y,x)\tilde{G}_1(y,x)$. Then

$$\begin{aligned}\frac{\partial}{\partial y}\,G_1(y,x) &= \tilde{G}_1(y,x)\frac{\partial}{\partial y}\,C_1(y,x) + C_1(y,x)\frac{\partial}{\partial y}\,\tilde{G}_1(y,x)\\ &= \tilde{G}_1(y,x)\frac{\partial}{\partial y}\,C_1(y,x) + kG_0(y,x)^{k-1}G_1(y,x).\end{aligned}$$

Therefore the differential equation for G_1 reduces to

$$\begin{aligned}\frac{\partial}{\partial y}\,C_1(y,x) &= (1-(k-1)yx^{k-1})^{k/(k-1)}G_0(y,x)^{2k-1}\\ &= (1-(k-1)yx^{k-1})^{k/(k-1)}x^{2k-1}(1-(k-1)yx^{k-1})^{-(2k-1)/(k-1)}\\ &= x^{2k-1}(1-(k-1)yx^{k-1})^{-1}.\end{aligned}$$

Consequently, $C_1(y,x) = (-1/(k-1))\,x^k\ln(1-(k-1)yx^{k-1}) + D_1(x)$. Due to the boundary condition $G_1(0,x) = 0$ we have $C_1(0,x) = 0$, whence $D_1(x) = 0$. In conclusion

$$G_1(y,x) = (-1/(k-1))\,x^k\ln(1-(k-1)yx^{k-1})(1-(k-1)yx^{k-1})^{-k/(k-1)}.$$

Now we assume that the hypothesis is true for $G_0(y,x),\ldots,G_n(y,x)$, $n \geq 1$, and we prove that it also holds for $G_{n+1}(y,x)$. In order to compute the right hand side of (20) we note that

$$\left[\sum_{r\geq 0} G_r(y,x)h^r\right]^m = \sum_{N\geq 0}\left(\sum_{\substack{(j_0,\ldots,j_N)\\ \sum j_i=m\\ \sum ij_i=N}}\binom{m}{j_0\ldots j_N}\prod_{i=0}^{N}G_i(y,x)^{j_i}\right)h^N.$$

Therefore we obtain from (20) that

$$\begin{aligned}&\frac{\partial}{\partial y}\,G_{n+1}(y,x)\\ &\quad= kG_0(y,x)^{k-1}G_{n+1}(y,x)\\ &\qquad+\underbrace{\sum_{\substack{(j_0,\ldots,j_n)\\ \sum j_i=k\\ \sum ij_i=n+1}}\binom{k}{j_0\ldots j_n}\prod_{i=0}^{n}G_i(y,x)^{j_i}}_{=:\phi_{n+1}(G_0,\ldots,G_n)}+\underbrace{\sum_{\substack{(j_0,\ldots,j_n)\\ \sum j_i=2k-1\\ \sum ij_i=n}}\binom{2k-1}{j_0\ldots j_n}\prod_{i=0}^{n}G_i(y,x)^{j_i}}_{=:\psi_{n+1}(G_0,\ldots,G_n)}\\ &\quad= kG_0(y,x)^{k-1}G_{n+1}(y,x) + \phi_{n+1}(G_0,\ldots,G_n) + \psi_{n+1}(G_0,\ldots,G_n).\end{aligned}$$

Similar to the case $n = 1$, the solutions of the homogeneous equation are $c(1-(k-1)yx^{k-1})^{-k/(k-1)}$ for $c \in \mathbb{C}$. By variation of constants we set

$G_{n+1}(y,x) = C_{n+1}(y,x)(1-(k-1)yx^{k-1})^{-k/(k-1)}$ which leads to the differential equation

$$\begin{aligned}&\frac{\partial}{\partial y} C_{n+1}(y,x)\\ &\quad = (1-(k-1)yx^{k-1})^{k/(k-1)} \left(\phi_{n+1}(G_0,\dots,G_n) + \psi_{n+1}(G_0,\dots,G_n)\right).\end{aligned}$$

From the induction hypothesis we determine the structure of the summands of ϕ_{n+1} and ψ_{n+1}. These summands are integer multiples of terms of the form

$$\begin{aligned}&\prod_{i=0}^{n} G_i(y,x)^{j_i}\\ &\quad = \prod_{i=0}^{n} \left(x^{[i]}(1-(k-1)yx^{k-1})^{-[i]/(k-1)} P_i(\ln(1-(k-1)yx^{k-1}))\right)^{j_i}\\ &\quad = x^{\sum j_i[i]}(1-(k-1)yx^{k-1})^{-\sum j_i[i]/(k-1)} \prod_{i=0}^{n} P_i(\ln(1-(k-1)yx^{k-1}))^{j_i}.\end{aligned}$$

Since P_i are polynomials of degree i, the product $\prod_{i=0}^{n} P_i$ is a polynomial of degree $\sum ij_i$. Moreover $\sum[i]j_i = (k-1)\sum ij_i + \sum j_i$. Hence, the summands of ϕ_{n+1} are multiples of

$$\begin{aligned}&x^{(n+1)(k-1)+k}(1-(k-1)yx^{k-1})^{-((n+1)(k-1)+k)/(k-1)}\\ &\qquad \cdot \tilde{P}_{n+1}(\ln(1-(k-1)yx^{k-1}))\\ &\quad = x^{[n+2]}(1-(k-1)yx^{k-1})^{-[n+2]/(k-1)} \tilde{P}_{n+1}(\ln(1-(k-1)yx^{k-1}))\end{aligned}$$

with suitable polynomials $\tilde{P}_{n+1}$ of degree $n+1$. Similarly the summands of ψ_{n+1} are multiples of

$$x^{[n+2]}(1-(k-1)yx^{k-1})^{-[n+2]/(k-1)} \bar{P}_{n+1}(\ln(1-(k-1)yx^{k-1}))$$

with suitable polynomials $\bar{P}_{n+1}$ of degree n. The coefficients of these terms are nonnegative integers. Now we prove that there exists at least one summand in ϕ_{n+1} with positive coefficients: If $n=1$, then consider $(j_0,j_1) = (k-2,2)$ which satisfies $j_i \geq 0$, $\sum j_i = k$ and $\sum ij_i = 2 = n+1$. If $n>1$, then consider $j_0 = k-2$, $j_1 = 1$, $j_n = 1$ and $j_i = 0$ for $1 < i < n$ which also satisfies $j_i \geq 0$, $\sum j_i = k$ and $\sum ij_i = n+1$. The corresponding multinomial coefficients $\binom{k}{j_0 \cdots j_n}$ are positive. In conclusion

$$\begin{aligned}\phi_{n+1} + \psi_{n+1} = x^{[n+2]}&(1-(k-1)yx^{k-1})^{-[n+2]/(k-1)}\\ &\cdot \hat{P}_{n+1}(\ln(1-(k-1)yx^{k-1}))\end{aligned}$$

with a suitable polynomial $\hat{P}_{n+1}$ of degree $n+1$. Therefore, we have to solve the differential equation

$$\frac{\partial}{\partial y} C_{n+1}(y,x) = x^{[n+2]}(1-(k-1)yx^{k-1})^{-(n+1)} \hat{P}_{n+1}(\ln(1-(k-1)yx^{k-1})).$$

Due to Lemma 32 there exists a polynomial P_{n+1} of degree $n+1$ so that

$$C_{n+1}(y,x) = x^{[n+2]}\frac{P_{n+1}(\ln(1-(k-1)yx^{k-1}))}{x^{k-1}(1-(k-1)yx^{k-1})^n} + D_{n+1}(x).$$

According to (B) we have $C_{n+1}(0,x) = 0$, whence $D_{n+1}(x) = 0$. In conclusion we obtain

$$\begin{aligned} G_{n+1}(y,x) &= C_{n+1}(y,x)(1-(k-1)yx^{k-1})^{-k/(k-1)} \\ &= x^{[n+2]}\frac{P_{n+1}(\ln(1-(k-1)yx^{k-1}))}{x^{k-1}(1-(k-1)yx^{k-1})^n}(1-(k-1)yx^{k-1})^{-k/(k-1)} \\ &= x^{[n+1]}(1-(k-1)yx^{k-1})^{-[n+1]/(k-1)}P_{n+1}(\ln(1-(k-1)yx^{k-1})). \end{aligned}$$

□

Remark 34. In the previous theorem the indeterminate h can be replaced by an arbitrary complex number h, since for each $r \geq 0$ the coefficient $G_r(y,x)$ belongs to $(\mathbb{C}[x])[\![y]\!]$ and $\operatorname{ord} G_r(y,x)$ is equal to r. Consequently, the family $(G_r(y,x))_{r\geq 0}$ is summable and, therefore, for any $h \in \mathbb{C}$ also $(h^r G_r(y,x))_{r\geq 0}$ is summable.

Hence, replacing the indeterminate by an arbitrary complex number makes sense and Theorem 33 describes the solution of ($\mathrm{D}_{\mathrm{formal}}$) for any generator $H(x)$ of an iteration group of type II in normal form.

Remark 35. So far the application of normal forms of ($\mathrm{AJ}_{\mathrm{formal}}$) did not lead to clear and easy to understand representations of the coefficients of the solution. Hence, we do not present this approach in the present paper.

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Received: March 26, 2009
Revised: December 21, 2009