Aequationes Math. 68 (2004) 200–229
0001-9054/04/030200-30
DOI 10.1007/s00010-004-2742-x

Aequationes Mathematicae

On the general solution of the system of cocycle equations without regularity conditions

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Dedicated to Professor János Aczél on the occasion of his eightieth birthday

Summary. We describe the general solution (α, β), where $\alpha = (\alpha(s,x))_{s\in\mathbb{C}}$ and $\beta = (\beta(s,x))_{s\in\mathbb{C}}$ are families of formal power series in $\mathbb{C}\llbracket x \rrbracket$, of the two so-called cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha(t,\pi(s,x)), \qquad s,t \in \mathbb{C} \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x)), \qquad s,t \in \mathbb{C} \tag{Co2}$$

together with the boundary condition

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

where $\pi = (\pi(s,x))_{s\in\mathbb{C}}$ is an iteration group in $\mathbb{C}\llbracket x \rrbracket$. Our method is based on the knowledge of the regular solutions of (Co1) and (Co2) and on a well-known and often used theorem concerning the algebraic relations between exponential functions and additive functions.

Mathematics Subject Classification (2000). Primary: 39B12, 39B50; Secondary: 13F25.

Keywords. The system of cocycle equations, embedding of a linear functional equation with respect to an iteration group, functional equations in rings of formal power series.

1. The problem

For the families $\alpha = (\alpha(s,x))_{s\in\mathbb{C}}$ and $\beta = (\beta(s,x))_{s\in\mathbb{C}}$ of formal power series

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \quad \beta(s,x) = \sum_{n\geq 0} \beta_n(s)x^n \in \mathbb{C}\llbracket x \rrbracket$$

with unknown coefficient functions $\alpha_n, \beta_n \colon \mathbb{C} \to \mathbb{C}$, $n \geq 0$, we study the system of so called cocycle equations

$$\alpha(s+t,x) = \alpha(s,x)\alpha(t,\pi(s,x)), \qquad s,t \in \mathbb{C} \tag{Co1}$$

$$\beta(s+t,x) = \beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x)), \qquad s,t \in \mathbb{C} \tag{Co2}$$

together with the boundary condition

$$\alpha(0,x) = 1, \qquad \beta(0,x) = 0, \tag{B1}$$

where $\pi = (\pi(s,x))_{s\in\mathbb{C}}$ is an iteration group, i.e. π is a solution of the translation equation

$$\pi(s+t,x) = \pi(t,\pi(s,x)), \qquad s,t \in \mathbb{C} \tag{T}$$

of the form

$$\pi(s,x) = \sum_{n\geq 1} \pi_n(s)x^n, \qquad s \in \mathbb{C}$$

with $\pi_n\colon \mathbb{C} \to \mathbb{C}$, $n \geq 1$, and $\pi_1(s) \neq 0$ for all $s \in \mathbb{C}$.

This problem has already been solved for analytic iteration groups and solutions α, β with entire coefficient functions when discussing covariant embeddings of a linear functional equation with respect to analytic iteration groups. (Cf. [4, 3].) These analytic solutions were computed by using differentiation (coefficientwise differentiation and integration, mixed chain rules, etc.).

For a foundation of the basic calculations with formal power series we refer the reader to [9] and to [2]. If $\psi(x) \in \mathbb{C}\,[\![x]\!]$ is of the form $\psi(x) = \sum_{n\geq k} \psi_n x^n$ with $\psi_k \neq 0$, then k is the order of ψ, which will be indicated as $\operatorname{ord}\psi(x) = k$.

Furthermore, the notion of *congruence modulo* x^r will be useful. We write $\varphi \equiv \psi \bmod x^r$ for formal power series $\varphi(x), \psi(x) \in \mathbb{C}\,[\![x]\!]$ if x^r is a divisor of the difference $\varphi(x) - \psi(x)$. In other words $\varphi(x) - \psi(x) = 0$, or its order is greater than or equal to r. The exponential series is given as

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!},$$

and the formal logarithm is the series defined by

$$\ln(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{n}.$$

Multiplicative powers of a formal series are usually indicated as $[\psi(x)]^n$ and not as $\psi(x)^n$.

In the meantime W. Jabłoński and L. Reich [10, 11] succeeded in the classification of all iteration groups π (without any regularity conditions). In the same way as with analytic iteration groups they distinguished two types of iteration groups:

Iteration groups of type I are of the form

$$\pi(s,x) = \pi_1(s)x + \sum_{\ell\geq 2} P_\ell(\pi_1(s))x^\ell, \qquad s \in \mathbb{C},$$

where $\pi_1\colon \mathbb{C} \to \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is a generalized exponential function, $\pi_1 \neq 1$, and where $P_\ell(y) \in \mathbb{C}[y]$ is a polynomial of formal degree equal to ℓ. At the moment we do not need the detailed structure, the general form, and the universal character (depending on certain parameters) of these polynomials P_ℓ. Iteration groups of type I have very simple normal forms, since for each π of type I there exists some $S(x) = x + s_2x^2 + \cdots \in \mathbb{C}\llbracket x \rrbracket$ such that

$$\pi(s,x) = S^{-1}(\pi_1(s)S(x)), \qquad s \in \mathbb{C}.$$

As it was shown in [4] it is enough to solve (Co1) and (Co2) for these normal forms $\pi(s,x) = \pi_1(s)x$. The analytic iteration groups of type I are obtained for $\pi_1(s) = e^{\mu s}$ with $\mu \neq 0$, a regular exponential function.

Iteration groups of type II can be described as

$$\pi(s,x) = x + \pi_k(s)x^k + \sum_{\ell > k} P_\ell(\pi_k(s))x^\ell, \qquad s \in \mathbb{C},$$

where $k \geq 2$, $\pi_k \colon \mathbb{C} \to \mathbb{C}$ is an additive function, $\pi_k \neq 0$, and where $P_\ell(y) \in \mathbb{C}[y]$ is a polynomial of formal degree equal to $\lfloor \frac{\ell-1}{k-1} \rfloor$. Here, too, we do not need the detailed structure, the general form, and the universal character of these polynomials P_ℓ. But the reader should be aware of the fact that even though we use the same letters these polynomials are different from the polynomials occurring in the description of iteration groups of type I. The analytic iteration groups of type II are obtained for $\pi_k(s) = c_k s$ with $c_k \neq 0$. In comparison with iteration groups of type I, these groups do not have so simple normal forms, and for that reason we do not use normal forms in our approach.

Our method in the present paper is based on the knowledge of the regular solutions of (Co1) and (Co2) as presented in [4], on a well-known and often used theorem [13] concerning the algebraic relations between exponential functions and additive functions, and on results about iteration groups in power series rings without regularity conditions [10, 11].

Finally, let us mention some facts concerning our notation. Throughout this paper the coefficients of a formal series or the coefficient functions of a family of series will be indicated with the same, but indexed, letter as the series or the family. Iteration groups are indicated with π, a particular analytic iteration group of type II with π^*. The polynomials P_ℓ describe the coefficient functions of an iteration group as polynomials in π_1 or π_k. The integer $k \geq 2$ is the index of the second nonzero coefficient function of an iteration group of type II. In the different theorems the polynomials Q_n usually depend on π_k and certain other coefficient functions. They are auxiliary polynomials, and in each situation they are a little bit differently defined. The families α and β denote solutions of (Co1) and (Co2). The families γ and Δ are closely related to α and β. For describing α we use series $E(x) \equiv 1 \bmod x$, and families $P(s,x)$. The families β are described with an additional series $F(x) \in \mathbb{C}\llbracket x \rrbracket$, and a family $Q(s,x)$. Here we tried to use the same notation as in the fundamental paper [4]. The reader should not mix these families $P(s,x)$ and $Q(s,x)$ with the earlier mentioned polynomials P and Q.

Knowing the solutions of the system of cocycle equations is an important step when discussing covariant embeddings of a linear functional equation. Such embeddings were studied in a very general setting by Z. Moszner in [12] and for functions defined on a real interval by G. Guzik in [5] and [7]. The first cocycle equation is also studied in [6] and [8]. It also appears as the triangular equation for instance in [1].

2. The general solution of (Co1)

As in the analytic case we can prove that for any solution α of (Co1) and (B1) the coefficient function α_0 satisfies $\alpha_0(s) \neq 0$ for all $s \in \mathbb{C}$. If we define $\hat{\alpha}(s,x)$ by

$$\alpha(s,x) = \alpha_0(s)\underbrace{\Big(1 + \sum_{n\geq 1} \frac{\alpha_n(s)}{\alpha_0(s)} x^n\Big)}_{=:\hat{\alpha}(s,x)}, \qquad s \in \mathbb{C},$$

then we obtain that α is a solution of (Co1) and (B1) if and only if α_0 is a generalized exponential function and $\hat{\alpha}$ is a solution of (Co1) and (B1). It can easily be verified that if $\hat{\alpha}(s,x) = 1 + \ldots$ is a solution of (Co1), then (B1) is also satisfied.

Finally introducing γ by

$$\gamma(s,x) := \ln(\hat{\alpha}(s,x)) = \sum_{n\geq 1} \gamma_n(s)x^n, \quad s \in \mathbb{C},$$

we obtain the following characterization:

Lemma 1. *The family $\hat{\alpha}$ is a solution of* (Co1) *if and only if γ is a solution of*

$$\gamma(s+t,x) = \gamma(s,x) + \gamma(t,\pi(s,x)), \qquad s,t \in \mathbb{C}. \qquad (\mathrm{Co1}')$$

Now it will be useful to distinguish between iteration groups of type I and type II.

2.1 The general solution of (Co1) for iteration groups of type I

In this section we always assume that the iteration group π is given in normal form, i.e. $\pi(s,x) = \pi_1(s)x$, where $\pi_1 \neq 1$ is a generalized exponential function.

First we determine necessary conditions on α for being a solution of (Co1). Thus we assume that $\hat{\alpha}$ is a solution of (Co1), and that $\gamma = \ln(\hat{\alpha})$ satisfies (Co1′).

Lemma 2. *The family γ is a solution of* (Co1′) *if and only if there exists $D(x) \in \mathbb{C}[\![x]\!]$ of order ≥ 1 such that*

$$\gamma(s,x) = D(\pi_1(s)x) - D(x), \qquad s \in \mathbb{C}.$$

Proof. Comparing coefficients in (Co1′), and using the fact that $\gamma_n(s+t) = \gamma_n(t+s)$ we obtain that

$$\gamma_n(s) + \gamma_n(t)\pi_1(s)^n = \gamma_n(t) + \gamma_n(s)\pi_1(t)^n, \qquad s,t \in \mathbb{C},\ n \geq 1.$$

Since $\pi_1 \neq 1$ is a generalized exponential function, there exists a series $(t_n)_{n\geq 1}$ of complex numbers such that $\pi_1(t_n)^n \neq 1$ for all $n \geq 1$. Thus

$$\gamma_n(s) = \frac{\gamma_n(t_n)}{\pi_1(t_n)^n - 1} \left(\pi_1(s)^n - 1\right), \qquad s \in \mathbb{C},\ n \geq 1,$$

which means

$$\gamma_n(s) = D_n(\pi_1(s)^n - 1), \qquad s \in \mathbb{C},\ n \geq 1,$$

with $D_n = \gamma_n(t_n)/(\pi_1(t_n)^n - 1) \in \mathbb{C}$. Consequently, $\gamma(s,x) = D(\pi_1(s)x) - D(x)$ with $D(x) = \sum_{n\geq 1} D_n x^n$. Conversely, each γ of this form is a solution if (Co1′). □

As an immediate consequence we derive that the general solution of (Co1) for iteration groups of type I can be described exactly in the same way as for analytic iteration groups of type I.

Corollary 3. *Assume that π is an iteration group of type I given in its normal form. The family α is a solution of* (Co1) *and* (B1)*, if and only if there exists some $E(x) \in \mathbb{C}\llbracket x \rrbracket$, $E(x) \equiv 1 \bmod x$, such that*

$$\alpha(s,x) = \alpha_0(s)\frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C},$$

where α_0 is a generalized exponential function.

Proof. Assuming that α is a solution of (Co1) and (B1), and writing it in the form $\alpha_0\hat{\alpha}$, we already know that α_0 is a generalized exponential function, and $\gamma(s,x) = \ln(\hat{\alpha}(s,x)) = D(\pi_1(s)x) - D(x)$ for some $D(x) \in \mathbb{C}\llbracket x \rrbracket$ of order ≥ 1. Thus, $D(x)$ can be substituted into any formal series and we obtain

$$\hat{\alpha}(s,x) = \exp(\gamma(s,x)) = \exp(D(\pi_1(s)x) - D(x)) = \frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C},$$

where $E(x) = \exp(D(x)) \equiv 1 \bmod x$.

Conversely, direct computations prove that each α of this form is a solution of (Co1) and (B1). □

Remark 4. Even if the iteration group π is not given in its normal form, each solution of (Co1) is necessarily of the above mentioned form.

Proof. Assume that $\pi(s,x) = S^{-1}(\tilde{\pi}(s,S(x)))$, $s \in \mathbb{C}$ with $S(x) = x + s_2x^2 + \ldots$, and $\tilde{\pi}(s,x) = \pi_1(s)x$. The same way as in Theorem 1.3 of [4] it is possible to prove that the general solution of (Co1) (i.e. the set of all solutions of (Co1)) is in one-to-one correspondence to the general solution $\tilde{\alpha}$ of

$$\tilde{\alpha}(s+t,x) = \tilde{\alpha}(s,x)\tilde{\alpha}(t,\tilde{\pi}(s,x)), \qquad s,t \in \mathbb{C} \tag{$\tilde{\text{C}}$o1}$$

via
$$\alpha(s, S^{-1}(y)) = \tilde{\alpha}(s, y), \qquad s \in \mathbb{C}.$$
Hence, since
$$\tilde{\alpha}(s, y) = \alpha_0(s)\frac{\tilde{E}(\tilde{\pi}(s, y))}{\tilde{E}(y)}, \qquad s \in \mathbb{C},$$
we obtain by an easy computation
$$\alpha(s, x) = \alpha_0(s)\frac{E(\pi(s, x))}{E(x)}, \qquad s \in \mathbb{C}$$
with $E(x) = (\tilde{E} \circ S)(x)$. □

2.2 The general solution of (Co1) for iteration groups of type II

In this section we write the iteration group in the form
$$\pi(s, x) = x + \pi_k(s)x^k + \sum_{\ell > k} P_\ell(\pi_k(s))x^\ell, \qquad s \in \mathbb{C},$$
where $k \geq 2$ and $\pi_k \neq 0$ is an additive function. Moreover, we also consider the corresponding analytic iteration group π^*, which is given by
$$\pi^*(s, x) = x + \pi_k^*(s)x^k + \sum_{\ell > k} P_\ell(\pi_k^*(s))x^\ell, \qquad s \in \mathbb{C},$$
with the same polynomials P_ℓ as in π, and where $\pi_k^*(s) = s$ for $s \in \mathbb{C}$.

The close connection between π and π^* is described by
$$\pi(s, x) = \pi^*(\pi_k(s), x), \qquad s \in \mathbb{C}.$$

For the proof of the next theorem and in order to prove that π^*, as introduced above, is always an iteration group, we need the following preparatory

Lemma 5. *Assume that $a \neq 0$ is an additive function from $\mathbb{C}$ to $\mathbb{C}$, and $R(y, z)$ is a polynomial in $\mathbb{C}[y, z]$. Then $R(a(s), a(t)) = 0$ for all $s, t \in \mathbb{C}$, if and only if $R = 0$.*

Proof. We only prove the nontrivial part of the assertion. Assume that $R(a(s), a(t)) = 0$ is satisfied for all $s, t \in \mathbb{C}$. We collect the summands of R with respect to powers of z, obtaining
$$R(y, z) = \sum_{i=0}^{d} r_i(z)y^i \quad \text{with} \quad r_i \in \mathbb{C}[z]$$
with some suitable d. First we fix an arbitrary element $t_0 \in \mathbb{C}$, and we get
$$0 = R(a(s), a(t_0)) = \sum_{i=0}^{d} r_i(a(t_0))a(s)^i.$$

Since a is a non-trivial additive function, $a(s)$ takes infinitely many values for $s \in \mathbb{C}$, whence $r_i(a(t_0)) = 0$ for $0 \le i \le d$. The element t_0 was arbitrarily chosen in $\mathbb{C}$, thus each $t \in \mathbb{C}$ must satisfy $r_i(a(t)) = 0$ for all i, which yields that $r_i = 0$ for $0 \le i \le d$. Hence $R = 0$. $\square$

Some technical details about multiplicative powers of $\pi(s,x)$ are collected in

Lemma 6. *Assume that*

$$\pi(s,x) = x + \pi_k(s)x^k + \sum_{\ell>k} P_\ell(\pi_k(s))x^\ell, \qquad s \in \mathbb{C}$$

is an iteration group of type II.

1. *For $n \ge 1$ the multiplicative n-th power of $\pi(s,x)$ is given by*

$$[\pi(s,x)]^n = x^n + n\pi_k(s)x^{n-1+k} + \sum_{\ell>n-1+k} \tilde{P}_\ell^{(n)}(\pi_k(s))x^\ell$$

where $\tilde{P}_\ell^{(n)}(\pi_k(s))$ are polynomials in $\pi_k(s)$.

2. *If $\varphi(t,s) = \sum_{n\ge 0} \varphi_n(t)x^n$, then*

$$\begin{aligned}\varphi(t,\pi(s,x)) = &\sum_{n=0}^{k-1} \varphi_n(t)x^n + \big(\varphi_k(t) + \pi_k(s)\varphi_1(t)\big)x^k \\ &+ \sum_{n>k}\Big(\varphi_n(t) + \pi_k(s)(n+1-k)\varphi_{n+1-k}(t) \\ &\qquad + Q_n(\pi_k(s),\varphi_1(t),\dots,\varphi_{n-k}(t))\Big)x^n\end{aligned}$$

with polynomials $Q_n(\pi_k(s),\varphi_1(t),\dots,\varphi_{n-k}(t))$ which are linear in $\varphi_j(t)$.

Proof. 1. For $n = 1$ the assertion is trivial. Assume that $n \ge 2$, and define the polynomial P_k by $P_k(y) = y$, then

$$\pi(s,x) = x + \sum_{\ell\ge k} P_\ell(\pi_k(s))x^\ell,$$

and

$$[\pi(s,x)]^n = \sum_{j=0}^{n} \binom{n}{j} x^j \left(\sum_{\ell\ge k} P_\ell(\pi_k(s))x^\ell\right)^{n-j}$$

$$= \sum_{j=0}^{n-2} \binom{n}{j} x^j \left(\sum_{\ell\ge k} P_\ell(\pi_k(s))x^\ell\right)^{n-j} + nx^{n-1}\sum_{\ell\ge k} P_\ell(\pi_k(s))x^\ell + x^n.$$

Obviously the order of

$$\binom{n}{j} x^j \left(\sum_{\ell \geq k} P_\ell(\pi_k(s)) x^\ell \right)^{n-j}$$

is $\geq j + k(n-j) = kn - (k-1)j$ for $0 \leq j \leq n-2$. Consequently

$$\begin{aligned} \operatorname{ord} \left(\sum_{j=0}^{n-2} \binom{n}{j} x^j \left(\sum_{\ell \geq k} P_\ell(\pi_k(s)) x^\ell \right)^{n-j} \right) &\geq kn - (k-1)(n-2) \\ &= n + 2(k-1) > n + k - 1. \end{aligned}$$

Since all the occurring coefficient functions are polynomials in $\pi_k(s)$ we deduce that

$$[\pi(s,x)]^n = x^n + n\pi_k(s) x^{n-1+k} + \sum_{\ell > n-1+k} \tilde{P}_\ell^{(n)}(\pi_k(s)) x^\ell$$

with suitable polynomials $\tilde{P}_\ell^{(n)}(y) \in \mathbb{C}[y]$.

2. Using the just derived representations of $[\pi(s,x)]^n$ we obtain

$$\begin{aligned} \varphi(t, \pi(s,x)) &= \sum_{n \geq 0} \varphi_n(t) [\pi(s,x)]^n \\ &= \varphi_0(t) + \sum_{n \geq 1} \varphi_n(t) \Big(x^n + n\pi_k(s) x^{n-1+k} + \sum_{\ell > n-1+k} \tilde{P}_\ell^{(n)}(\pi_k(s)) x^\ell \Big). \end{aligned}$$

Collecting terms of order n yields the assertion. □

Remark 7. Assume that π is an iteration group of type II with

$$\pi(s,x) = x + \pi_k(s) x^k + \sum_{\ell > k} P_\ell(\pi_k(s)) x^\ell, \qquad s \in \mathbb{C},$$

then π^* as defined above is also an iteration group of type II.

Proof. Since π is a solution of the translation equation (T) we have

$$\begin{aligned} & x + \pi_k(s+t) x^k + \sum_{\ell > k} P_\ell(\pi_k(s+t)) x^\ell \\ &= \pi(s,x) + \pi_k(t) [\pi(s,x)]^k + \sum_{\ell > k} P_\ell(\pi_k(t)) [\pi(s,x)]^\ell \end{aligned}$$

for all $s, t \in \mathbb{C}$. Expanding the right-hand side we finally derive

$$\begin{aligned} & x + (\pi_k(s) + \pi_k(t)) x^k + \sum_{\ell > k} P_\ell(\pi_k(s) + \pi_k(t)) x^\ell \\ &= x + (\pi_k(s) + \pi_k(t)) x^k + \sum_{\ell > k} R_\ell(\pi_k(s), \pi_k(t)) x^\ell \end{aligned}$$

with $R_\ell(y,z) \in \mathbb{C}[y,z]$ a polynomial, for $\ell > k$. Comparison of coefficients yields

$$P_\ell(\pi_k(s)+\pi_k(t)) = R_\ell(\pi_k(s),\pi_k(t)), s,t \in \mathbb{C},\ \ell > k.$$

Since π_k is a nontrivial additive function, according to Lemma 5 we are allowed to replace $\pi_k(s)$ and $\pi_k(t)$ by indeterminates S and T obtaining

$$P_\ell(S+T) = R_\ell(S,T), \qquad \ell > k.$$

Finally substituting s for S and t for T we get

$$P_\ell(s+t) = R_\ell(s,t), \qquad \ell > k.$$

This means that

$$\begin{aligned} &x + (\pi_k^*(s)+\pi_k^*(t))x^k + \sum_{\ell>k} P_\ell(\pi_k^*(s)+\pi_k^*(t))x^\ell \\ &= x + (\pi_k^*(s)+\pi_k^*(t))x^k + \sum_{\ell>k} R_\ell(\pi_k^*(s),\pi_k^*(t))x^\ell \end{aligned}$$

for $s,t \in \mathbb{C}$, thus π^* is a solution of (T). Then it is clear that π^* is of type II. □

Theorem 8. *The family α is a solution of* (Co1) *and* (B1) *where π is an iteration group of type II, if and only if there exists some $E(x) \in \mathbb{C}[\![x]\!]$, $E(x) \equiv 1 \bmod x$, such that*

$$\alpha(s,x) = \alpha_0(s)P(s,x)\frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C},$$

where α_0 is a generalized exponential function, and where

$$P(s,x) = \prod_{n=1}^{k-1} \exp\left(\kappa_n \int_0^\tau [\pi^*(\sigma,x)]^n d\sigma\Big|_{\tau=\pi_k(s)}\right),$$

with $\kappa_1,\dots,\kappa_{k-1} \in \mathbb{C}$. The coefficients of $P(s,x)$ and the coefficients of $\alpha(s,x)$ are polynomials in $\pi_k(s)$.

Proof. First we assume that $\alpha = \alpha_0\hat{\alpha}$ is a solution of (Co1) and (B1), whence, $\gamma = \ln(\hat{\alpha})$ is a solution of (Co1′). According to Lemma 6 the coefficient of x^n in the expansion of

$$\gamma(t,\pi(s,x)) = \sum_{n\ge1}\gamma_n(t)\left[x + \pi_k(s)x^k + \sum_{\ell>k} P_\ell(\pi_k(s))x^\ell\right]^n$$

equals

$$\begin{cases} \gamma_n(t) & \text{if } n<k, \\ \gamma_k(t) + \pi_k(s)\gamma_1(t) & \text{if } n=k, \\ \gamma_n(t) + \pi_k(s)(n+1-k)\gamma_{n+1-k}(t) + Q_n(\pi_k(s),\gamma_1(t),\dots,\gamma_{n-k}(t)) & \text{if } n>k, \end{cases}$$

with certain polynomials Q_n. Comparing coefficients in (Co1′) we derive

$$\begin{aligned}\gamma_n(s+t) &= \gamma_n(s)+\gamma_n(t), \qquad n<k\\ \gamma_k(s+t) &= \gamma_k(s)+\gamma_k(t)+\pi_k(s)\gamma_1(t),\\ \gamma_n(s+t) &= \gamma_n(s)+\gamma_n(t)+\pi_k(s)(n+1-k)\gamma_{n+1-k}(t)\\ &\qquad +Q_n(\pi_k(s),\gamma_1(t),\dots,\gamma_{n-k}(t)), \qquad n>k.\end{aligned}$$

Using $\gamma_k(s+t)=\gamma_k(t+s)$, we obtain from the above functional equation for γ_k that

$$\pi_k(s)\gamma_1(t)=\pi_k(t)\gamma_1(s), \qquad s,t\in\mathbb{C}.$$

Choosing $t_0\in\mathbb{C}$ such that $\pi_k(t_0)\neq 0$ we furthermore get

$$\gamma_1(s)=\frac{\gamma_1(t_0)}{\pi_k(t_0)}\pi_k(s), \qquad s\in\mathbb{C}.$$

Using in a similar way $\gamma_n(s+t)=\gamma_n(t+s)$ for $n>k$ we find that

$$\begin{aligned}\gamma_n(s)=\frac{1}{n\pi_k(t_0)}\Big(& n\pi_k(s)\gamma_n(t_0)+Q_{n-1+k}(\pi_k(s),\gamma_1(t_0),\dots,\gamma_{n-1}(t_0))\\ &-Q_{n-1+k}(\pi_k(t_0),\gamma_1(s),\dots,\gamma_{n-1}(s))\Big), \quad s\in\mathbb{C},\ n\geq 2.\end{aligned}$$

Thus, by induction on n, we get that each $\gamma_n(s)$ can be expressed as a polynomial $\Psi_n(\pi_k(s))$ in $\pi_k(s)$. Finally putting $\hat{\alpha}=\exp(\gamma)$, we get

$$\hat{\alpha}(s,x)=1+\sum_{n\geq 1}\Phi_n(\pi_k(s))x^n, \qquad s\in\mathbb{C}$$

with polynomials $\Phi_n(y)\in\mathbb{C}[y]$.

From this representation of $\hat{\alpha}$ we derive, that if π_k is an entire function, then $\hat{\alpha}$ is an analytic solution of (Co1). Using this form of $\hat{\alpha}$, comparison of coefficients in (Co1) yields

$$\begin{aligned}&1+\sum_{n\geq 1}\Phi_n(\pi_k(s+t))x^n\\ &=\left(1+\sum_{n\geq 1}\Phi_n(\pi_k(s))x^n\right)\left(1+\sum_{n\geq 1}\Phi_n(\pi_k(t))[\pi(s,x)]^n\right),\end{aligned}$$

whence

$$\Phi_n(\pi_k(s)+\pi_k(t))=R_n(\pi_k(s),\pi_k(t)), \qquad s,t\in\mathbb{C},\ n\geq 1$$

for certain polynomials $R_n(y,z)\in\mathbb{C}[y,z]$. For each $n\geq 1$ this is a polynomial relation in $\pi_k(s)$ and $\pi_k(t)$. Since π_k is a non-trivial additive function, and these relations hold for all $s,t\in\mathbb{C}$, according to Lemma 5 we are allowed to replace $\pi_k(s)$ and $\pi_k(t)$ in these polynomial relations, by indeterminates S and T, which yields

$$\Phi_n(S+T)=R_n(S,T), \qquad n\geq 1.$$

Now we replace S by $\pi_k^*(s)$ and T by $\pi_k^*(t)$, obtaining

$$\Phi_n(\pi_k^*(s)+\pi_k^*(t)) = R_n(\pi_k^*(s),\pi_k^*(t)), \qquad s,t\in\mathbb{C},\ n\geq 1.$$

This means that $\alpha^*(s,x)=1+\sum_{n\geq 1}\Phi_n(\pi_k^*(s))x^n$ is an analytic solution of the equation

$$\alpha^*(s+t,x)=\alpha^*(s,x)\alpha^*(t,\pi^*(s,x)), \qquad s,t\in\mathbb{C} \tag{Co1*}$$

for the analytic iteration group π^*. According to Theorem 2.6 of [4] there exist $E(x)=1+\cdots\in\mathbb{C}\,[\![x]\!]$ and $\kappa_1,\ldots,\kappa_{k-1}\in\mathbb{C}$, such that

$$\alpha^*(s,x)=P^*(s,x)\frac{E(\pi^*(s,x))}{E(x)}, \qquad s\in\mathbb{C}$$

with

$$P^*(s,x)=\prod_{n=1}^{k-1}\exp\left(\kappa_n\int_0^s[\pi^*(\sigma,x)]^n d\sigma\right).$$

If we now use the fact that the coefficient functions of each $\pi^*(s,x)$ are polynomials in $\pi_k^*(s)=s$, and if we carry out the coefficientwise integration, substitution into exp and the multiplications, we derive that the coefficient functions of $P^*(s,x)$ are polynomials in $\pi_k^*(s)=s$.

Finally, replacing $\pi_k^*(s)$ by the indeterminate S and then substituting $\pi_k(s)$ for S we obtain

$$\begin{aligned}\hat\alpha(s,x)&=1+\sum_{n\geq 1}\Phi_n(\pi_k(s))x^n=\alpha^*(\pi_k(s),x)\\ &=P^*(\pi_k(s),x)\frac{E(\pi^*(\pi_k(s),x))}{E(x)}=P(s,x)\frac{E(\pi(s,x))}{E(x)}, \qquad s\in\mathbb{C}\end{aligned}$$

with $P(s,x)=P^*(\pi_k(s),x)$. Consequently, α is of the given form. Conversely, each α of this form is a solution of (Co1). □

Remark 9. From [4] we furthermore obtain that in the representation of α given in Corollary 3 or Theorem 8 the series E and the family $P(s,x)$ are uniquely determined by α.

If furthermore π is an iteration group of type II, using results of [4] we obtain that

$$\exp\left(\kappa_n\int_0^\tau \pi^*(\sigma,x)^n d\sigma\Big|_{\tau=\pi_k(s)}\right)=1+\kappa_n\pi_k(s)x^n+\ldots, \qquad n\geq 1,$$

and that the coefficient functions of both $P(s,x)$ and $[P(s,x)]^{-1}$ are polynomials in $\pi_k(s)$. There it was also shown that

$$\frac{E(\pi(s,x))}{E(x)}\equiv 1 \bmod x^k.$$

3. The general solution of (Co2)

We assume that α is a solution of (Co1) and (B1) represented as in Corollary 3 or Theorem 8. Then the multiplicative inverse of α exists. Instead of β it is more convenient to investigate

$$\Delta(s,x) = \frac{\beta(s,x)}{\alpha(s,x)E(x)}, \qquad s \in \mathbb{C}.$$

Again we distinguish between iteration groups of type I and type II.

3.1 The general solution of (Co2) for iteration groups of type I

Assume that $\pi(s,x) = \pi_1(s)x$ for $s \in \mathbb{C}$, where $\pi_1 \neq 1$ is a generalized exponential function.

Lemma 10. *Assume that α is a solution of* (Co1) *and* (B1) *represented as in Corollary 3. If π is an iteration group of type I, then (α, β) is a solution of* (Co2) *if and only if (α, Δ) is a solution of*

$$\Delta(s+t,x) = \Delta(s,x) + \alpha_0(s)^{-1}\Delta(t,\pi(s,x)), \qquad s,t \in \mathbb{C}. \qquad (\mathrm{Co2}')$$

The proof follows by simple calculations from Corollary 3 and (Co2).

Theorem 11. *Assume that α is a solution of* (Co1) *and* (B1) *represented as in Corollary 3, and that π is an iteration group of type I given in its normal form.*

- *If $\alpha_0 \neq \pi_1^n$ for all $n \geq 0$, then the pair (α, β) is a solution of* (Co2) *if and only if*

$$\beta(s,x) = \alpha_0(s)E(\pi(s,x))\Big[F(x) - \alpha_0(s)^{-1}F(\pi(s,x))\Big], \qquad s \in \mathbb{C},$$

  where $F(x) \in \mathbb{C}[\![x]\!]$.
- *If $\alpha_0 = \pi_1^{n_0}$ for some $n_0 \geq 0$, then the pair (α, β) is a solution of* (Co2) *if and only if*

$$\beta(s,x) = \alpha_0(s)E(\pi(s,x))\Big[A(s)x^{n_0} + F(x) - \alpha_0(s)^{-1}F(\pi(s,x))\Big], \qquad s \in \mathbb{C}$$

  where A is an arbitrary additive function and $F(x) \in \mathbb{C}[\![x]\!]$.

Proof. Assume that (α, β) is a solution of (Co2), then (α, Δ) is a solution of (Co2$'$). Writing $\Delta(s,x) = \sum_{n\geq 0}\Delta_n(s)x^n$, comparing coefficients in (Co2$'$), and using the fact that $\Delta_n(s+t) = \Delta_n(t+s)$ we obtain

$$\Delta_n(s) + \alpha_0(s)^{-1}\pi_1(s)^n\Delta_n(t) = \Delta_n(t) + \alpha_0(t)^{-1}\pi_1(t)^n\Delta_n(s), \qquad s,t \in \mathbb{C},\ n \geq 0.$$

Case 1. If $\alpha_0 \neq \pi_1^n$ for all $n \geq 0$, then for each $n \geq 0$ there exists $t_n \in \mathbb{C}$ such that $\alpha_0(t_n)^{-1}\pi_1(t_n)^n \neq 1$ and

$$\begin{aligned}\Delta_n(s) &= \frac{\Delta_n(t_n)}{1-\alpha_0(t_n)^{-1}\pi_1(t_n)^n}\left(1-\alpha_0(s)^{-1}\pi_1(s)^n\right)\\ &= F_n\left(1-\alpha_0(s)^{-1}\pi_1(s)^n\right), \qquad s\in\mathbb{C},\ n\geq 0,\end{aligned}$$

with $F_n = \Delta_n(t_n)/(1-\alpha_0(t_n)^{-1}\pi_1(t_n)^n) \in \mathbb{C}$. Hence

$$\Delta(s,x) = F(x) - \alpha_0(s)^{-1}F(\pi_1(s)x), \qquad s\in\mathbb{C},$$

with $F(x) = \sum_{n\geq 0} F_n x^n$, and

$$\beta(s,x) = \alpha_0(s)E(\pi(s,x))\Big[F(x)-\alpha_0(s)^{-1}F(\pi(s,x))\Big], \qquad s\in\mathbb{C}.$$

Conversely, for each β of this form the pair (α,β) is a solution of (Co2).

Case 2. If $\alpha_0 = \pi_1^{n_0}$ for some $n_0 \geq 0$, then n_0 is uniquely determined, for otherwise we would have a relation $\pi_1(s)^m = 1$ with $m \neq 0$ for all $s \in \mathbb{C}$. This contradicts the fact, that π_1 takes infinitely many values. In the same way as in the first case, we get

$$\Delta_n(s) = F_n(1-\alpha_0(s)^{-1}\pi_1(s)^n), \qquad s\in\mathbb{C},\ n\neq n_0$$

with $F_n \in \mathbb{C}$. Moreover, the coefficient function Δ_{n_0} is additive. Hence

$$\Delta(s,x) = \Delta_{n_0}(s)x^{n_0} + F(x) - \alpha_0(s)^{-1}F(\pi_1(s)x), \qquad s\in\mathbb{C},$$

with $F(x) = \sum_{n\geq 0} F_n x^n$ (where the coefficient F_{n_0} is not determined by Δ), and

$$\beta(s,x) = \alpha_0(s)E(\pi(s,x))\Big[\Delta_{n_0}(s)x^{n_0} + F(x)-\alpha_0(s)^{-1}F(\pi(s,x))\Big], \qquad s\in\mathbb{C}.$$

Conversely, for each β of the form

$$\beta(s,x) = \alpha_0(s)E(\pi(s,x))\Big[A(s)x^{n_0} + F(x)-\alpha_0(s)^{-1}F(\pi(s,x))\Big], \qquad s\in\mathbb{C}$$

where A is an additive function and $F(x) \in \mathbb{C}[\![x]\!]$, the pair (α,β) is a solution of (Co2). □

Remark 12. Assume that $\pi(s,x) = S^{-1}(\tilde\pi(s,S(x)))$, $s\in\mathbb{C}$, with $S(x) = x + s_2x^2 + \ldots$, and $\tilde\pi(s,x) = \pi_1(s)x$.

If $\alpha_0 \neq \pi_1^n$ for all $n \geq 0$, then the general solution of (Co2) is of the same form as given above.

If $\alpha_0 = \pi_1^{n_0}$, for some $n_0 \geq 0$, then (α,β) is a solution of (Co2) if and only if

$$\beta(s,x) = \alpha_0(s)E(\pi(s,x))\Big[A(s)[S(x)]^{n_0} + F(x)-\alpha_0(s)^{-1}F(\pi(s,x))\Big], \qquad s\in\mathbb{C}.$$

Proof. We only prove the last assertion. Assume that $(\tilde\alpha,\tilde\beta)$ is a solution of (Co1), (B1), and (Co2) for the iteration group $\tilde\pi(s,x) = \pi_1(s)x$ in normal form with

$\alpha_0 = \pi_1^{n_0}$, for some $n_0 \geq 0$. Then

$$\tilde{\alpha}(s,x) = \alpha_0(s)\frac{\tilde{E}(\tilde{\pi}(s,x))}{\tilde{E}(x)}, s \in \mathbb{C}$$

and

$$\tilde{\beta}(s,x) = \alpha_0(s)\tilde{E}(\tilde{\pi}(s,x))\Big[A(s)x^{n_0} + \tilde{F}(x) - \alpha_0(s)^{-1}\tilde{F}(\tilde{\pi}(s,x))\Big], \qquad s \in \mathbb{C}$$

for certain $\tilde{E}(x), \tilde{F}(x) \in \mathbb{C}[\![x]\!]$, α_0 a generalized exponential function, and A an additive function. Putting $y = S(x)$, we obtain $S(\pi(s,x)) = \tilde{\pi}(s,y)$ and similar as in Theorem 1.3 of [4]

$$\begin{aligned}\beta(s,x) &= \beta(s,S^{-1}(y)) = \tilde{\beta}(s,y)\\ &= \alpha_0(s)\tilde{E}(\tilde{\pi}(s,y))\Big[A(s)y^{n_0} + \tilde{F}(y) - \alpha_0(s)^{-1}\tilde{F}(\tilde{\pi}(s,y))\Big]\\ &= \alpha_0(s)\tilde{E}(S(\pi(s,x)))\Big[A(s)[S(x)]^{n_0} + \tilde{F}(S(x)) - \alpha_0(s)^{-1}\tilde{F}(S(\pi(s,x)))\Big]\\ &= \alpha_0(s)E(\pi(s,x))\Big[A(s)[S(x)]^{n_0} + F(x) - \alpha_0(s)^{-1}F(\pi(s,x))\Big], \qquad s \in \mathbb{C}\end{aligned}$$

for $E(x) = (\tilde{E} \circ S)(x)$ and $F(x) = (\tilde{F} \circ S)(x)$. □

3.2 The general solution of (Co2) for iteration groups of type II

Lemma 13. *Assume that α is a solution of* (Co1) *and* (B1) *represented as in Theorem 8. If π is an iteration group of type II, then (α, β) is a solution of* (Co2) *if and only if (α, Δ) is a solution of*

$$\Delta(s+t,x) = \Delta(s,x) + \frac{\alpha_0(s)^{-1}}{P(s,x)}\Delta(t,\pi(s,x)), \qquad s,t \in \mathbb{C}. \qquad \text{(Co2}''\text{)}$$

We still need two preparatory lemmata, describing relations between non-trivial generalized exponential and non-trivial additive functions.

Lemma 14. *Assume that $a \neq 0$ is an additive function from $\mathbb{C}$ to $\mathbb{C}$, that $e \neq 1$ is a generalized exponential function from $\mathbb{C}$ to $\mathbb{C}^*$, and that $P(y), Q(y) \in \mathbb{C}[y]$ are polynomials. Then*

$$P(a(s)) + Q(a(s))e(s) = 0, \qquad s \in \mathbb{C} \qquad (*)$$

if and only if $P = Q = 0$.

This follows directly from Theorem 6 of [13], since 1 and e are distinct exponential functions (in [13] they are called multiplicative functions), and since a is a non-zero additive function. For the convenience of the reader we give a proof adapted to the situation of the lemma.

Proof. We only prove the nontrivial part of the assertion. Assume that $(*)$ is satisfied. If both $P(x) = p$ and $Q(x) = q$ are constant, then $p + qe(s) = 0$ for all $s \in \mathbb{C}$. If $q = 0$, then necessarily $p = 0$ and we are done. If $q \neq 0$, then $e(s) = -p/q$ which is not a non-trivial generalized exponential function. Thus, this situation cannot occur.

Now we will continue the proof by induction over $n = \deg(P) + \deg(Q)$. The case $n = 0$ was just proved. We assume that $n > 0$, and if there are polynomials $\tilde{P}, \tilde{Q}$ with the property $(*)$ and $\deg(\tilde{P}) + \deg(\tilde{Q}) < n$, then $\tilde{P} = \tilde{Q} = 0$.

Since $e \neq 1$, there exists some $t_0 \in \mathbb{C}$ such that $\lambda := e(t_0) \neq 1$. Moreover, we denote $a(t_0)$ by μ. From $(*)$ we obtain

$$P(a(s + t_0)) + Q(a(s + t_0))e(s + t_0) = 0, \qquad s \in \mathbb{C}$$

and consequently

$$P(a(s) + \mu) + Q(a(s) + \mu)e(s)\lambda = 0, \qquad s \in \mathbb{C}. \tag{$**$}$$

Multiplying $(*)$ by λ and subtracting this from $(**)$ we get

$$P(a(s) + \mu) - \lambda P(a(s)) + \lambda\Big(Q(a(s) + \mu) - Q(a(s))\Big)e(s) = 0, \qquad s \in \mathbb{C}$$

which can be written as

$$\tilde{P}(a(s)) + \tilde{Q}(a(s))e(s) = 0, \qquad s \in \mathbb{C}$$

with $\tilde{P}(y) = P(y + \mu) - \lambda P(y)$ and $\tilde{Q}(y) = \lambda\,(Q(y + \mu) - Q(y))$.

Now we claim that $\deg(\tilde{P}) + \deg(\tilde{Q}) < n$, whence $\tilde{P} = \tilde{Q} = 0$. If $Q(y) = \sum_{i=0}^{m} q_i y^i$ with $q_m \neq 0$, then

$$\tilde{Q}(y)/\lambda = Q(y + \mu) - Q(y) = \sum_{i=0}^{m-1} q_i\left((y + \mu)^i - y^i\right) + q_m \sum_{j=0}^{m-1} \binom{m}{j} \mu^{m-j} y^j$$

which is of degree strictly less than $m = \deg(Q)$. Similar computations show that $\deg(\tilde{P}) \leq \deg(P)$, whence we conclude by the induction assumption that $\tilde{P} = \tilde{Q} = 0$.

Finally we have to show that $P = Q = 0$. Assuming that $P \neq 0$, it is of the form $P(y) = \sum_{i=0}^{m} p_i y^i$ with $m \geq 0$ and $p_m \neq 0$. But then the coefficient of y^m in $\tilde{P}$ is $(1 - \lambda)p_m \neq 0$ which is a contradiction. Thus $P = 0$ and then according to $(*)$ also $Q = 0$. □

Lemma 15. *Assume that $a \neq 0$ is an additive function from $\mathbb{C}$ to $\mathbb{C}$, that $e \neq 1$ is a generalized exponential function from $\mathbb{C}$ to $\mathbb{C}^*$, and that $\Phi(x_1, x_2, x_3, x_4)$ is a polynomial in $\mathbb{C}[x_1, x_2, x_3, x_4]$ which is of degree at most 1 in x_1 and x_2. Then $\Phi(e(s), e(t), a(s), a(t)) = 0$ for all $s, t \in \mathbb{C}$, if and only if $\Phi = 0$.*

Proof. We only prove the nontrivial part of the assertion. Assuming that

$$\Phi(e(s), e(t), a(s), a(t)) = 0$$

for all $s, t \in \mathbb{C}$ and collecting with respect to powers of $e(s)$ and $a(s)$, we derive

$$\sum_{j=0}^{n} \Big(p_j(e(t), a(t)) + q_j(e(t), a(t)) e(s) \Big) a(s)^j = 0, \qquad s, t \in \mathbb{C},$$

for some $n \in \mathbb{N}$ with polynomials $p_j(y, z), q_j(y, z) \in \mathbb{C}[y, z]$. For fixed $t = t_0 \in \mathbb{C}$

$$\sum_{j=0}^{n} (\tilde{p}_j + \tilde{q}_j e(s))\, a(s)^j = 0, \qquad s \in \mathbb{C}$$

is satisfied with $\tilde{p}_j = p_j(e(t_0), a(t_0))$ and $\tilde{q}_j = q_j(e(t_0), a(t_0))$. In other words

$$P(a(s)) + Q(a(s)) e(s) = 0, \qquad s \in \mathbb{C}$$

for $P(y) = \sum_{j=0}^{n} \tilde{p}_j y^j$ and $Q(y) = \sum_{j=0}^{n} \tilde{q}_j y^j$. From the last lemma we deduce that $P = Q = 0$, whence $\tilde{p}_j = \tilde{q}_j = 0$ for $0 \leq j \leq n$. This means $p_j(e(t_0), a(t_0)) = q_j(e(t_0), a(t_0)) = 0$ for $0 \leq j \leq n$. Since t_0 was an arbitrary element of $\mathbb{C}$ the last equality must be satisfied for any t, thus

$$p_j(e(t), a(t)) = q_j(e(t), a(t)) = 0, \qquad t \in \mathbb{C},\ 0 \leq j \leq n.$$

Since Φ is of degree at most 1 in $e(s)$ and $e(t)$, the polynomials p_j and q_j for $0 \leq j \leq n$ can be written as

$$\begin{aligned} p_j(e(t), a(t)) &= p_{j,1}(a(t)) + p_{j,2}(a(t)) e(t) \\ q_j(e(t), a(t)) &= q_{j,1}(a(t)) + q_{j,2}(a(t)) e(t) \end{aligned}$$

with suitable polynomials $p_{j,1}(y), p_{j,2}(y), q_{j,1}(y), q_{j,2}(y) \in \mathbb{C}[y]$. Again, according to the previous lemma $p_{j,1}(y) = p_{j,2}(y) = q_{j,1}(y) = q_{j,2}(y) = 0$, thus $p_j = q_j = 0$ for $0 \leq j \leq n$, and consequently $\Phi = 0$. □

Since the general solution of (Co2) is much more complicated for iteration groups of type II, we discuss the different cases which can occur in different theorems.

Theorem 16. *Assume that α is a solution of* (Co1) *and* (B1) *represented as in Theorem 8, that $\alpha_0 \neq 1$, and that π is an iteration group of type II. Then the pair (α, β) is a solution of* (Co2) *if and only if*

$$\beta(s, x) = \alpha_0(s) P(s, x) E(\pi(s, x)) \Big[F(x) - \alpha_0(s)^{-1} \frac{F(\pi(s, x))}{P(s, x)} \Big], \qquad s \in \mathbb{C},$$

for some $F(x) \in \mathbb{C}[\![x]\!]$.

Proof. Assume that (α, β) is a solution of (Co2), then (α, Δ) is a solution of (Co2″). Introducing coefficient functions Δ_n by $\Delta(s, x) = \sum_{n \geq 0} \Delta_n(s) x^n$ we prove that each Δ_n is a polynomial in the exponential function α_0^{-1} and in the additive function π_k which is of formal degree 1 in α_0^{-1}.

Case 1. Assume that $P(s,x)=1$, then (Co2$''$) reduces to

$$\Delta(s+t,x)=\Delta(s,x)+\alpha_0(s)^{-1}\Delta(t,\pi(s,x)), \qquad s,t\in\mathbb{C}. \tag{Co2$'''$}$$

Expanding the right-hand side of this equation according to Lemma 6, comparison of coefficients yields for all $s,t\in\mathbb{C}$

$$\begin{aligned}
&\Delta_n(s+t)=\Delta_n(s)+\alpha_0(s)^{-1}\Delta_n(t), \qquad n<k\\
&\Delta_n(s+t)=\Delta_n(s)+\alpha_0(s)^{-1}\Big(\Delta_n(t)+Q_n(\pi_k(s),\Delta_1(t),\dots,\Delta_{n+1-k}(t))\Big),\\
&\hfill n\geq k,
\end{aligned}$$

where Q_n is a polynomial linear in $\Delta_j(t)$ for $1\leq j\leq n+1-k$. We choose some $t_0\in\mathbb{C}$ such that $\alpha_0(t_0)\neq 1$. Since $\Delta_n(s+t)=\Delta_n(t+s)$ we obtain

$$\Delta_n(s)=\frac{1}{1-\alpha_0(t_0)^{-1}}\Delta_n(t_0)(1-\alpha_0(s)^{-1}), \qquad s\in\mathbb{C},\ n<k,$$

which is a polynomial of formal degree 1 in $\alpha_0(s)^{-1}$. Moreover

$$\begin{aligned}
\Delta_n(s)=\frac{1}{1-\alpha_0(t_0)^{-1}}&\Big(\Delta_n(t_0)(1-\alpha_0(s)^{-1})\\
&+\alpha_0(t_0)^{-1}Q_n(\pi_k(t_0),\Delta_1(s),\dots,\Delta_{n+1-k}(s))\\
&-\alpha_0(s)^{-1}Q_n(\pi_k(s),\Delta_1(t_0),\dots,\Delta_{n+1-k}(t_0))\Big),\\
&\hfill s\in\mathbb{C},\ n\geq k.
\end{aligned}$$

Thus, by induction over n, and by the linearity of Q_n in $\Delta_j(s)$, we obtain

$$\Delta_n(s)=R_n(\alpha_0(s)^{-1},\pi_k(s)), \qquad s\in\mathbb{C},\ n\geq 0,$$

where $R_n(y,z)\in\mathbb{C}[y,z]$ is a polynomial of degree at most 1 in y.

Case 2. If $P(s,x)\neq 1$, then it follows from Remark 9 that $P(s,x)=1+\kappa_r\pi_k(s)x^r+\dots$, with $\kappa_r\neq 0$ and $1\leq r<k$. Moreover, the coefficient functions of $[P(s,x)]^{-1}=1-\kappa_r\pi_k(s)x^r+\dots$ are polynomials in $\pi_k(s)$. Expanding $\Delta(t,\pi(s,x))$ according to Lemma 6, and multiplying it with $[P(s,x)]^{-1}$, we obtain by comparison of coefficients in (Co2$''$) that for all $s,t\in\mathbb{C}$

$$\begin{aligned}
&\Delta_n(s+t)=\Delta_n(s)+\alpha_0(s)^{-1}\Delta_n(t), \qquad n<r\\
&\Delta_n(s+t)=\Delta_n(s)+\alpha_0(s)^{-1}\Big(\Delta_n(t)+Q_n(\pi_k(s),\Delta_0(t),\dots,\Delta_{n-r}(t))\Big),\\
&\hfill n\geq r,
\end{aligned}$$

where Q_n is a polynomial linear in $\Delta_j(t)$ for $0\leq j\leq n-r$. We choose some $t_0\in\mathbb{C}$ such that $\alpha_0(t_0)\neq 1$. Since $\Delta_n(s+t)=\Delta_n(t+s)$ we obtain

$$\Delta_n(s)=\frac{1}{1-\alpha_0(t_0)^{-1}}\Big(\Delta_n(t_0)(1-\alpha_0(s)^{-1})\Big), \qquad s\in\mathbb{C},\ n<r,$$

and

$$\Delta_n(s) = \frac{1}{1-\alpha_0(t_0)^{-1}} \Big(\Delta_n(t_0)(1-\alpha_0(s)^{-1}) \\ +\alpha_0(t_0)^{-1} Q_n(\pi_k(t_0), \Delta_0(s), \dots, \Delta_{n-r}(s)) \\ -\alpha_0(s)^{-1} Q_n(\pi_k(s), \Delta_0(t_0), \dots, \Delta_{n-r}(t_0))\Big), \\ s \in \mathbb{C},\ n \geq r.$$

Also in the second case,

$$\Delta_n(s) = R_n(\alpha_0(s)^{-1}, \pi_k(s)), \qquad s \in \mathbb{C},\ n \geq 0,$$

where $R_n(y, z) \in \mathbb{C}[y, z]$ is a polynomial of degree at most 1 in y.

Hence, in both cases we have

$$\Delta(s, x) = \sum_{n \geq 0} R_n(\alpha_0(s)^{-1}, \pi_k(s))x^n, \qquad s \in \mathbb{C}.$$

Inserting this form into (Co2$''$) we get

$$\sum_{n \geq 0} R_n(\alpha_0(s+t)^{-1}, \pi_k(s+t))x^n$$

$$= \sum_{n \geq 0} R_n(\alpha_0(s)^{-1}, \pi_k(s))x^n + \alpha_0(s)^{-1}[P(s, x)]^{-1} \sum_{n \geq 0} R_n(\alpha_0(t)^{-1}, \pi_k(t))[\pi(s, x)]^n.$$

We write the right-hand side of this equation as

$$\sum_{n \geq 0} \Phi_n(\alpha_0(s)^{-1}, \alpha_0(t)^{-1}, \pi_k(s), \pi_k(t))x^n, \qquad s, t \in \mathbb{C},$$

where $\Phi_n(x_1, x_2, x_3, x_4) \in \mathbb{C}[x_1, x_2, x_3, x_4]$ is a polynomial of degree at most 1 in x_1 and x_2. Hence, for each $n \geq 0$ the polynomial relation

$$R_n(\alpha_0(s)^{-1}\alpha_0(t)^{-1}, \pi_k(s) + \pi_k(t)) = \Phi_n(\alpha_0(s)^{-1}, \alpha_0(t)^{-1}, \pi_k(s), \pi_k(t)),\ s, t \in \mathbb{C}$$

is satisfied. According to Lemma 15, we are allowed to replace the values of $\alpha_0(s)^{-1}$, $\alpha_0(t)^{-1}$, $\pi_k(s)$, $\pi_k(t)$ in these relations by indeterminates U, V, S, T, respectively. This yields

$$R_n(UV, S+T) = \Phi_n(U, V, S, T), \qquad n \geq 0.$$

Finally, we substitute for the indeterminates U, V, S, T the values of regular exponential and additive functions namely e^s, e^t, s, t, respectively. If we define

$$\Delta^*(s, x) := \sum_{n \geq 0} R_n(e^{-s}, s)x^n, \qquad s \in \mathbb{C},$$

then Δ^* satisfies

$$\Delta^*(s+t, x) = \Delta^*(s, x) + e^{-s} \frac{\Delta^*(t, \pi^*(s, x))}{P^*(s, x)}, \qquad s, t \in \mathbb{C} \tag{Co2**}$$

for the analytic iteration group π^* introduced in section 2.2. Since the functions $s \mapsto e^s$ and $s \mapsto s$ are regular, and since the coefficient functions of Δ^* are polynomials in these functions, Δ^* is a solution with entire coefficient functions of (Co2**) with respect to the analytic iteration group π^*. From Theorem 2.8 of [4] we derive that there exists $F(x) \in \mathbb{C}[\![x]\!]$ such that

$$\Delta^*(s,x) = F(x) - e^{-s}\frac{F(\pi^*(s,x))}{P^*(s,x)}, \qquad s \in \mathbb{C}.$$

In this equation we replace, according to Lemma 14, the non-trivial exponential and additive functions by indeterminates, namely e^{-s} by U and s by S, whence

$$\sum_{n\geq 0} R_n(U,S)x^n = F(x) - U\frac{F(\pi^*(S,x))}{P^*(S,x)}.$$

Finally replacing U by $\alpha_0(s)^{-1}$ and S by $\pi_k(s)$ we derive

$$\begin{aligned}\Delta(s,x) &= \sum_{n\geq 0} R_n(\alpha_0(s)^{-1}, \pi_k(s))x^n \\ &= F(x) - \alpha_0(s)^{-1}\frac{F(\pi^*(\pi_k(s),x))}{P^*(\pi_k(s),x)} \\ &= F(x) - \alpha_0(s)^{-1}\frac{F(\pi(s,x))}{P(s,x)}, \qquad s \in \mathbb{C}.\end{aligned}$$

Consequently, β is of the given form. Conversely, for each β of this form the pair (α, β) is a solution of (Co2). □

Now we consider the case $\alpha_0 = 1$. Then (Co2$''$) reduces to

$$\Delta(s+t,x) = \Delta(s,x) + \frac{1}{P(s,x)}\Delta(t,\pi(s,x)), \qquad s,t \in \mathbb{C}. \tag{Co2$''''$}$$

In the next theorem we analyze the situations that $P(s,x) = 1 + \kappa_r\pi_k(s)x^r + \dots$ with either $\kappa_r \neq 0$ and $r < k-1$, or $r = k-1$ and $\kappa_{k-1} \notin \mathbb{N}_0$. These are the generic cases.

Theorem 17. *Assume that α is a solution of* (Co1) *and* (B1) *represented as in Theorem 8, that $\alpha_0 = 1$, and that π is an iteration group of type II. In the generic cases the pair (α,β) is a solution of* (Co2) *if and only if*

$$\beta(s,x) = P(s,x)E(\pi(s,x))\Big[F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x)\Big], \qquad s \in \mathbb{C},$$

where $F(x) \in \mathbb{C}[\![x]\!]$ and

$$Q(s,x) = \sum_{n=0}^{r-1}\int_0^\tau \frac{\ell_n[\pi^*(\sigma,x)]^n}{P^*(\sigma,x)E(\pi^*(\sigma,x))}\,d\sigma\Big|_{\tau=\pi_k(s)},$$

with $\ell_0,\dots,\ell_{r-1} \in \mathbb{C}$, and where π^* and P^* are introduced in section 2.2.

Proof. First we prove that

$$\Delta(s,x) = \sum_{n\geq 0} \Phi_n(\pi_k(s))x^n, \qquad s\in\mathbb{C}$$

with $\Phi_n(y)\in\mathbb{C}[y]$. Afterwards again we investigate the corresponding equation for the analytic iteration group π^*, use the particular form of these analytic solutions, and rewrite it by replacing π_k^* by π_k in order to derive solutions of (Co2$''''$).

Since we had assumed that

$$P(s,x) = 1+\kappa_r\pi_k(s)x^r+\dots, \qquad 1\leq r<k,\ \kappa_r\neq 0,\ s\in\mathbb{C},$$

its multiplicative inverse is given by

$$[P(s,x)]^{-1} = 1-\kappa_r\pi_k(s)x^r+\dots, \qquad s\in\mathbb{C}.$$

This together with Lemma 6 allows to expand the right-hand side of (Co2$''''$).

Case 1. If $r<k-1$, then from (Co2$''''$) we have for all $s,t\in\mathbb{C}$

$$\begin{aligned}
\Delta_n(s+t) &= \Delta_n(s)+\Delta_n(t), \qquad n<r,\\
\Delta_r(s+t) &= \Delta_r(s)+\Delta_r(t)-\kappa_r\pi_k(s)\Delta_0(t),\\
\Delta_n(s+t) &= \Delta_n(s)+\Delta_n(t)-\kappa_r\pi_k(s)\Delta_{n-r}(t)\\
&\qquad +Q_n(\pi_k(s),\Delta_0(t),\dots,\Delta_{n-r-1}(t)), \qquad n>r,
\end{aligned}$$

where Q_n is a polynomial linear in $\Delta_j(t)$ for $0\leq j\leq n-r-1$. We choose some $t_0\in\mathbb{C}$ such that $\pi_k(t_0)\neq 0$. Since $\Delta_n(s+t)=\Delta_n(t+s)$ we obtain from the first r equations that Δ_n is additive for $n<r$. From the coefficients of x^r we derive

$$\Delta_0(s) = \frac{1}{\pi_k(t_0)}\Delta_0(t_0)\pi_k(s), \qquad s\in\mathbb{C}.$$

From the remaining equations we get

$$\begin{aligned}
\Delta_{n-r}(s) = \frac{1}{\kappa_r\pi_k(t_0)}\Big(&\kappa_r\Delta_{n-r}(t_0)\pi_k(s)\\
&+Q_n(\pi_k(t_0),\Delta_0(s),\dots,\Delta_{n-r-1}(s))\\
&-Q_n(\pi_k(s),\Delta_0(t_0),\dots,\Delta_{n-r-1}(t_0))\Big),\ s\in\mathbb{C},\ n>r.
\end{aligned}$$

Hence, by induction we find polynomials $\Phi_n(y)\in\mathbb{C}[y]$ for $n\geq 0$ such that

$$\Delta(s,x) = \sum_{n\geq 0}\Phi_n(\pi_k(s))x^n, \qquad s\in\mathbb{C}.$$

Case 2. Assume that $r=k-1$ and $\kappa_r\notin\mathbb{N}_0$. Here the computations are similar to the first case, but for $n>k-1$ a further term occurs when comparing coefficients in (Co2''''). To be more precise, for all $s, t \in \mathbb{C}$ we have

$$\begin{aligned}
\Delta_n(s+t) &= \Delta_n(s) + \Delta_n(t), \qquad n < k-1,\\
\Delta_{k-1}(s+t) &= \Delta_{k-1}(s) + \Delta_{k-1}(t) - \kappa_{k-1}\pi_k(s)\Delta_0(t),\\
\Delta_n(s+t) &= \Delta_n(s) + \Delta_n(t) + (n-k+1-\kappa_{k-1})\pi_k(s)\Delta_{n-k+1}(t)\\
&\quad + Q_n(\pi_k(s), \Delta_0(t), \dots, \Delta_{n-k}(t)), \qquad n > k-1,
\end{aligned}$$

where Q_n is a polynomial linear in $\Delta_j(t)$ for $0 \le j \le n-k$. We choose some $t_0 \in \mathbb{C}$ such that $\pi_k(t_0) \ne 0$. Since $\Delta_n(s+t) = \Delta_n(t+s)$ we obtain from the equation for Δ_{k-1} that

$$\Delta_0(s) = \frac{1}{\pi_k(t_0)}\Delta_0(t_0)\pi_k(s), \qquad s \in \mathbb{C}.$$

From the equations for Δ_n with $n > k-1$ we get for $s \in \mathbb{C}$

$$\begin{aligned}
\Delta_{n-k+1}(s) = \frac{1}{(n-k+1-\kappa_{k-1})\pi_k(t_0)}\Big(&(n-k+1-\kappa_{k-1})\Delta_{n-k+1}(t_0)\pi_k(s)\\
&+Q_n(\pi_k(s), \Delta_0(t_0), \dots, \Delta_{n-k}(t_0))\\
&-Q_n(\pi_k(t_0), \Delta_0(s), \dots, \Delta_{n-k}(s))\Big).
\end{aligned}$$

Thus also in this case, by induction we find polynomials $\Phi_n(y) \in \mathbb{C}[y]$ for $n \ge 0$ such that

$$\Delta(s,x) = \sum_{n\ge0} \Phi_n(\pi_k(s))x^n, \qquad s \in \mathbb{C}.$$

From this representation of Δ we derive in both cases, that if π_k is an entire function, then Δ is an analytic solution of (Co2). Inserting this form of Δ into (Co2''''), we obtain

$$\begin{aligned}
\sum_{n\ge0} \Phi_n(\pi_k(s+t))x^n &= \sum_{n\ge0} \Phi_n(\pi_k(s))x^n + [P(s,x)]^{-1}\sum_{n\ge0}\Phi_n(\pi_k(t))[\pi(s,x)]^n\\
&= \sum_{n\ge0} R_n(\pi_k(s), \pi_k(t))x^n, \qquad s,t \in \mathbb{C},
\end{aligned}$$

with certain polynomials $R_n(y,z) \in \mathbb{C}[y,z]$. Comparison of coefficients yields

$$\Phi_n(\pi_k(s) + \pi_k(t)) = R_n(\pi_k(s), \pi_k(t)), \qquad s,t \in \mathbb{C},\ n \ge 0.$$

For $n \ge 0$ these are polynomial relations in $\pi_k(s)$ and $\pi_k(t)$. Since π_k is a non-trivial additive function, and these relations hold for all $s, t \in \mathbb{C}$, according to Lemma 5 we are allowed to replace $\pi_k(s)$ and $\pi_k(t)$ by indeterminates S and T, which yields

$$\Phi_n(S+T) = R_n(S,T), \qquad n \ge 0.$$

Now we replace S by $\pi_k^*(s)$ and T by $\pi_k^*(t)$, obtaining

$$\Phi_n(\pi_k^*(s) + \pi_k^*(t)) = R_n(\pi_k^*(s), \pi_k^*(t)), \qquad s,t \in \mathbb{C},\ n \ge 0.$$

This means that $\Delta^*(s,x) := \sum_{n\geq 0} \Phi_n(\pi_k^*(s))x^n$ is an analytic solution of the equation

$$\Delta^*(s+t,x) = \Delta^*(s,x) + \frac{1}{P^*(s,x)}\Delta^*(t,\pi^*(s,x)), \qquad s,t \in \mathbb{C} \qquad (\text{Co2}^{***})$$

for the analytic iteration group π^*. According to Theorem 2.8 of [4] there exist $F(x) \in \mathbb{C}[\![x]\!]$ and $\ell_0,\dots,\ell_{r-1} \in \mathbb{C}$ such that

$$\Delta^*(s,x) = F(x) - \frac{F(\pi^*(s,x))}{P^*(s,x)} + Q^*(s,x), \qquad s \in \mathbb{C},$$

with

$$Q^*(s,x) = \sum_{n=0}^{r-1} \int_0^s \frac{\ell_n[\pi^*(\sigma,x)]^n}{P^*(\sigma,x)E(\pi^*(\sigma,x))}\, d\sigma.$$

In this representation of Δ^* we again replace the non-trivial additive function $\pi_k^*(s)$ by the indeterminate S, whence

$$\sum_{n\geq 0} \Phi_n(S)x^n = F(x) - \frac{F(\pi^*(S,x))}{P^*(S,x)} + Q^*(S,x).$$

Finally replacing S by $\pi_k(s)$ we derive

$$\begin{aligned}\Delta(s,x) &= \sum_{n\geq 0} \Phi_n(\pi_k(s))x^n \\ &= F(x) - \frac{F(\pi^*(\pi_k(s),x))}{P^*(\pi_k(s),x)} + Q^*(\pi_k(s),x) \\ &= F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x), \qquad s \in \mathbb{C}.\end{aligned}$$

Consequently, β is of the given form. Conversely, for each β of this form the pair (α,β) is a solution of (Co2). □

Finally we have to discuss the two non-generic situations where $P(s,x) = 1$ or $P(s,x) = 1 + \kappa_{k-1}\pi_k(s)x^{k-1} + \dots$ with $\kappa_{k-1} = n_1 \in \mathbb{N}$. In both cases we will realize that an additional additive function occurs in the general solution of Δ.

Theorem 18. *Assume that α is a solution of* (Co1) *and* (B1) *represented as in Theorem 8, that $\alpha_0 = 1$, and that π is an iteration group of type II.*

- *If $P(s,x) = 1$, then (α,β) is a solution of* (Co2) *if and only if*

$$\beta(s,x) = E(\pi(s,x))\Big[A(s) + F(x) - F(\pi(s,x)) + Q(s,x)\Big], \qquad s \in \mathbb{C},$$

where A is an additive function, $F(x) \in \mathbb{C}[\![x]\!]$, and

$$Q(s,x) = \sum_{n=0}^{k-1} \int_0^\tau \frac{\ell_n[\pi^*(\sigma,x)]^n}{E(\pi^*(\sigma,x))}\, d\sigma\Big|_{\tau=\pi_k(s)},$$

with $\ell_0,\dots,\ell_{k-1} \in \mathbb{C}$ and where π^* was introduced in section 2.2.

- *If $P(s,x) = 1 + \kappa_{k-1}\pi_k(s)x^{k-1} + \dots$ with $\kappa_{k-1} = n_1 \in \mathbb{N}$, then the pair (α, β) is a solution of* (Co2) *if and only if*

$$\beta(s,x) = P(s,x)E(\pi(s,x))\Big[A(s)\delta(x) + F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x)\Big],$$

for $s \in \mathbb{C}$, where A is an additive function, $\delta(x) = x^{n_1} + \sum_{n>n_1}\delta_n x^n$ is a formal series, such that $A(s)\delta(x)$ is a solution of (Co2$''$), *$F(x) \in \mathbb{C}[\![x]\!]$, and*

$$Q(s,x) = \left(\sum_{n=0}^{k-2}\int_0^\tau \frac{\ell_n[\pi^*(\sigma,x)]^n}{P^*(\sigma,x)E(\pi^*(\sigma,x))}\,d\sigma \right.$$
$$\left. + \int_0^\tau \frac{\ell_{n_1+k-1}[\pi^*(\sigma,x)]^{n_1+k-1}}{P^*(\sigma,x)}\,d\sigma\right)\Bigg|_{\tau=\pi_k(s)},$$

with $\ell_0, \dots, \ell_{k-2}, \ell_{n_1+k-1} \in \mathbb{C}$ and where π^* and P^* were introduced in section 2.2.

Proof. First we assume that (α, β) is a solution of (Co2), whence (α, Δ) is a solution of (Co2$''$) and we determine necessary conditions on the coefficient functions of Δ.

Case 1. If $P(s,x) = 1$, then (Co2$''$) reduces to

$$\Delta(s+t,x) = \Delta(s,x) + \Delta(t,\pi(s,x)), \qquad s,t \in \mathbb{C}, \qquad (\text{Co2}''''')$$

which is similar to (Co1$'$). The only difference to Theorem 8 is that Δ has a coefficient function Δ_0. We deduce from (Co2$'''''$) that Δ_0 is additive and $\Delta_n(s) = \Psi_n(\pi_k(s))$ for $s \in \mathbb{C}$, $n \ge 1$, with $\Psi_n(y) \in \mathbb{C}[y]$, analogous to Theorem 8. Thus

$$\Delta(s,x) = \Delta_0(s) + \sum_{n\ge 1}\Psi_n(\pi_k(s))x^n, \qquad s \in \mathbb{C}.$$

Moreover, we see that $\hat{\Delta}(s,x) := \sum_{n\ge1}\Psi_n(\pi_k(s))x^n$ is also a solution of (Co2$'''''$).

Case 2. If $P(s,x) = 1 + \kappa_{k-1}\pi_k(s)x^{k-1} + \dots$ with $\kappa_{k-1} = n_1 \in \mathbb{N}$, then similar computations as in the second case of Theorem 17 yield that Δ_n is additive for $n < k-1$, that

$$\Delta_0(s) = \frac{1}{\pi_k(t_0)}\Delta_0(t_0)\pi_k(s), \qquad s \in \mathbb{C}$$

for some $t_0 \in \mathbb{C}$ with $\pi_k(t_0) \ne 0$, and that for $n > k-1$, $n \ne n_1 + k - 1$

$$\begin{aligned}\Delta_{n-k+1}(s) = \frac{1}{(n-k+1-n_1)\pi_k(t_0)}\Big(&(n-k+1-n_1)\Delta_{n-k+1}(t_0)\pi_k(s)\\ &+Q_n(\pi_k(s), \Delta_0(t_0), \dots, \Delta_{n-k}(t_0))\\ &-Q_n(\pi_k(t_0), \Delta_0(s), \dots, \Delta_{n-k}(s))\Big)\end{aligned} \qquad (\Delta)$$

is satisfied for all $s \in \mathbb{C}$. Moreover, if $n_1 < k-1$ then Δ_{n_1} is additive, otherwise it satisfies an equation of the form

$$\begin{aligned}\Delta_{n_1}(s+t) &= \Delta_{n_1}(s) + \Delta_{n_1}(t) - n_1\pi_k(s)\Delta_0(t), \qquad \text{if } n_1 = k-1\\ \Delta_{n_1}(s+t) &= \Delta_{n_1}(s) + \Delta_{n_1}(t) + (1-k)\pi_k(s)\Delta_{n_1-k+1}(t)\\ &\quad + Q_{n_1}(\pi_k(s), \Delta_0(t), \ldots, \Delta_{n_1-k}(t)), \qquad \text{if } n_1 > k-1.\end{aligned}$$

With the next arguments, which refer to both case 1 and case 2, we want to prove that in the situation of the present theorem for any solution (α, Δ) of (Co2$''$) there exists a solution $(\alpha, \tilde{\Delta})$ of (Co2$''$) such that the coefficient functions $\tilde{\Delta}_n(s)$ are just polynomials in $\pi_k(s)$, and such that

$$\tilde{\Delta}_n(t_0) = \Delta_n(t_0), \qquad n \geq 0$$

for some fixed $t_0 \in \mathbb{C}$ with $\pi_k(t_0) \neq 0$.

Let α, β, Δ describe a solution of (Co2) or (Co2$''$) and assume that α^*, β^*, Δ^* are the corresponding solutions of

$$\beta^*(s+t, x) = \beta^*(s, x)\alpha^*(t, \pi^*(s, x)) + \beta^*(t, \pi^*(s, x)), \qquad s, t \in \mathbb{C} \qquad (\text{Co2}^*)$$

and (Co2***) for the corresponding analytic iteration group π^* introduced in section 2.2. Then

$$\alpha^*(s, x) = P^*(s, x)\frac{E(\pi^*(s, x))}{E(x)}, \qquad s \in \mathbb{C}$$

and according to Theorem 2.5 of [4]

$$\Delta^*(s, x) = \frac{\beta^*(s, x)}{\alpha^*(s, x)E(x)} = \int_0^s \frac{\ell(\pi^*(\sigma, x))}{P^*(\sigma, x)E(\pi^*(\sigma, x))}\, d\sigma, \qquad s \in \mathbb{C},$$

with $\ell(x) = \sum_{n\geq 0} \ell_n x^n \in \mathbb{C}[\![x]\!]$. Since $P^*(s, x)E(\pi^*(s, x)) \equiv 1 \bmod x$ we derive

$$\frac{\ell(\pi^*(\sigma, x))}{P^*(\sigma, x)E(\pi^*(\sigma, x))} = \sum_{n\geq 0}^{k-2} \ell_n x^n + \sum_{n\geq k-1} (\ell_n + \varphi_n(\sigma, \ell_0, \ldots, \ell_{n+1-k}))\, x^n$$

with polynomials φ_n. Consequently,

$$\Delta^*(s, x) = \sum_{n\geq 0}^{k-2} \ell_n s x^n + \sum_{n\geq k-1} (\ell_n s + \tilde{\varphi}_n(s, \ell_0, \ldots, \ell_{n+1-k}))\, x^n, \qquad s \in \mathbb{C} \quad (\Delta^*)$$

with suitable polynomials $\tilde{\varphi}_n$. If we put

$$\tilde{\Delta}(s, x) := \Delta^*(\pi_k(s), x), \qquad s \in \mathbb{C},$$

then for any choice of $\ell(x) \in \mathbb{C}[\![x]\!]$, the pair $(\alpha, \tilde{\Delta})$ is a solution of (Co2$''$). Now we choose some $t_0 \in \mathbb{C}$ such that $\pi_k(t_0) \neq 0$. Then for each $n \geq 0$ it is possible to determine $\ell_n \in \mathbb{C}$ so that $\tilde{\Delta}_n(t_0) = \Delta_n(t_0)$, namely

$$\ell_n = \begin{cases} \Delta_n(t_0)/\pi_k(t_0) & \text{for } n \leq k-2\\ (\Delta_n(t_0) - \varphi_n(t_0, \ell_0, \ldots, \ell_{n+1-k}))/\pi_k(t_0) & \text{for } n \geq k-1.\end{cases}$$

This way we found a family $\tilde{\Delta}$, whose coefficient functions are polynomials in π_k such that $\tilde{\Delta}(t_0, x) = \Delta(t_0, x)$ for some $t_0 \in \mathbb{C}$ such that $\pi_k(t_0) \neq 0$. The family $\tilde{\Delta}$ is uniquely determined once t_0 is chosen.

Finally we discuss again the two cases mentioned at the beginning of the proof.

Case 1. If $P(s, x) = 1$, then all the coefficient functions of $\hat{\Delta}(s, x)$ are polynomials in π_k and $\hat{\Delta}_0 = 0$. Moreover, the coefficient functions $\hat{\Delta}_n$ are uniquely determined. Furthermore, there exists exactly one $\tilde{\Delta}$ whose coefficient functions are polynomials in π_k such that $\tilde{\Delta}(t_0, x) = \Delta(t_0, x)$ for some $t_0 \in \mathbb{C}$ with $\pi_k(t_0) \neq 0$ and such that $(\alpha, \tilde{\Delta})$ is a solution of (Co2$''$). Hence $\hat{\Delta} = \tilde{\Delta}$. According to Theorem 2.8 of [4] there exist $F(x) \in \mathbb{C}\llbracket x \rrbracket$ and $\ell_0, \dots, \ell_{k-1} \in \mathbb{C}$ such that

$$\Delta^*(s, x) = F(x) - F(\pi^*(s, x)) + Q^*(s, x), \qquad s \in \mathbb{C},$$

where

$$Q^*(s, x) = \sum_{n=0}^{k-1} \int_0^s \frac{\ell_n [\pi^*(\sigma, x)]^n}{E(\pi^*(\sigma, x))} \, d\sigma.$$

Thus

$$\hat{\Delta}(s, x) = \tilde{\Delta}(s, x) = \Delta^*(\pi_k(s), x) = F(x) - F(\pi^*(\pi_k(s), x)) + Q^*(\pi_k(s), x) =$$
$$F(x) - F(\pi(s, x)) + Q(s, x), \qquad s \in \mathbb{C}.$$

From this representation we immediately get the desired form of β. Conversely, for each β of this form the pair (α, β) is a solution of (Co2).

Case 2. Assume that (α, Δ) and $(\alpha, \tilde{\Delta})$ are two solutions of (Co2$''$) where the coefficient functions of $\tilde{\Delta}$ are polynomials in π_k and where $\tilde{\Delta}(t_0, x) = \Delta(t_0, x)$ for some $t_0 \in \mathbb{C}$ with $\pi_k(t_0) \neq 0$. Since the coefficient functions Δ_n for $n < n_1$ are uniquely determined by (Δ) (following from (Co2$''$)) we have

$$\Delta_n(s) = \tilde{\Delta}_n(s), \qquad s \in \mathbb{C},\ n < n_1.$$

Both Δ_{n_1} and $\tilde{\Delta}_{n_1}$ satisfy the same (inhomogeneous) Cauchy functional equation, whence there exists an additive function A such that

$$\Delta_{n_1}(s) = \tilde{\Delta}_{n_1}(s) + A(s), \qquad s \in \mathbb{C}$$

with $A(t_0) = 0$. For $n \geq n_1$ we prove by induction that there exists some $\delta_n \in \mathbb{C}$ such that

$$\Delta_n(s) = \tilde{\Delta}_n(s) + \delta_n A(s), \qquad s \in \mathbb{C}.$$

For $n = n_1$ we have $\delta_{n_1} = 1$. Assume that $n > n_1$ and that our claim is true for all indices less than n. Here it is important to remember that the polynomials $Q_n(\pi_k(s), \Delta_0(t), \dots, \Delta_{n-k}(t))$ occurring in the expansion of $\Delta_n(s + t)$ are linear in $\Delta_j(t)$. To be more precise,

$$Q_n(\pi_k(s), \Delta_0(t), \dots, \Delta_{n-k}(t)) = \sum_{j=0}^{n-k} \Delta_j(t) q_{n,j}(\pi_k(s)), \qquad s, t \in \mathbb{C}$$

with polynomials $q_{n,j}(y) \in \mathbb{C}[y]$ for $0 \le j \le n-k$ and $n \ge k$. Applying the induction assumption on the formula (Δ) for Δ_n we get

$$\begin{aligned}
\Delta_n(s) = \frac{1}{(n-n_1)\pi_k(t_0)} \Big(& (n-n_1)\Delta_n(t_0)\pi_k(s) \\
& + Q_{n+k-1}(\pi_k(s), \Delta_0(t_0), \ldots, \Delta_{n-1}(t_0)) \\
& - Q_{n+k-1}(\pi_k(t_0), \Delta_0(s), \ldots, \Delta_{n-1}(s)) \Big) \\
= \frac{1}{(n-n_1)\pi_k(t_0)} \Big(& (n-n_1)\Delta_n(t_0)\pi_k(s) \\
& + Q_{n+k-1}(\pi_k(s), \Delta_0(t_0), \ldots, \Delta_{n-1}(t_0)) \\
& - Q_{n+k-1}(\pi_k(t_0), \Delta_0(s), \ldots, \Delta_{n_1-1}(s), \\
& \qquad \tilde{\Delta}_{n_1}(s) + A(s), \ldots, \tilde{\Delta}_{n-1}(s) + \delta_{n-1}A(s)) \Big) \\
= \frac{1}{(n-n_1)\pi_k(t_0)} \Big(& (n-n_1)\tilde{\Delta}_n(t_0)\pi_k(s) \\
& + Q_{n+k-1}(\pi_k(s), \tilde{\Delta}_0(t_0), \ldots, \tilde{\Delta}_{n-1}(t_0)) \\
& - Q_{n+k-1}(\pi_k(t_0), \tilde{\Delta}_0(s), \ldots, \tilde{\Delta}_{n-1}(s)) + \tilde{\delta}_n A(s) \Big)
\end{aligned}$$

for a suitable $\tilde{\delta}_n \in \mathbb{C}$. This means

$$\Delta_n(s) = \tilde{\Delta}_n(s) + \delta_n A(s), \qquad s \in \mathbb{C}$$

for a suitable $\delta_n \in \mathbb{C}$. Consequently, we have shown that

$$\Delta(s,x) = \tilde{\Delta}(s,x) + A(s) \sum_{n \ge n_1} \delta_n x^n, \qquad s \in \mathbb{C}.$$

By construction

$$\tilde{\Delta}(s,x) = \Delta^*(\pi_k(s), x), \qquad s \in \mathbb{C}$$

and according to Theorem 2.8 of [4] there exist $F(x) \in \mathbb{C}[\![x]\!]$ and $\ell_0, \ldots, \ell_{k-2}$, $\ell_{n_1+k-1} \in \mathbb{C}$ such that

$$\Delta^*(s,x) = F(x) - \frac{F(\pi^*(s,x))}{P^*(s,x)} + Q^*(s,x), \qquad s \in \mathbb{C},$$

with

$$Q^*(s,x) = \sum_{n=0}^{k-2} \int_0^s \frac{\ell_n [\pi^*(\sigma,x)]^n}{P^*(\sigma,x) E(\pi^*(\sigma,x))}\, d\sigma + \int_0^s \frac{\ell_{n_1+k-1} [\pi^*(\sigma,x)]^{n_1+k-1}}{P^*(\sigma,x)}\, d\sigma.$$

For that reason we have

$$\begin{aligned}\Delta(s,x) &= \tilde{\Delta}(s,x) + A(s)\sum_{n\geq n_1}\delta_n x^n\\ &= A(s)\sum_{n\geq n_1}\delta_n x^n + \Delta^*(\pi_k(s),x)\\ &= A(s)\sum_{n\geq n_1}\delta_n x^n + F(x) - \frac{F(\pi^*(\pi_k(s),x))}{P^*(\pi_k(s),x)} + Q^*(\pi_k(s),x)\\ &= A(s)\sum_{n\geq n_1}\delta_n x^n + F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x), \qquad s\in\mathbb{C}.\end{aligned}$$

Since both (α,Δ) and $(\alpha,\tilde{\Delta})$ are solutions of (Co2$''$), also $A(s)\delta(x)$ is a solution of (Co2). Consequently, β is of the given form.

Conversely, for each β of the form

$$\beta(s,x) = P(s,x)E(\pi(s,x))\Big[A(s)\delta(x) + F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x)\Big], \qquad s\in\mathbb{C},$$

where A is an additive function, $\delta(x) = x^{n_1} + \sum_{n>n_1}\delta_n x^n$ is a formal series, such that $A(s)\delta(x)$ is a solution of (Co2$''$), $F(x)\in\mathbb{C}[\![x]\!]$, and $Q(s,x)$ equals

$$\left(\sum_{n=0}^{k-2}\int_0^\tau \frac{\ell_n[\pi^*(\sigma,x)]^n}{P^*(\sigma,x)E(\pi^*(\sigma,x))}\,d\sigma + \int_0^\tau \frac{\ell_{n_1+k-1}[\pi^*(\sigma,x)]^{n_1+k-1}}{P^*(\sigma,x)}\,d\sigma\right)\Bigg|_{\tau=\pi_k(s)},$$

with $\ell_0,\dots,\ell_{k-2},\ell_{n_1+k-1}\in\mathbb{C}$ and where π^* and P^* were introduced in section 2.2, the pair (α,β) is a solution of (Co2). □

Remark 19. In the second case of the preceding theorem we saw that the coefficient function Δ_{n_1} of a solution (α,Δ) of (Co2$''$) is not uniquely determined by (Co2$''$). If $(\alpha,\tilde{\Delta})$ is also a solution of (Co2$''$) then there exists an additive function A such that $\Delta_{n_1} = \tilde{\Delta}_{n_1} + A$. In our construction we had assumed, moreover, that $A(t_0)=0$ for some $t_0\in\mathbb{C}$ with $\pi_k(t_0)\neq 0$. This is not a severe restriction on A, since it can always be written as the sum of two additive functions, namely

$$A(s) = \frac{A(t_0)}{\pi_k(t_0)}\pi_k(s) + \left(A(s) - \frac{A(t_0)}{\pi_k(t_0)}\pi_k(s)\right), \qquad s\in\mathbb{C}.$$

The first one is a scalar multiple of π_k, the second one is the difference of two additive functions, and by construction, for $s=t_0$ it admits the value 0.

Adding an arbitrary additive function A to $\tilde{\Delta}_{n_1}$ changes the value of $\tilde{\Delta}(t_0,x)$. This, of course, leads to the construction of another family $\Delta^*(s,x)$. According to (Δ^*) we always find a coefficient $\ell_{n_1}\in\mathbb{C}$ such that $\Delta^*_{n_1}(t_0) = \tilde{\Delta}_{n_1}(t_0) + A(t_0)$. This proves, that we did not forget any solutions in the proof.

The phenomenon of adding a further additive function A multiplied with a series δ seemingly did not occur in the situation of regular solutions. This is due to the fact that all regular additive functions are of the form $\mathbb{C}\ni s\mapsto cs$ with $c\in\mathbb{C}$. Moreover, $ct_0=0$ with $t_0\neq 0$ yields immediately $c=0$.

In the situation $P(s,x) = 1 + n_1\pi_k(s)x^{k-1} + \dots$ with $n_1 \in \mathbb{N}$ we have to describe the solutions of (Co2$''$) which are of the form $A(s)\delta(x)$ with $\delta(x) = x^{n_1} + \sum_{n>n_1} \delta_n x^n \in \mathbb{C}[\![x]\!]$.

Theorem 20. *Assume that π is an iteration group of type II, and $P(s,x) = 1 + n_1\pi_k(s)x^{k-1} + \dots$ with $n_1 \in \mathbb{N}$.*

- *Let $\Delta(s,x) := A(s)\delta(x)$, where A is an arbitrary additive function and $\delta(x) = x^{n_1} + \sum_{n>n_1} \delta_n x^n \in \mathbb{C}[\![x]\!]$. If $A = 0$, then Δ is a solution of* (Co2$''$) *for any δ. If $A \neq 0$, then Δ is a solution of* (Co2$''$) *if and only if*
$$P(s,x) = \frac{\delta(\pi(s,x))}{\delta(x)}, \qquad s \in \mathbb{C}. \tag{P}$$
- *For each iteration group π of type II, there exists exactly one $\delta(x) \in \mathbb{C}[\![x]\!]$, such that* (P) *is satisfied.*

Proof. If $A = 0$, then $\Delta(s,x) = A(s)\delta(x) = 0$ is a solution of (Co2$''$).

Assume that $A \neq 0$. Using the additivity of A, the family Δ is a solution of (Co2$''$) if and only if
$$A(t)\delta(x) = \frac{1}{P(s,x)}A(t)\delta(\pi(s,x)), \qquad s,t \in \mathbb{C}.$$
Since $A \neq 0$, there exists some $t_0 \in \mathbb{C}$ such that $A(t_0) \neq 0$, thus in the last equation we are allowed to cancel $A(t)$, and we get
$$\delta(x) = \frac{1}{P(s,x)}\delta(\pi(s,x)), \qquad s \in \mathbb{C}.$$
Since $\operatorname{ord}(\delta(x)) = n_1 = \operatorname{ord}(\delta(\pi(s,x)))$ the quotient $\delta(\pi(s,x))/\delta(x)$ is also a family of formal series, and we finally obtain that Δ satisfies (Co2$''$), if and only if (P) is fulfilled.

For each $\delta(x) = x^{n_1} + \sum_{n>n_1} \delta_n x^n$ there exists exactly one series $E(x) = 1 + \ldots \in \mathbb{C}[\![x]\!]$ such that $\delta(x) = x^{n_1}E(x)$. Thus, (P) is satisfied if and only if
$$P(s,x) = \frac{[\pi(s,x)]^{n_1}}{x^{n_1}}\frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C}.$$

Easy computations show that $[\pi(s,x)]^{n_1}/x^{n_1}$ and also its multiplicative inverse are solutions of (Co1). Since moreover,
$$\frac{[\pi(s,x)]^{n_1}}{x^{n_1}} = \left[\frac{\pi(s,x)}{x}\right]^{n_1} = [1 + \pi_k(s)x^{k-1} + \dots]^{n_1} = 1 + n_1\pi_k(s)x^{k-1} + \dots$$
the equation (P) is satisfied if and only if
$$P(s,x)[1 + n_1\pi_k(s)x^{k-1} + \dots]^{-1} = \frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C}.$$
The left-hand side of this equation is a solution of (Co1) which is congruent 1 modulo x^k. According to Theorem 8, solutions of this kind can be expressed

as $E(\pi(s,x))/E(x)$ with a uniquely determined series $E(x) = 1 + \cdots \in \mathbb{C}[\![x]\!]$. Consequently, there exists exactly one series E and thus exactly one family δ satisfying (P). □

Remark 21. In [4] we had tried to express the solutions of (Co1) with as few integrals as possible. For iteration groups of type II we just showed that if $\alpha(s,x) \equiv \alpha_0(s) \bmod x^k$ then α can be expressed as

$$\alpha(s,x) = \alpha_0(s)\frac{E(\pi(s,x))}{E(x)}, \qquad s \in \mathbb{C}$$

with $E(x) \equiv 1 \bmod x$. From the last theorem we deduce that each solution α of (Co1) with $\alpha(s,x) \equiv \alpha_0(s)(1+n_1\pi_k(s)x^{k-1}) \bmod x^k$ and $n_1 \in \mathbb{N}$ can be expressed as

$$\alpha(s,x) = \alpha_0(s)\frac{\delta(\pi(s,x))}{\delta(x)}, \qquad s \in \mathbb{C}$$

with a suitable $\delta(x) = x^{n_1} + \sum_{n>n_1} \delta_n x^n$.

4. Conclusion

This way we described how to compute rather in an explicit way the solutions of the two cocycle equations for iteration groups without any regularity condition. In a forthcoming paper we will describe a more abstract way for doing this by investigating formal cocycle equations. It was interesting to realize, that the structure of the general solutions corresponds in a natural way to the structure of the analytic solutions.

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Manuscript received: October 27, 2003 and, in final form, May 25, 2004.