Endofunctions of given cycle type

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Abstract

Iteration of an endofunction f on a finite set X defines cycles of f. To a given set L of lengths and to a given function $m\colon L \to \mathbb{N}_0$, the number of all those functions having $m(l)$ cycles of length $l \in L$ and possibly other cycles of length $l \notin L$ will be computed. Furthermore by introducing group actions the number of patterns of these functions can be derived from the Cauchy-Frobenius Lemma. We compare these solutions with the results derived from combinatorial species theory.

An *endofunction* on the set X is a function f with domain and range X. The term *endofunction* comes from *species theory*. See for instance [2, 3, 4, 5, 10]. Bijective endofunctions are usually called *permutations*. We are only dealing with endofunctions on a finite set X, so without loss of generality each n-set (i. e. a set of cardinality n) can be replaced by $\boldsymbol{n} := \{1, 2, \ldots, n\}$.

Denoting the k-th iterate of an endofunction f by f^k we define f to have a *cycle of length* k, if and only if there is some $i \in \boldsymbol{n}$ such that $f^k(i) = i$ and $f^l(i) \neq i$ for all $1 \leq l < k$, and the elements of $\{i, f(i), \ldots, f^{k-1}(i)\}$ form a *cycle of* f *(of length* k*)*. Furthermore f restricted to a cycle is a cyclic permutation of the elements of this cycle. It is obvious that each endofunction on a finite set must have at least one cycle. The *cycle type* of an endofunction f can be considered as a multi-set $\mathcal{L} := \{l^{m(l)} \mid l \in \boldsymbol{n}\}$, where $m(l) \in \mathbb{N}_0$ is the number of cycles of f of length l. Let s be the number of elements lying in cycles of f, then

$$s = \sum_{l \in \boldsymbol{n}} l \cdot m(l) \quad \text{and} \quad 1 \leq s \leq n.$$

Renaming the elements of $\boldsymbol{n}$ leads to the following *group action* of the *symmetric group* $S_{\boldsymbol{n}}$ on the set of all endofunctions:

$$S_{\boldsymbol{n}} \times \boldsymbol{n}^{\boldsymbol{n}} \to \boldsymbol{n}^{\boldsymbol{n}}, \qquad (\pi, f) \mapsto \pi \circ f \circ \pi^{-1} =: \pi f \pi^{-1}, \tag{1}$$

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which describes the relabelling of f (or more precisely of its sagittal graph) along the bijection π. It is obvious that f and $\pi f \pi^{-1}$ are of the same cycle type. The $S_{\boldsymbol{n}}$-orbit of f is the set

$$S_{\boldsymbol{n}}(f) = \left\{\pi f \pi^{-1} \mid \pi \in S_{\boldsymbol{n}}\right\}.$$

The set of orbits under the action of $S_{\boldsymbol{n}}$ will be denoted by $S_{\boldsymbol{n}} \backslash\backslash \boldsymbol{n}^{\boldsymbol{n}}$. These orbits of endofunctions will be called *classes of endofunctions*, *mapping types*, *unlabelled endofunctions* or *isomorphism types of endofunctions*.

Reading S. Beckett's "Watt" [1] the authors came across the following problem: *There are 5 people sitting around a table trying to form pairs by glancing at each other. In how many situations do they form at least one pair?*

This problem can be solved by finding the number of endofunctions on $\mathbf{5}$, having no cycles of length 1, and having one or two cycles of length 2. Another way of solving it can be described as subtracting the number of endofunctions having no cycles of length 1 and no cycles of length 2 from the number of endofunctions having no cycles of length 1. The general aim of this paper is the following: For a given set L of cycle lengths and a multi-set $\mathcal{L} = \{l^m(l) \mid l \in L\}$ enumerate all labelled and unlabelled endofunctions on $\boldsymbol{n}$ having exactly $m(l)$ cycles of length $l \in L$ and possibly other cycles of length $l \notin L$.

One possibility to enumerate endofunctions (or classes of endofunctions) of given type is, to identify them with *functional digraphs* (with loops permitted) which can be counted by using the *generating function* of the numbers of *labelled* (or *unlabelled*) *rooted trees* as was pointed out by F. Harary. In [12] he first proves that a finite digraph is functional, if and only if each of its maximal connected components consists of exactly one directed cycle, and when deleting all the edges in this cycle each of the remaining vertices is the *root* of a *directed rooted tree*. The same idea is applied in species theory (cf. [2]). A *species* is a rule S which associates to each finite set X a finite set $\mathsf{S}[X]$ of S-structures, and to each bijection $\beta : X \to Y$ a mapping $\mathsf{S}[\beta] : \mathsf{S}[X] \to \mathsf{S}[Y]$ which satisfies

$$\mathsf{S}[\alpha \circ \beta] = \mathsf{S}[\alpha] \circ \mathsf{S}[\beta] \text{ and } \mathsf{S}[\mathrm{id}_X] = \mathrm{id}_{\mathsf{S}[X]}$$

for all bijections $\beta : X \to Y$ and $\alpha : Y \to Z$. The mapping $\mathsf{S}[\beta]$ is called the transport of the S-structures along β. Furthermore we associate with each species three generating functions: The *exponential generating function* of all S-structures

$$\mathsf{S}(x) = \sum_{n \geq 0} |\mathsf{S}[\boldsymbol{n}]| \frac{x^n}{n!},$$

the *type generating function*

$$\widetilde{\mathsf{S}}(x) = \sum_{n \geq 0} |S_{\boldsymbol{n}} \backslash\backslash \mathsf{S}[\boldsymbol{n}]| \, x^n,$$

where $S_{\boldsymbol{n}} \backslash\backslash \mathsf{S}[\boldsymbol{n}]$ is the set of all $S_{\boldsymbol{n}}$-orbits on $\mathsf{S}[\boldsymbol{n}]$, and the *cycle index series* Z_{S} which is the sum of the cycle index polynomials of the stabilizers of the orbits of structures in $\mathsf{S}[\boldsymbol{n}]$ for all $n \geq 0$. The *cycle index* $Z(G, X)$ of a group G acting on a finite set X is a standard tool of the *Pólya Enumeration Theory* (cf. [7, 13, 14, 16]). Especially we will need the cycle index of the natural action of $S_{\boldsymbol{n}}$ on $\boldsymbol{n}$ which is the polynomial

$$Z(S_{\boldsymbol{n}}, \boldsymbol{n}) = \sum_{\lambda \vdash\!\dashv n} \prod_{l=1}^{n} \frac{1}{l^{\lambda_l} \lambda_l!} x_l^{\lambda_l} \in \mathbb{Q}[x_1, \ldots, x_n],$$

where the sum must be taken over all *cycle types* $\lambda \vdash\!\dashv n$ of n, which means that $\lambda = (\lambda_1, \ldots, \lambda_n)$ fulfils $\sum_l l \cdot \lambda_l = n$.

Let $\mathsf{End}_{\mathcal{L},\bar{L}}$ denote the species of endofunctions having exactly $m(l)$ cycles of length $l \in L$ and possibly other cycles of length $l \notin L$. Then $\mathsf{End}_{\mathcal{L},\bar{L}}$ can be written as the product of two species

$$\mathsf{End}_{\mathcal{L},\bar{L}} = \mathsf{Per}_{\mathcal{L}}(\mathsf{Rtr}) \cdot \mathsf{Per}_{\bar{L}}(\mathsf{Rtr}), \tag{2}$$

where $\mathsf{Per}_{\mathcal{L}}$ denotes the species of permutations of cycle type $\mathcal{L}$, $\mathsf{Per}_{\bar{L}}$, that of permutations having no cycles of length $l \in L$, and Rtr, that of rooted trees. These species of permutations can be decomposed in the form

$$\mathsf{Per}_{\mathcal{L}} = \prod_{l \in L} \mathsf{Set}_{m(l)}(\mathsf{Cyc}_l), \qquad \mathsf{Per}_{\bar{L}} = \prod_{l \notin L} \mathsf{Set}(\mathsf{Cyc}_l), \tag{3}$$

where Set denotes the species of sets, Set_m, that of sets of cardinality m, and Cyc_l, that of cyclic permutations of length l. The following formulae hold:

$$\mathsf{Set}(x) = \sum_{n \geq 0} \frac{x^n}{n!}, \quad \widetilde{\mathsf{Set}}(x) = \frac{1}{1-x}, \quad Z_{\mathsf{Set}}(x_1, x_2, x_3 \ldots) = \exp(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + \ldots),$$

$$\mathsf{Set}_m(x) = \frac{x^m}{m!}, \quad \widetilde{\mathsf{Set}_m}(x) = x^m, \quad Z_{\mathsf{Set}_m}(x_1, x_2, x_3 \ldots) = Z(S_{\boldsymbol{m}}, \boldsymbol{m}),$$

$$\mathsf{Cyc}_l(x) = \frac{x^l}{l}, \quad \widetilde{\mathsf{Cyc}_l}(x) = x^l, \quad Z_{\mathsf{Cyc}_l}(x_1, x_2, x_3 \ldots) = Z(C_{\boldsymbol{l}}, \boldsymbol{l}) = \frac{1}{l} \sum_{d|l} \varphi(d) x_d^{n/d},$$

where φ is the *Euler* φ-function, and $C_{\boldsymbol{l}}$ is a cyclic group of order l, e. g. $C_{\boldsymbol{l}} = \langle (1, 2, \ldots, l) \rangle$.

The species of rooted trees is also well known. When deleting all the edges incident with the root, and when considering the vertices previously connected with the root as roots of smaller rooted trees, then it is obvious that Rtr fulfils the following functional equation:

$$\mathsf{Rtr} = \mathsf{Sin} \cdot \mathsf{Set}(\mathsf{Rtr}),$$

where Sin is the species *characteristic of singletons*. Since $\mathsf{Sin}(x) = \widetilde{\mathsf{Sin}}(x) = x$ the following equations hold for the exponential generating function $\mathsf{Rtr}(x)$ of rooted trees and the type counting generating function $\widetilde{\mathsf{Rtr}}(x)$

$$\mathsf{Rtr}(x) = x \cdot \exp(\mathsf{Rtr}(x)) \qquad \widetilde{\mathsf{Rtr}}(x) = x \cdot \exp(\sum_{n=1}^{\infty} \frac{\widetilde{\mathsf{Rtr}}(x^n)}{n})$$

which were already given by G. Pólya in [16]. They can be used for recursively computing the numbers of labelled or unlabelled rooted trees. (See for instance formula (4.1.44) of [2].)

Theorem 1: *The exponential generating function* $\mathsf{End}_{\mathcal{L},\bar{L}}(x)$ *of the numbers of labelled endofunctions and the generating function* $\widetilde{\mathsf{End}_{\mathcal{L},\bar{L}}}(x)$ *of the numbers of unlabelled endofunctions having exactly* $m(l)$ *cycles of length* $l \in L$ *and possibly other cycles of length* $l \notin L$ *can be computed according to the decompositions (2) and (3).*

To find the exact numbers of structures over n elements involves coefficient extraction and various other techniques.

Next we want to describe another approach, which is using the *Cauchy–Frobenius Lemma* and the *Principle of Inclusion and Exclusion* in order to enumerate such endofunctions. In the following two theorems we are going to show that the problem of enumerating (classes of) endofunctions in $\mathsf{End}_{\mathcal{L},\bar{L}}$ defined on an n-set, we will write $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$, can be replaced by the determination of the number of (classes of) functions defined on a restricted domain having *no* cycles of length l for all $l \in L$.

Theorem 2: *Given a set* L *of cycle lengths and a function* $m: L \to \mathbb{N}_0$*. Then the cardinality of* $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$ *for*

$$n \geq s := \sum_{l \in L} l \cdot m(l)$$

is given by

$$\left|\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]\right| = \binom{n}{s} \frac{s!}{\prod_{l \in L} l^{m(l)} m(l)!} \left|\boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}\right|,$$

where $\boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}$ *denotes the set of all functions* g *from* $\boldsymbol{n} \setminus \boldsymbol{s}$ *to* $\boldsymbol{n}$ *having* no *cycles of length* $l \in L$.

Proof: It is easy to prove that the function Φ defined below is a bijection.

$$\Phi: \mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}] \to \left\{(A, \sigma, g) \mid A \in \binom{\boldsymbol{n}}{s},\ \sigma \in S_A \text{ of cycle type } \boldsymbol{\mathcal{L}},\ g \in \boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus A}\right\}. \quad (4)$$

(In order to make the notation clearer, the set of all s-subsets of $\boldsymbol{n}$ is indicated by $\binom{\boldsymbol{n}}{s}$.) An endofunction f is mapped under Φ onto the triple (A_f, σ_f, g_f), where A_f is the s-subset of $\boldsymbol{n}$ on which f determines $m(l)$ cycles of length l for all $l \in L$. The permutation σ_f is the restriction $f|_{A_f}$, which is indeed a permutation of cycle type $\boldsymbol{\mathcal{L}}$. Finally $g_f := f|_{\boldsymbol{n}\setminus A_f}$ is a function from $\boldsymbol{n} \setminus A_f$ to $\boldsymbol{n}$ having no cycles of length $l \in L$. So the cardinality of $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$ is the cardinality of $\Phi(\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}])$ which is

$$\binom{n}{s} \frac{s!}{\prod_{l \in L} l^{m(l)} m(l)!} \left|\boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}\right|.$$

$\square$

Since the group action of (1) can be restricted to an action of $S_{\boldsymbol{n}}$ on the set $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$ we also want to enumerate the number of $S_{\boldsymbol{n}}$-orbits on $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$. Let $\mathcal{F} = \mathcal{F}(\sigma, L)$ be the set $\Phi^{-1}(\left\{(\boldsymbol{s}, \sigma, g) \mid g \in \boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}\right\})$, where Φ is defined in (4), and where σ is a permutation of $\boldsymbol{s}$ of cycle type $\mathcal{L}$. For $f \in \mathcal{F}$ let $\mathcal{G}$ be the subgroup of those $\pi \in S_{\boldsymbol{n}}$ for which $\pi f \pi^{-1}$ is also in $\mathcal{F}$. Investigating this group $\mathcal{G}$ we will realize that $\mathcal{G}$ does not depend on the special choice of $f \in \mathcal{F}$. Since $A_{\pi f \pi^{-1}} = \pi A_f = \pi \boldsymbol{s}$, the restriction $\pi|_{\boldsymbol{s}}$ must be a permutation of $\boldsymbol{s}$. Furthermore for $\pi \in \mathcal{G}$ we have $\sigma = \pi f \pi^{-1}|_{\boldsymbol{s}} = \pi|_{\boldsymbol{s}} \sigma \pi^{-1}|_{\boldsymbol{s}}$, so

$$\mathcal{G} = \{\pi \in S_{\boldsymbol{n}} \mid \pi|_{\boldsymbol{s}} \in \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma)\} \approx \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \oplus S_{\boldsymbol{n}\setminus\boldsymbol{s}}.$$

This direct sum of the *centralizer* $\mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma)$ of σ and a symmetric group $S_{\boldsymbol{n}\setminus\boldsymbol{s}}$ is a permutation representation of the direct product $\mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \times S_{\boldsymbol{n}\setminus\boldsymbol{s}}$ acting on $\boldsymbol{n}$ defined by

$$(\pi_1, \pi_2)(i) = \begin{cases} \pi_1(i) & \text{if } i \le s \\ \pi_2(i) & \text{if } i > s \end{cases}$$

for $(\pi_1, \pi_2) \in \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \times S_{\boldsymbol{n}\setminus\boldsymbol{s}}$. Since σ is of cycle type $\mathcal{L}$, the centralizer of σ is *similar* to the *direct sum* of *plethysms* of cyclic and symmetric groups,

$$\mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \approx \bigoplus_{l \in L} (C_{\boldsymbol{l}} \odot S_{\boldsymbol{m(l)}}).$$

This plethysm is a group action of the *wreath product*

$$C_{\boldsymbol{l}} \wr S_{\boldsymbol{m(l)}} := \left\{(\psi, \pi) \mid \psi \in C_{\boldsymbol{l}}^{\boldsymbol{m(l)}}, \pi \in S_{\boldsymbol{m(l)}}\right\}$$

on the set $\boldsymbol{l} \cdot \boldsymbol{m(l)}$. From that it is clear that the centralizer of σ consists of

$$\prod_{l \in L} l^{m(l)} m(l)!$$

elements. (For more details about the plethysm of two group actions see [14].)

For each $f \in \mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$ the intersection of $\mathcal{F}$ and the orbit $S_{\boldsymbol{n}}(f)$ is not empty. Especially this intersection $S_{\boldsymbol{n}}(f) \cap \mathcal{F}$ equals the orbit $\mathcal{G}(\bar{f})$ where $\bar{f}$ is an arbitrary element of $S_{\boldsymbol{n}}(f) \cap \mathcal{F}$. A bijection between the set of $S_{\boldsymbol{n}}$-orbits on $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$ and the set of $\mathcal{G}$-orbits on $\mathcal{F}$ is given by

$$S_{\boldsymbol{n}} \backslash\backslash \mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}] \to \mathcal{G} \backslash\backslash \mathcal{F}, \qquad S_{\boldsymbol{n}}(f) \mapsto S_{\boldsymbol{n}}(f) \cap \mathcal{F}.$$

So

$$\left|S_{\boldsymbol{n}} \backslash\backslash \mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]\right| = |\mathcal{G} \backslash\backslash \mathcal{F}| = \frac{1}{|\mathcal{G}|} \sum_{\pi \in \mathcal{G}} |\mathcal{F}_\pi|,$$

where $\mathcal{F}_\pi$ denotes the set of all fixed points of π in $\mathcal{F}$, i. e.

$$\mathcal{F}_\pi = \left\{f \in \mathcal{F} \mid \pi f \pi^{-1} = f\right\}.$$

Each endofunction $f \in \mathcal{F} = \mathcal{F}(\sigma, L)$ can be identified with a pair $(f|_{\boldsymbol{s}}, f|_{\boldsymbol{n}\setminus\boldsymbol{s}}) = (\sigma, f|_{\boldsymbol{n}\setminus\boldsymbol{s}}) \in \{\sigma\} \times \boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}$, and $\mathcal{G} = \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \oplus S_{\boldsymbol{n}\setminus\boldsymbol{s}}$ acts on $\mathcal{F}$ in the following way:

$$\mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \oplus S_{\boldsymbol{n}\setminus\boldsymbol{s}} \times (\{\sigma\} \times \boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}) \to \{\sigma\} \times \boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}, \qquad ((\pi_1, \pi_2), (\sigma, g)) \mapsto (\sigma, (\pi_1, \pi_2) \circ g \circ \pi_2^{-1}).$$

Then $\left|\mathcal{F}_{(\pi_1,\pi_2)}\right|$, the number of fixed points of the permutation $(\pi_1, \pi_2) \in \mathcal{G}$ in $\mathcal{F}$, is

$$\left|\left\{g \in \boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}} \mid (\pi_1, \pi_2) \circ g \circ \pi_2^{-1} = g\right\}\right|.$$

So we have proved the following theorem:

Theorem 3: *The number of $S_{\boldsymbol{n}}$-orbits on $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$ is given by*

$$\left|S_{\boldsymbol{n}} \backslash\backslash \mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]\right| = \frac{1}{\prod_{l \in L} l^{m(l)} m(l)!} \frac{1}{(n-s)!} \sum_{\pi_1 \in \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma)} \sum_{\pi_2 \in S_{\boldsymbol{n}\setminus\boldsymbol{s}}} \left|\mathcal{F}_{(\pi_1,\pi_2)}\right|,$$

where σ is any permutation of $\boldsymbol{s}$ of cycle type $\boldsymbol{\mathcal{L}}$ and where $\mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma)$ denotes the centralizer of σ.

Now the main problem is the determination of $\left|\mathcal{F}_{(\pi_1,\pi_2)}\right|$ for $(\pi_1, \pi_2) \in \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \times S_{\boldsymbol{n}\setminus\boldsymbol{s}}$. In order to avoid confusion between cycles of a function and cycles of a permutation π the latter will be called π-cycles. It is well known (cf. [3, 4, 5, 6]) how to construct all functions $g \in \boldsymbol{n}^{\boldsymbol{n}\setminus\boldsymbol{s}}$, which are fixed points of (π_1, π_2). (From now on the set of these functions will be denoted by $\boldsymbol{n}_{(\pi_1,\pi_2)}^{\boldsymbol{n}\setminus\boldsymbol{s}}$.) Such a function g maps the elements of a π_2-cycle of length k onto the elements of a π_1-cycle or π_2-cycle of length l, where l is a divisor of k. After determining g on one point of the π_2-cycle g is determined on the whole π_2-cycle, since $g\pi_2^j = (\pi_1^j, \pi_2^j)g$. Now it is obvious that only those elements of $\boldsymbol{n} \setminus \boldsymbol{s}$, which lie in a π_2-cycle that is mapped onto a π_2-cycle of the same length, can lie in a cycle of the function $g \in \boldsymbol{n}_{(\pi_1,\pi_2)}^{\boldsymbol{n}\setminus\boldsymbol{s}}$. Let $g \in \boldsymbol{n}^{\boldsymbol{n}\setminus\boldsymbol{s}}$ be a fixed point of (π_1, π_2) and let $C_1, \ldots, C_t$ be π_2-cycles of length i. The set of elements of C_k will be indicated as $\bar{C}_k$. If g maps $\bar{C}_k$ (bijectively) onto $\bar{C}_{k+1}$ for $1 \leq k < t$ and $\bar{C}_t$ onto $\bar{C}_1$, then we will call $(C_1, \ldots, C_t)$ a *big (t, i)-cycle* of g. We have already seen that cycles of g can only occur on big (t, i)-cycles of g. In the next Lemma (cf. [5, 6]) we compute the length of such cycles which are defined on a big (t, i)-cycle of g.

Lemma 4: *For $(\pi_1, \pi_2) \in \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \times S_{\boldsymbol{n}\setminus\boldsymbol{s}}$ let $(C_1, \ldots, C_t)$ be a big (t, i)-cycle of $g \in \boldsymbol{n}_{(\pi_1,\pi_2)}^{\boldsymbol{n}\setminus\boldsymbol{s}}$, where the π_2-cycle C_1 is given by $C_1 = (c_1, \ldots, c_i)$. Suppose furthermore that g satisfies $g^t(c_1) = c_{1+j}$, where $j \in \{0, 1, \ldots, i-1\}$. Then all $t \cdot i$ elements of $\bigcup_{k=1}^t \bar{C}_k$ lie in $\gcd(i, j)$ cycles of length*

$$\frac{t \cdot i}{\gcd(i, j)}$$

of the function g.

So far we have demonstrated that the length of cycles of $g \in \boldsymbol{n}^{\boldsymbol{n}}_{(\pi_1,\pi_2)}$, defined on a big (t,i)-cycle of g, is a multiple of t. Or in other words, cycles of length l occur only on big (t,i)-cycles of g, where t is a divisor of l, and l/t is a divisor of i. Let $\pi_2 \in S_{\boldsymbol{n}\setminus\boldsymbol{s}}$ be a permutation of cycle type $(\lambda_1, \lambda_2, \ldots)$. We want to denote the λ_i π_2-cycles of length i by $C_1^i, \ldots, C_{\lambda_i}^i$. Let X_i be the union

$$X_i := \bigcup_{j=1}^{\lambda_i} \bar{C}_j^i,$$

then the set of all functions g from $\boldsymbol{n} \setminus \boldsymbol{s}$ to $\boldsymbol{n}$ can be described as a cartesian product

$$\boldsymbol{n}^{\boldsymbol{n}\setminus\boldsymbol{s}} = \boldsymbol{n}^{\bigcup_{i=1}^{n-s} X_i} = \mathop{\times}_{i=1}^{n-s} \boldsymbol{n}^{X_i},$$

and

$$\Psi\colon \boldsymbol{n}^{\boldsymbol{n}\setminus\boldsymbol{s}} \to \mathop{\times}_{i=1}^{n-s} \boldsymbol{n}^{X_i}, \qquad g \mapsto \big(g|_{X_1}, \ldots, g|_{X_{n-s}}\big)$$

is a bijection between $\boldsymbol{n}^{\boldsymbol{n}\setminus\boldsymbol{s}}$ and $\times_{i=1}^{n-s} \boldsymbol{n}^{X_i}$. Since X_i is a union of π_2-cycles, $\pi_2|_{X_i}$ is a permutation of X_i and for $g \in \boldsymbol{n}^{X_i}$ the composition $(\pi_1,\pi_2)g{\pi_2}^{-1}|_{X_i}$ is again in $\boldsymbol{n}^{X_i}$. Using $\boldsymbol{n}^{X_i}_{(\pi_1,\pi_2)}$ as a short notation for the set $\big\{g \in \boldsymbol{n}^{X_i} \mid (\pi_1,\pi_2)g{\pi_2}^{-1}|_{X_i} = g\big\}$, then $\Psi(\boldsymbol{n}^{\boldsymbol{n}\setminus\boldsymbol{s}}_{(\pi_1,\pi_2)}) = \times_{i=1}^{n-s} \boldsymbol{n}^{X_i}_{(\pi_1,\pi_2)}$. If $\boldsymbol{n}^{X_i}_{\bar{L},(\pi_1,\pi_2)}$ denotes the set of all functions $g \in \boldsymbol{n}^{X_i}_{(\pi_1,\pi_2)}$ which have no cycles of length $l \in L$, then $\Psi(\mathcal{F}_{(\pi_1,\pi_2)}) = \times_{i=1}^{n-s} \boldsymbol{n}^{X_i}_{\bar{L},(\pi_1,\pi_2)}$. So we conclude that

$$\big|\mathcal{F}_{(\pi_1,\pi_2)}\big| = \left|\mathop{\times}_{i=1}^{n-s} \boldsymbol{n}^{X_i}_{\bar{L},(\pi_1,\pi_2)}\right| = \prod_{i=1}^{n-s} \left|\boldsymbol{n}^{X_i}_{\bar{L},(\pi_1,\pi_2)}\right|.$$

As usual in such computations the cardinality of $\mathcal{F}_{(\pi_1,\pi_2)}$ depends only on the cycle types of π_1 and π_2 but not on the special choice of π_1 and π_2. For computing this cardinality we can assume that L consists only of cycle lengths $l \leq n-s$. Furthermore define a set $D = D(L)$ of positive integers by

$$D := \left(\bigcup_{l \in L} D(l)\right),$$

where $D(l)$ is the set of all divisors of l.

Theorem 5: *The number of fixed points of $(\pi_1,\pi_2) \in \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \times S_{\boldsymbol{n}\setminus\boldsymbol{s}}$ in $\mathcal{F}$ is given by*

$$\big|\mathcal{F}_{(\pi_1,\pi_2)}\big| = \prod_{i=1}^{n-s} \left(\sum_{j=0}^{\lambda_i(\pi_2)} j!\tilde{Z}(S_{\boldsymbol{j}}, L, i) \binom{\lambda_i(\pi_2)}{j} \left(\sum_{k|i} k(\lambda_k(\pi_1) + \lambda_k(\pi_2)) \right)^{\lambda_i(\pi_2)-j} \right), \quad (5)$$

where $\tilde{Z}(S_{\boldsymbol{j}}, L, i)$ is defined by

$$\tilde{Z}(S_{\boldsymbol{j}}, L, i) := Z\left(S_{\boldsymbol{j}}, \boldsymbol{j} \;\middle|\; x_k = \begin{cases} (-1) i^{k-1} \psi(L, k, i) & \text{if } k \in D \\ 0 & \text{if } k \notin D \end{cases}\right).$$

This means that we have to replace each variable x_k in the cycle index $Z(S_{\boldsymbol{j}}, \boldsymbol{j})$ by the given expression. The numbers $\psi(L, t, i)$ are cardinalities

$$\psi(L, t, i) := \left|\left\{ j \in \{0, 1, \ldots, i-1\} \;\middle|\; \frac{i \cdot t}{\gcd(i, j)} = l \in L \right\}\right| = \sum_{\substack{l \in L \\ t | l | t \cdot i}} \varphi(l/t) = \sum_{\substack{d | i \\ d \cdot t \in L}} \varphi(d),$$

where φ is the Euler φ-function. *Finally the cycle types of the permutations π_i are indicated by $(\lambda_1(\pi_i), \lambda_2(\pi_i), \ldots)$ for $i = 1, 2$.*

Proof: For $t \in D$ and for each t-set $A \subseteq \boldsymbol{\lambda_i(\pi_2)}$ let B_A^i be the set of all those functions $g \in \boldsymbol{n}_{(\pi_1, \pi_2)}^{X_i}$ with the property that there is a cyclic arrangement of the C_j^i for $j \in A$ which forms a big (t, i)-cycle of g on which g defines cycles of length $l \in L$. Then

$$\boldsymbol{n}_{\bar{L}, (\pi_1, \pi_2)}^{X_i} = \boldsymbol{n}_{(\pi_1, \pi_2)}^{X_i} \setminus \bigcup_{A \in \mathcal{A}} B_A^i,$$

where

$$\mathcal{A} = \bigcup_{t \in D} \binom{\boldsymbol{\lambda_i(\pi_2)}}{t}.$$

Application of the *Principle of Inclusion and Exclusion* yields

$$\left|\boldsymbol{n}_{\bar{L}, (\pi_1, \pi_2)}^{X_i}\right| = \sum_{Y \in 2^{\mathcal{A}}} (-1)^{|Y|} \left|\bigcap_{A \in Y} B_A^i\right|, \tag{6}$$

where $2^{\mathcal{A}}$ denotes the power set of $\mathcal{A}$. If there are $A, A' \in Y$, $A \neq A'$, such that $A \cap A' \neq \emptyset$, then $\bigcap_{A \in Y} B_A^i = \emptyset$; so all elements of Y must be pairwise disjoint. Each such Y consisting of pairwise disjoint subsets of $\boldsymbol{\lambda_i(\pi_2)}$ defines a function $r \colon D \to \mathbb{N}_0$ by $r(t) := |\{A \in Y \mid |A| = t\}|$, which satisfies $\sum r(t) t \leq \lambda_i(\pi_2)$. The cardinality of $\bigcap_{A \in Y} B_A^i$ can be computed by

$$\prod (t-1)!^{r(t)} \left(i^{t-1} \psi(L, t, i)\right)^{r(t)} \left(\sum_{j | i} j (\lambda_j(\pi_1) + \lambda_j(\pi_2))\right)^{\lambda_i(\pi_2) - \sum r(t) t},$$

where all sums and products which are not specified run over all $t \in D$. There are $(t-1)!$ different ways to arrange t objects in cyclic order. On a big (t, i)-cycle a function g can be defined in $i^{t-1} \psi(L, t, i)$ ways, such that g has cycles of length $l \in L$. This can be proved in the following way: As was already mentioned above, a function $g \in \boldsymbol{n}_{(\pi_1, \pi_2)}^{X_i}$ is defined on the whole cycle after determining g on one point of the cycle. So there are i^{t-1} possibilities to define g on $\bar{C}_1, \bar{C}_2, \ldots, \bar{C}_{t-1}$ (if we use the notation of Lemma 4). According to the definition of g on $\bar{C}_t$ the function g has cycles of length l on the big (t,i)-cycle $(C_1, C_2, \ldots, C_t)$, if and only if t is a divisor of l and there exists $j \in \{0, 1, \ldots, i-1\}$, such that $ti/\gcd(i,j) = l$, so there are $\psi(L,t,i)$ possibilities to define g on $\bar{C}_t$. Now the function g is defined on $\sum r(t)t$ π_2-cycles of length i. The remaining π_2-cycles of length i, these are $\lambda_i(\pi_2) - \sum r(t)t$ π_2-cycles, must be mapped under g onto π_1 or π_2-cycles of length dividing i, which leads to

$$\left(\sum_{j|i} j(\lambda_j(\pi_1) + \lambda_j(\pi_2))\right)^{\lambda_i(\pi_2) - \sum r(t)t}$$

possibilities. So the cardinality $\left|\bigcap_{A \in Y} B_A^i\right|$ depends only on the function r, but does not depend on the special choice of the elements of Y (i. e. for all sets Y, which define the same function r, we compute the same value of $\left|\bigcap_{A \in Y} B_A^i\right|$.) For that reason we have to determine the number of those $Y \in 2^{\mathcal{A}}$, which define the same function $r \colon D \to \mathbb{N}_0$, where r satisfies $\sum r(t)t \leq \lambda_i(\pi_2)$. There are

$$\binom{\lambda_i(\pi_2)}{\sum r(t)t}$$

possibilities to choose $\sum r(t)t$ π_2-cycles of length i from $\lambda_i(\pi_2)$ π_2-cycles of length i. The set of chosen π_2-cycles can be partitioned into $r(t)$ subsets consisting of t π_2-cycles each (for all $t \in D$) in

$$\frac{(\sum r(t)t)!}{\prod t!^{r(t)} r(t)!}$$

ways. For making notation easier let λ_j be $\lambda_j(\pi_2)$ and let λ'_j stand for $\lambda_j(\pi_1) + \lambda_j(\pi_2)$. Then from (6) we derive that the cardinality $\left|\boldsymbol{n}^{X_i}_{\bar{L},(\pi_1,\pi_2)}\right|$ is given by

$$\begin{aligned}
&\sum_{\substack{r \in \mathbb{N}_0^D \\ \sum r(t)t \leq \lambda_i}} (-1)^{\sum r(t)} \binom{\lambda_i}{\sum r(t)t} \frac{(\sum r(t)t)!}{\prod t^{r(t)} r(t)!} \prod \left(i^{t-1}\psi(L,t,i)\right)^{r(t)} \left(\sum_{j|i} j\lambda'_j\right)^{\lambda_i - \sum r(t)t} \\
&= \sum_{j=0}^{\lambda_i} \sum_{\substack{\mu \vdash j \\ k \notin D \Rightarrow \mu_k = 0}} \left(\frac{1}{\prod_k k^{\mu_k} \mu_k!} (-1)^{\sum_k \mu_k} \prod_k \left(i^{k-1}\psi(L,k,i)\right)^{\mu_k}\right) \binom{\lambda_i}{j} j! \left(\sum_{k|i} k\lambda'_k\right)^{\lambda_i - j} \\
&= \sum_{j=0}^{\lambda_i} j! Z\left(S_{\boldsymbol{j}}, \boldsymbol{j} \;\middle|\; x_k = \begin{cases} (-1)i^{k-1}\psi(L,k,i) & \text{if } k \in D \\ 0 & \text{else} \end{cases}\right) \binom{\lambda_i}{j} \left(\sum_{k|i} k\lambda'_k\right)^{\lambda_i - j},
\end{aligned}$$

where all sums and products which are not specified in the first line run over all $t \in D$. The second sum in the second line must be taken over all cycle types $\mu \vdash j$ of j, with the additional property that μ_k must be 0 for all $k \notin D$. (A function $r\colon D \to \mathbb{N}_0$ with $\sum_{t\in D} r(t)t = j$ defines a cycle-type $\mu \vdash j$ by $\mu_t = r(t)$ for $t \in D$ and $\mu_k = 0$ for all $k \notin D$.) So the proof is finished. $\square$

The set $\boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}$ turns out to be the set $\mathcal{F}_{(\mathrm{id},\mathrm{id})}$. Specializing formula (5) gives

Corollary 6: *The number of functions $g\colon \boldsymbol{n}\setminus\boldsymbol{s} \to \boldsymbol{n}$ having no cycles of length $l \in L$ is given by:*

$$\left|\boldsymbol{n}_{\bar{L}}^{\boldsymbol{n}\setminus\boldsymbol{s}}\right| = \sum_{j=0}^{n-s} j!Z\left(S_{\boldsymbol{j}},\boldsymbol{j} \;\middle|\; x_k = \begin{cases} -1 & \text{if } k \in L \\ 0 & \text{if } k \notin L \end{cases}\right)\binom{n-s}{j} n^{n-s-j}.$$

Returning to Beckett's novel "Watt" there are 5 people glancing at each other. Among the 1024 possibilities to do this, there are $1024 - 444 = 580$ situations in which at least 2 of them are watching one another. Among the 13 mapping patterns on 5 points which have no 1-cycle, there are 5 patterns which have no 2-cycle as well. Below you can see these 13 mapping patterns together with the number of different labellings of each pattern.

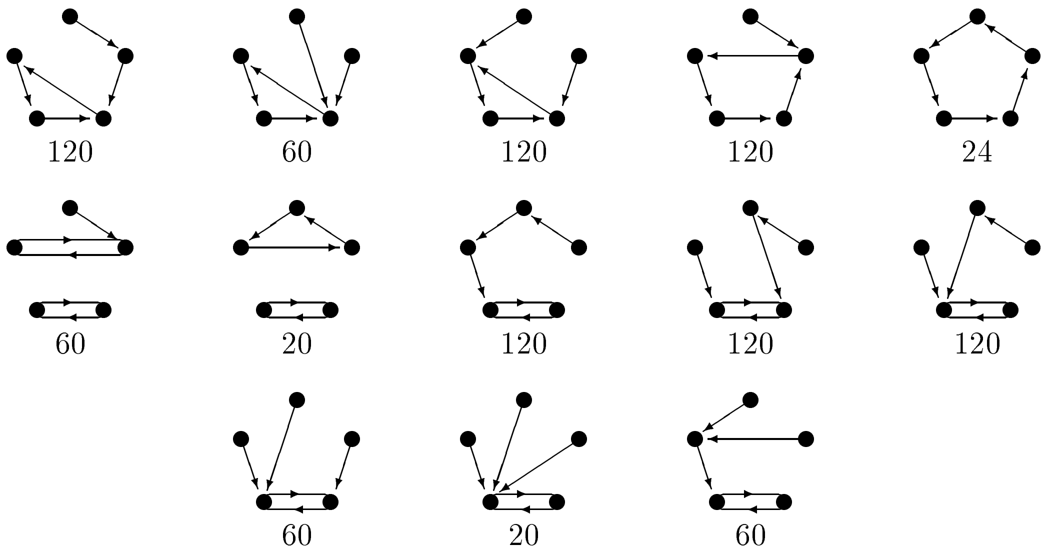

Figure 1: Classes of endofunctions on **5** without cycles of length one.

In situations when all occurring cycles are described by $\mathcal{L}$, i. e. no further cycles may (or can) appear (so we can assume that $L = \boldsymbol{n}$), then Theorem 5 specializes to:

Corollary 7: *The number of fixed points of $(\pi_1, \pi_2) \in \mathrm{Stab}_{S_{\boldsymbol{s}}}(\sigma) \times S_{\boldsymbol{n}\setminus\boldsymbol{s}}$ in $\mathcal{F} = \mathcal{F}(\sigma, \boldsymbol{n})$ is given by* $\left|\mathcal{F}_{(\pi_1,\pi_2)}\right| =$

$$= \prod_{i=1}^{n-s}\left(\left(\sum_{k|i} k(\lambda_k(\pi_1)+\lambda_k(\pi_2))\right)^{\lambda_i(\pi_2)} - i\lambda_i(\pi_2)\left(\sum_{k|i} k(\lambda_k(\pi_1)+\lambda_k(\pi_2))\right)^{\lambda_i(\pi_2)-1}\right)$$

$$= \prod_{i=1}^{n-s} \left(\left(\sum_{k|i} k(\lambda_k(\pi_1) + \lambda_k(\pi_2)) \right)^{\lambda_i(\pi_2)-1} \left(\sum_{k|i} k(\lambda_k(\pi_1) + \lambda_k(\pi_2)) - i\lambda_i(\pi_2) \right) \right).$$

Proof: If $d \mid i$ and $k \leq \lambda_i(\pi_2)$ then $d \cdot k \leq i \cdot \lambda_i(\pi_2) \leq n$ so that

$$\psi(L, k, i) = \sum_{d|i} \varphi(d) = i$$

since we can assume that $L = \boldsymbol{n}$. From that we deduce that

$$\tilde{Z}(S_{\boldsymbol{j}}, L, i) = Z(S_{\boldsymbol{j}}, \boldsymbol{j} \mid x_k = -i^k) = \begin{cases} 1 & \text{if } j = 0 \\ -i & \text{if } j = 1 \\ 0 & \text{else.} \end{cases}$$

In the cases $j = 0$ or 1 we have $Z(S_{\boldsymbol{0}}, \boldsymbol{0}) = 1$ and $Z(S_{\boldsymbol{1}}, \boldsymbol{1}) = x_1$, so nothing is to prove. Let $j \geq 2$ then

$$Z(S_{\boldsymbol{j}}, \boldsymbol{j} \mid x_k = -i^k) = \frac{1}{j!} \sum_{\pi \in S_{\boldsymbol{j}}} \prod_{k=1}^{j} (-i^k)^{\lambda_k(\pi)} =$$

$$\frac{1}{j!} \sum_{\pi \in S_{\boldsymbol{j}}} (-1)^{\sum_{k=1}^{j} \lambda_k(\pi)} \cdot i^{\sum_{k=1}^{j} k\lambda_k(\pi)} = i^j \frac{1}{j!} \sum_{\pi \in S_{\boldsymbol{j}}} (-1)^{c(\pi)}, \tag{7}$$

where $c(\pi)$ is the number of cycles in the cycle decomposition of π. Since the sign of π can be computed as $(-1)^{j-c(\pi)}$ all even permutations give rise to a summand $(-1)^j i^j$ in (7) and all odd permutations contribute to a summand $(-1)^{j+1} i^j$ such that (7) vanishes for $j \geq 2$. □

Finally we want to investigate some special cases:

Examples 8:

- In the case $\sum_{l \in L} lm(l) = n$ the set $\mathsf{End}_{\mathcal{L},\bar{L}}[\boldsymbol{n}]$ is the set of all permutations in $S_{\boldsymbol{n}}$ of cycle-type $\boldsymbol{\mathcal{L}}$.

- In the case $L = \emptyset$ we are counting endofunctions without prescribed cycle-type. The results for unlabelled endofunctions specialize to formulae given by R. L. Davis [6] and by N. G. de Bruijn in [8]. Since endofunctions can be considered as permutations of rooted trees the species of endofunctions is given by

  $$\mathsf{End} = \mathsf{Per}(\mathsf{Rtr})$$

  so that

  $$\widetilde{\mathsf{End}}(x) = \widetilde{\mathsf{Per}(\mathsf{Rtr})}(x) = Z_{\mathsf{Per}}(\widetilde{\mathsf{Rtr}}(x), \widetilde{\mathsf{Rtr}}(x^2), \ldots) = \prod_{n \geq 1} \frac{1}{1 - \widetilde{\mathsf{Rtr}}(x^n)}. \tag{8}$$

- Sets of *contractions* on $\boldsymbol{n}$ (or *forests of rooted trees*) are endofunctions on $\boldsymbol{n}$ all of whose cycles are of length one. For the special case $L = \{2, 3, \ldots, n\}$ and $m(l) = 0$ for all $l \in L$ we can enumerate the number of all unlabelled sets of contractions from Theorem 3 and Theorem 5. In this situation

$$\psi(L, k, i) = \begin{cases} i & \text{if } k > 1 \\ i - 1 & \text{if } k = 1 \end{cases}$$

such that

$$\widetilde{Z}(S_{\boldsymbol{j}}, L, i) = \frac{1}{j!}(1 - j \cdot i)$$

and the number of these unlabelled structures on $\boldsymbol{n}$ is given by

$$\frac{1}{n!} \sum_{\pi \in S_{\boldsymbol{n}}} \prod_{\substack{i=1 \\ \lambda_i(\pi) > 0}}^{n} \left(1 + \sum_{k|i} k \lambda_k(\pi) \right)^{\lambda_i(\pi) - 1} \left(1 + \sum_{\substack{k|i \\ k \neq i}} k \lambda_k(\pi) \right).$$

This formula is essentially the same expression as (13) in [4]. Let Con denote the species of sets of contractions then we have

$$\mathsf{Con} = \mathsf{Set}(\mathsf{Rtr}), \text{ so } \mathsf{Sin} \cdot \mathsf{Con} = \mathsf{Rtr}$$

and consequently

$$\widetilde{\mathsf{Con}}(x) = \frac{\widetilde{\mathsf{Rtr}}(x)}{x}. \tag{9}$$

- Roughly speaking an endofunction having no cycles of length 1 is a *functional digraph*. From Theorem 3 and Theorem 5 we compute for the special case of $L = \{1\}$ and $m(1) = 0$ the number of unlabelled functional digraphs on n points as

$$\frac{1}{n!} \sum_{\pi \in S_{\boldsymbol{n}}} \prod_{i=1}^{n} \left(\sum_{j|i} j \lambda_j(\pi) - 1 \right)^{\lambda_i(\pi)}.$$

Denoting the species of functional digraphs by Fun then

$$\mathsf{End} = \mathsf{Con} \cdot \mathsf{Fun}$$

which implies

$$\widetilde{\mathsf{Fun}}(x) = \frac{\widetilde{\mathsf{End}}(x)}{\widetilde{\mathsf{Con}}(x)} = \frac{x \cdot \widetilde{\mathsf{End}}(x)}{\widetilde{\mathsf{Rtr}}(x)}. \tag{10}$$

The following formulae follow easily from (2) and (3). Since they were proved before combinatorial species theory was invented we want to present them using the cycle index notation. In [12] F. Harary took a graph theoretic approach for the enumeration of functional digraphs and classes of endofunctions (when allowing loops to occur) by successive applications of *Pólyas Enumeration Theorem*. Harary computed the generating function of all unlabelled functional digraphs as

$$\widetilde{\mathsf{Fun}}(x) = \sum_{\text{all } m(l)=0}^{\infty} \prod_{l=2}^{\infty} Z(S_{\boldsymbol{m(l)}}[C_l], \boldsymbol{m(l)} \times \boldsymbol{l} \mid x_i = \mathsf{Rtr}(x^i)) =$$

$$= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \sum_{l=2}^{\infty} Z(C_l, \boldsymbol{l} \mid x_i = \mathsf{Rtr}(x^{im}))\right),$$

where the *composition* $S_{\boldsymbol{m(l)}}[C_l]$ is a permutation representation of the *wreath product* $C_l \wr S_{\boldsymbol{m(l)}}$ on $\boldsymbol{m(l)} \times \boldsymbol{l}$, which is similar to the plethysm $C_l \odot S_{\boldsymbol{m(l)}}$ acting on $\boldsymbol{l} \cdot \boldsymbol{m(l)}$. So they have the same cycle indices. In the same way the generating function $\widetilde{\mathsf{End}}(x)$ of the numbers of classes of endofunctions is given by summing for $l = 1, \ldots, \infty$. In [17] R. C. Read gave some simplification of Harary's formula, which lead to (8) and (10). In [9] the same generating function for the classes of endofunctions is found by factorizing words over a totally ordered alphabet into Lyndon words. This method was generalized by V. Strehl [18] by introducing a cycle counting parameter.

As a special case of Harary's formula we have the following result: The number of unlabelled endofunctions on $\boldsymbol{n}$ having exactly $m(l)$ cycles of length $l \in L$ and no further cycles is the coefficient of x^n in

$$\prod_{l \in L} Z(S_{\boldsymbol{m(l)}}[C_l], \boldsymbol{m(l)} \times \boldsymbol{l} \mid x_i = \mathsf{Rtr}(x^i)).$$

- The following formula for the number of functions $g\colon \boldsymbol{n} \setminus \boldsymbol{s} \to \boldsymbol{n}$ having no cycles

$$\left|\boldsymbol{n}_{\bar{\boldsymbol{n}}}^{\boldsymbol{n}\setminus\boldsymbol{s}}\right| = n^{n-s} - (n-s)n^{n-s-1} = sn^{n-s-1}$$

follows directly from Corollary 6 by using the same methods as for proving that (7) vanishes for $j > 1$. Combining this result with Theorem 2 we have shown that the number of all endofunctions on $\boldsymbol{n}$ having exactly $m(l)$ cycles of length $l \in L$ and no further cycles is given by

$$|\mathsf{End}_{\mathcal{L},\bar{\boldsymbol{n}}}[\boldsymbol{n}]| = \binom{n}{s} \frac{s!}{\prod_{l \in L} l^{m(l)} m(l)!} sn^{n-s-1}.$$

This formula corresponds to formula 3.3.13 in [11]. Moreover these numbers follow easily from Lagrange inversion: They are special cases of (3.1.47) in [2] for $r_s(n) = n^s$, $f_s = 1$ if we only want to enumerate the forest of s-rooted trees, or f_s equals the number of permutations on $\boldsymbol{s}$ of cycle type $\mathcal{L}$ for endofunctions of type $\mathcal{L}$ respectively.

- For $L = \boldsymbol{n}$, $m(1) = 1$ and $m(l) = 0$ for $l > 1$ we compute the number of all unlabelled rooted trees on n points by

$$\frac{1}{(n-1)!} \sum_{\pi \in S_{\boldsymbol{n-1}}} \prod_{\substack{i=1 \\ \lambda_i(\pi)>0}}^{n-1} \left(1 + \sum_{k|i} k\lambda_k(\pi)\right)^{\lambda_i(\pi)-1} \left(1 + \sum_{\substack{k|i \\ k \neq i}} k\lambda_k(\pi)\right).$$

  From (9) it is clear that this formula is similar to the formula for enumerating unlabelled sets of contractions. An essentially equivalent formula for the numbers of rooted trees kept fixed under a given permutation is given in (3.2.61) of [2]. It was proved by multidimensional Lagrange inversion (cf. [15].) In [3, 4] you can find a bijective proof of this formula.

  Using formula (4.1.45) of [2] together with these numbers it is possible to enumerate unlabelled trees on n vertices.

Most of these formulae are implemented in SYMMETRICA [19], a computer algebra system devoted to combinatorics and representation theory of finite symmetric groups and related groups. It would be interesting to compare the computational complexity of using the bottom up approach by the Cauchy-Frobenius lemma with the complexity of the top down approach of species theory.

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