Isometry Classes of Indecomposable Linear Codes

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Abstract

In the constructive theory of linear codes, we can restrict attention to the isometry classes of indecomposable codes, as it was shown by Slepian. We describe these classes as orbits and we demonstrate how they can be enumerated using cycle index polynomials and the tools already incorporated in SYMMETRICA, a computer algebra package devoted to representation theory and combinatorics of symmetric groups and of related classes of groups. Moreover, we describe how systems of representatives of these classes can be evaluated using double coset methods.

1 Notation

Suppose we are given a permutation group G on a set X. It induces interesting structures both on X and on G that are closely related (for more details cf. [7]). To begin with, there are the *orbits* of the elements $x \in X$: $G(x) := \{gx \mid g \in G\} \subseteq X$, two of which are either identical or disjoint. Therefore, the set

$$G\backslash\backslash X := \{G(x) \mid x \in X\}.$$

of *all the orbits* is a set-partition of X. To the orbits there correspond the *stabilizers* (which are in fact subgroups): $G_x := \{g \in G \mid gx = x\}$, and the close relationship between orbits and stabilizers is the existence of the following natural bijection between an orbit of an element and the set of left cosets of its stabilizer:

$$G(x) \to G/G_x, \quad gx \mapsto gG_x.$$

The *cycle index* of G is the following polynomial $Z(G)$ in the indeterminates $x_1, x_2, \ldots, x_{|X|}$ over $\mathbb{Q}$, defined by

$$Z(G) := \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{|X|} x_i^{a_i(g)},$$

where $(a_1(g), \ldots, a_{|X|}(g))$ is the cycle type of the permutation $g \in G$. This means, g decomposes into $a_i(g)$ disjoint cycles of length i for $i = 1, \ldots, |X|$. All elements of a conjugacy class have the same cycle type, so the cycle index can be computed in the following way:

$$Z(G) = \frac{1}{|G|} \sum_{C \in \mathcal{C}} |C| \prod_{i=1}^{|X|} x_i^{a_i(g_C)}, \tag{1}$$

where $\mathcal{C}$ is the system of conjugacy classes of G and where g_C is a representative of the conjugacy class C.
Let G and H denote permutation groups on X and Y, respectively. The *wreath product* $H \wr_X G$ is a permutation group on the set $Y^X := \{f: X \to Y\}$, defined by

$$H \wr_X G := H^X \times G = \{(\psi, g) \mid \psi \in H^X, g \in G\}$$

with multiplication $(\psi, g)(\psi', g') = (\psi\psi'_g, gg')$, where $\psi\psi'_g(x) := \psi(x)\psi'_g(x)$ and $\psi'_g(x) := \psi'(g^{-1}x)$. In the case when $G \leq S_n$ and $X = n := \{0, 1, \ldots, n-1\}$ we write $H \wr G$ instead of $H \wr_n G$.
The wreath product $H \wr_X G$ acts in a natural way on Y^X. The effect of the permutation $(\psi, g) \in H \wr_X G$ on $f \in Y^X$ is

$$(\psi, g)(f) =: \widetilde{f}, \text{ where } \widetilde{f}(x) = \psi(g) f(g^{-1}x). \tag{2}$$

The following lemma ([10, 9]) reduces the action of a wreath product to the action of the group G on the set of all functions from X into the set of all orbits of H on Y:

1.1 Lehmann's Lemma: *If G and H denote permutation groups on X and Y, respectively, then G acts on $(H\backslash\backslash Y)^X$ in the following way:*

$$g(f) := f \circ g^{-1}.$$

Moreover, the mapping

$$\Phi: H \wr_X G\backslash\backslash Y^X \to G\backslash\backslash(H\backslash\backslash Y)^X, (H \wr_X G(f)) \mapsto G(F)$$

is a bijection if $F \in (H\backslash\backslash Y)^X$ is given by $F(x) = H(f(x))$.
There are many enumerative and constructive results dealing with various group actions on the set Y^X induced by permutation groups on the domain X and on the range Y. For example, $S_n \times H$ acts upon Y^n by

$$(\pi, h)(f) := h \circ f \circ \pi^{-1}.$$

The corresponding generating function for the numbers of $S_n \times H$-orbits on Y^n is given by (see [1]):

$$\sum_{n=0}^{\infty} |(S_n \times H)\backslash\backslash Y^n|\, x^n = Z(H)\Big|_{x_i = \sum_{j=0}^{\infty} x^{ij}} = Z(H)\Big|_{x_i = \frac{1}{1-x^i}}. \tag{3}$$

The group action of above can be restricted to the set of all injective mappings from n to Y, the corresponding generating function is

$$\sum_{n=0}^{\infty} \left|(S_n \times H)\backslash\backslash Y^n_{\mathrm{inj}}\right| x^n = Z(H)\big|_{x_i=1+x^i}. \tag{4}$$

2 Isometry classes of linear codes

A *linear* (n,k)*-code* over the Galois field $GF(q)$ is a k-dimensional subspace of the vector space $Y^X := GF(q)^n$. As usual codewords will be written as rows $x = (x_0,\dots,x_{n-1})$. A $k \times n$-matrix Γ over $GF(q)$ is called *a generator matrix* of the linear (n,k)-code C, if and only if the rows of Γ form a basis of C, so that $C = \{x \cdot \Gamma \mid x \in GF(q)^k\}$. Two linear (n,k)-codes C_1, C_2 are called *equivalent*, if and only if there is an isometry (with respect to the Hamming metric) which maps C_1 onto C_2. Using the notion of finite group actions one can easily express equivalence of codes in terms of the wreath product action introduced above: C_1 and C_2 are equivalent, if and only if there exist $(\psi,\pi) \in GF(q)^* \wr S_n$ (where $GF(q)^*$ denotes the multiplicative group of the Galois field) such that $(\psi,\pi)(C_1) = C_2$.
The *complete monomial group* $GF(q)^* \wr S_n$ of degree n over $GF(q)^*$ acts on $GF(q)^n$ as it was described above (see equation (2)) in the more general case of $H \wr_X G$ on Y^X:

$$(\psi,\pi)(f)(x) = \psi(x) f(\pi^{-1}x). \tag{5}$$

In order to apply the results of the theory of finite group actions, this equivalence relation for linear (n,k)-codes is translated into an equivalence relation for generator matrices of linear codes, and these generator matrices are considered to be functions $\Gamma\colon n \to GF(q)^k \setminus \{0\}$ where $\Gamma(i)$ is the i-th column of the generator matrix Γ. (We exclude 0-columns for obvious reasons.)

2.1 Theorem *The matrices corresponding to the two functions Γ_1 and Γ_2 from n to $GF(q)^k \setminus \{0\}$ are generator matrices of two equivalent codes, if and only if Γ_1 and Γ_2 lie in the same orbit of the following action of $GL_k(q) \times GF(q)^* \wr S_n$ as permutation group on $(GF(q)^k \setminus \{0\})^n$:*

$$(A,(\psi,\pi))(\Gamma) = A\psi(\cdot)\Gamma(\pi^{-1}\cdot),$$

or, more explicitly,

$$(A,(\psi,\pi))(\Gamma)(i) := A\psi(i)\Gamma(\pi^{-1}(i)).$$

Following Slepian, we use the following notation:

$T_{nkq} :=$ the number of orbits of functions $\Gamma\colon n \to GF(q)^k \setminus \{0\}$ under the group action of 2.1, i.e. $T_{nkq} = \left|(GL_k(q) \times GF(q)^* \wr S_n)\backslash\backslash(GF(q)^k \setminus \{0\})^n\right|$.

$\bar{T}_{nkq}$:= the number of orbits of functions $\Gamma: n \to GF(q)^k \setminus \{0\}$ under the group action of 2.1, such that for all $i, j \in n$, $i \neq j$ and all $\alpha \in GF(q)^*$ the value of $\Gamma(i)$ is different from $\alpha\Gamma(j)$. (In the case $q = 2$, this is the number of injective functions G.)

S_{nkq} := the number of equivalence classes of linear (n, k)-codes over $GF(q)$ with no columns of zeros. (A linear (n, k)-code has columns of zeros, if and only if there is some $i \in n$ such that $x_i = 0$ for all codewords x, and so we should exclude such columns.)

$\bar{S}_{nkq}$:= the number of classes of injective linear (n, k)-codes over $GF(q)$ with no columns of zeros. (A linear (n, k)-code is called injective, if and only if for all $i, j \in n$, $i \neq j$ and $\alpha \in GF(q)^*$ there is some codeword x such that $x_i \neq \alpha x_j$.)

R_{nkq} := the number of classes of indecomposable linear (n, k)-codes over $GF(q)$ with no columns of zeros. (The definition of an indecomposable code will be given later.)

$\bar{R}_{nkq}$:= the number of classes of indecomposable, injective linear (n, k)-codes over $GF(q)$ with no columns of zeros.

W_{nkq} := be the number of classes of linear (n, k)-codes over $GF(q)$ with columns of zeros allowed.

The following formulae hold:

$$W_{nkq} = \sum_{i=k}^{n} S_{ikq}, \; S_{nkq} = T_{nkq} - T_{n,k-1,q}, \; \bar{S}_{nkq} = \bar{T}_{nkq} - \bar{T}_{n,k-1,q}. \qquad (6)$$

As initial values we have $S_{n1q} = 1$ for $n \in \mathbb{N}$, $\bar{S}_{11q} = 1$ and $\bar{S}_{n1q} = 0$ for $n > 1$. It is important to realize that

- T_{nkq} is the number of orbits of functions from n to $GF(q)^k \setminus \{0\}$ without any restrictions on the rank of the induced matrix.
- All matrices which are induced from functions Γ of the same orbit have the same rank.
- The number of orbits of functions Γ which induce matrices of rank less or equal $k - 1$ is $T_{n,k-1,q}$. (This proposition holds for $\bar{T}_{nkq}$ as well.)

In the next section we will show that the R_{nkq} or $\bar{R}_{nkq}$ can be computed from the S_{nkq} or $\bar{S}_{nkq}$ respectively, so the main problem is the computation of the T_{nkq} or $\bar{T}_{nkq}$.

In the case $q = 2$ the wreath product $GF(q)^* \wr S_n$ becomes the group S_n, and so there is the group $GL_k(2)$ acting on $GF(2)^k \setminus \{0\}$ and the symmetric group S_n acting on n. Applying the formulae (3) and (4) we get

$$\sum_{n=0}^{\infty} T_{nk2} x^n = Z(GL_k(2))\big|_{x_i = \sum_{j=0}^{\infty} x^{ij}} = Z(GL_k(2))\big|_{x_i = \frac{1}{1-x^i}} \tag{7}$$

and

$$\sum_{n=0}^{\infty} \bar{T}_{nk2} x^n = Z(GL_k(2))\big|_{x_i = 1 + x^i}. \tag{8}$$

In the case $q \neq 2$ the wreath product $GF(q)^* \wr S_n$ acts both on range and domain of the functions Γ. Applying LEHMANN's Lemma 1.1 there is the bijection

$$\Phi: GF(q)^* \wr S_n \backslash\backslash (GF(q)^k \setminus \{0\})^n \to S_n \backslash\backslash \left(GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})\right)^n,$$

$$GF(q)^* \wr S_n(\Gamma) \mapsto S_n(\bar{\Gamma})$$

where

$$\bar{\Gamma}: n \to GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\}), i \mapsto GF(q)^*(\Gamma(i))$$

and S_n acts on $\left(GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})\right)^n$ by $\pi(\bar{\Gamma}) = \bar{\Gamma} \circ \pi^{-1}$. Using this bijection we have to investigate the following action of $S_n \times GL_k(q)$:

$$(\pi, A)(\bar{\Gamma}) = A\bar{\Gamma}\pi^{-1},$$

where $GL_k(q)$ acts on $GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})$ by $A(GF(q)^*(v)) = GF(q)^*(Av)$. The set of the $GF(q)^*$-orbits $GF(q)^* \backslash\backslash (GF(q)^k \setminus \{0\})$ is the $(k-1)$-dimensional projective space:

$$GF(q)^* \backslash\backslash GF(q)^k = PG_{k-1}(q)$$

and the representation of $GL_k(q)$ as a permutation group is the projective linear group $PGL_k(q)$.
This proves in fact the following to be true:

2.2 Theorem *The isometry classes of linear (n,k)-codes over $GF(q)$ are the orbits of $GL_k(q) \times S_n$ on the set of mappings $PG_{k-1}(q)^n$. This set of orbits is equal to the set of orbits of $GL_k(q)$ on the set $S_n \backslash\backslash PG_{k-1}(q)^n$, which can be represented by a complete set of mappings of different content, if the content of $f \in PG_{k-1}(q)^n$ is defined to be the sequence of orders of inverse images $|f^{-1}(x)|$.*
Thus the set of isometry classes of linear (n,k)-codes over $GF(q)$ is equal to the set of orbits of $GL_k(q)$ on the set of mappings $f \in PG_{k-1}(q)$ of different content that form $k \times n$-matrices of rank k.
The particular classes of elements with orders of inverse images $|f^{-1}(x)| \leq 1$ are the classes consisting of Hamming codes.

Knowing the cycle index of $PGL_k(q)$ acting on $PG_{k-1}(q)$ the equations (3) and (4) can be applied again.
In [13] SLEPIAN explained how the cycle index of $GL_k(2)$ can be computed using results of ELSPAS [3]. The first author [4] generalized this concept for computing the cycle indices of $GL_k(q)$ and $PGL_k(q)$ acting on $GF(q)^k$ or $PG_{k-1}(q)$ respectively. The steps of the method used were the following ones:

1. Determination of the conjugacy classes of $GL_k(q)$ by applying the theory of normal forms of matrices (or vector space endomorphisms). This theory can be found in many textbooks of algebra.

2. Determination of the order of the conjugacy classes, which can be found in DICKSON, GREEN or KUNG [2, 5, 8].

3. Determination of the cycle type of a linear map or of a projectivity respectively. Since normal forms of regular matrices are strongly connected with companion and hypercompanion matrices (see [6]) of monic, irreducible polynomials over $GF(q)$ it is important to know the *exponent* or *subexponent* of such polynomials (see [11, 6]). The exponent of such a polynomial $f(x) \in GF(q)[x]$ is defined to be

$$\exp(f(x)) := \min \{n \in \mathbb{N} \mid f(x) \mid x^n - 1\}$$

   and the subexponent is

$$\text{subexp}(f(x)) := \min \{n \in \mathbb{N} \mid \exists \alpha \in GF(q)^* : f(x) \mid x^n - \alpha\}.$$

   This element $\alpha \in F_q^*$ is uniquely defined, and it is called the *integral element* of $f(x)$. The exponent of $f(x)$ can be used to compute the cycle type of the companion or hypercompanion matrices of a monic, irreducible polynomial $f(x)$, and by a direct product formula for cycle indices the cycle types of the normal forms in $GL(k, F_q)$ can be derived. Using the subexponent of $f(x)$ and defining a formula similar to the direct product formula of cycle indices, which depends on the integral element of $f(x)$ as well, the cycle type of a projectivity can be computed.

4. Determination of the cycle index by applying formula (1).

These cycle indices are now available in the computer algebra package SYMMETRICA. Tables obtained this way are shown lower down.

3 Indecomposability

In order to minimize the number of orbits that must be enumerated or represented, and following SLEPIAN again, we can restrict attention to *indecomposable* linear (n, k)-codes. Let C_1 be a linear (n_1, k_1)-code over $GF(q)$ with generator matrix Γ_1 and let C_2 be a linear (n_2, k_2)-code over $GF(q)$ with generator matrix Γ_2, then the code C with generator matrix

$$\Gamma := \left(\begin{array}{c|c} \Gamma_1 & 0 \\ \hline 0 & \Gamma_2 \end{array} \right)$$

is called the *direct sum* of the codes C_1 and C_2, and it will be denoted by $C = C_1 \oplus C_2$. A code C is called *decomposable*, if and only if it is equivalent to a code which is the direct sum of two or more linear codes. Otherwise it is called *indecomposable*.

In [13] SLEPIAN proves that every decomposable linear (n,k)-code is equivalent to a direct sum of indecomposable codes, and that this decomposition is unique up to equivalence and order of the summands. SLEPIAN used a generating function scheme for computing the numbers R_{nk2} and $\bar{R}_{nk2}$. However after constructing these codes the authors realized that in some situations this formula doesn't work correctly. For that reason we are giving another formula to determine the numbers R_{nkq} and $\bar{R}_{nkq}$. For the rest of this section let $n \geq 2$.

3.1 Theorem *The number R_{nkq} is equal to*

$$S_{nkq} - \sum_a \sum_b \prod_{\substack{j=1 \\ a_j \neq 0}}^{n-1} \left(\sum_{\substack{c=(c_1,\ldots,c_{a_j}) \in \mathbb{N}^{a_j} \\ j \geq c_1 \geq \ldots \geq c_{a_j} \geq 1,\ \sum c_i = b_j}} U(j,a,c) \right),$$

where

$$U(j,a,c) = \prod_{i=1}^{j} Z(S_{\nu(i,a_j,c)})\big|_{x_\ell = R_{jiq}}, \quad \nu(i,a_j,c) = |\{1 \leq l \leq a_j \mid c_l = i\}|,$$

and where the first sum is taken over the cycle types $a = (a_1, \ldots, a_{n-1})$ of n, (which means that $a_i \in \mathbb{N}_0$ and $\sum i a_i = n$) such that $\sum a_i \leq k$, while the second sum is over the $(n-1)$-tuples $b = (b_1, \ldots, b_{n-1}) \in \mathbb{N}_0^{n-1}$, for which $a_i \leq b_i \leq i a_i$, and $\sum b_i = k$. In the same way the $\bar{R}_{nkq}$ can be computed from the $\bar{S}_{nkq}$. The numerical results show that for fixed q and n the sequence of R_{nkq} is unimodal and symmetric. (It is easy to prove that this sequence must be symmetric, but the proof of the unimodality is still open.)

Proof: In order to evaluate the values R_{nkq} the number of all classes of decomposable linear (n,k)-codes must be subtracted from S_{nkq}. In other words one has to find all possibilities of decomposing a linear (n,k)-code into a direct sum of indecomposable linear (n_i, k_i)-codes such that

$$\sum_{i=1}^{l} n_i = n, \qquad \sum_{i=1}^{l} k_i = k, \qquad 1 \leq k_i \leq n_i, \qquad 2 \leq l \leq k. \tag{9}$$

According to SLEPIANs theorem the (n_i, k_i)-codes can be arranged such that $n_1 \geq n_2 \geq \ldots \geq n_l$ and in the case that $n_i = n_{i+1}$ the inequality $k_i \geq k_{i+1}$ holds. At first all partitions of n into at least two parts and into at most k parts must be found. Let $n = n_1 + n_2 + \ldots + n_l$ be such a partition with $n_i \geq 1$ and $2 \leq l \leq k$. Then the cycle type of λ is $(a_1, a_2, \ldots, a_{n-1})$, where $a_i = |\{1 \leq j \leq l \mid n_j = i\}|$. Two decomposable codes which determine two different partitions of n are not equivalent. In the second step to each partition of n one has to find all sequences $(k_1, \ldots, k_l)$ such that (9) is fulfilled, and such that codes belonging to different sequences are not equivalent. In order to do this we start with such a sequence $(k_1, \ldots, k_l)$ and define

$$b_i := \sum_{j:n_j=i} k_j,$$

then

$$\sum_{i=1}^{n-1} b_i = \sum_{i=1}^{l} k_i = k \text{ and } a_i \leq b_i \leq i a_i.$$

Two decomposable codes which on the one hand belong to one partition of n but on the other hand define two different vectors b and b' are not in the same equivalence class. Now the other way round we start with a sequence $(b_1, \ldots, b_{n-1})$ and try to determine all sequences $(k_1, \ldots, k_l)$ which define non equivalent codes such that $b_i = \sum_{j:n_j=i} k_j$. Again by SLEPIAN's theorem we must find all partitions of $b_j \neq 0$ (this implies $a_j \neq 0$) into exacyly a_j parts of the form

$$b_j = \sum_{i=1}^{a_j} c_i, \qquad j \geq c_1 \geq \ldots \geq c_{a_j} \geq 1.$$

In the last step $U(j, a, c)$ must be evaluated. This is the the number of classes of linear $(j \cdot a_j, b_j)$-codes, which have a decomposition into a direct sum of indecomposable (j, c_i)-codes for $1 \leq i \leq a_j$ to a given partition c of b_j into exactly a_j parts. We already know that there are $R_{j,c_i,q}$ classes of indecomposable linear (j, c_i)-codes. In the case that all the c_i are distinct, this number is given by

$$\prod_{i=1}^{a_j} R_{j,c_i,q}.$$

Otherwise there exist i, l such that $i < l$ and $c_i = c_l$. Then $c_i = c_{i+1} = \ldots = c_l$ and permuting the corresponding summands in the direct sum may lead to equivalent codes. For $1 \leq i \leq j$ let $\nu(i) = \nu(i, a_j, c)$ be the cardinality of $\{t \mid c_t = i\}$. Now there is a bijection between the classes of codes, which are a direct sum of $\nu(i)$ (j, i)-codes and the orbits of the symmetric group $S_{\nu(i)}$ on the set of all mappings from $\nu(i)$ into a set of R_{jiq} elements, where the action of $S_{\nu(i)}$ is given by:

$$S_{\nu(i)} \times {R_{jiq}}^{\nu(i)} \to {R_{jiq}}^{\nu(i)}, \quad (\pi, f) \mapsto f \circ \pi^{-1}.$$

By PÓLYAs theorem [12] one has to compute the cycle index $S_{\nu(i)}$ and each variable must be substituted by R_{jiq}. Doing this for all possible values of i we get

$$U(j,a,c) = \prod_{i=1}^{j} Z(S_{\nu(i)})\big|_{x_\ell = R_{jiq}}.$$

Using the computer algebra system SYMMETRICA, among many other ones the tables of numbers of indecomposable codes which are shown below were computed.

4 Construction

Methods for constructing representatives of classes of linear (n,k)-codes obviously do not reach as far as the enumerative methods, but the use of computers allows to get a complete overview of linear (n,k)-codes over $GF(q)$ for quite a number of parameter triples (n,k,q). We have seen before that the isometry classes can be described as orbits of $GL_k(q)$ on sets of mappings into the projective space. A very interesting and helpful constructive method for discrete structures which can be defined as orbits of finite groups on finite sets is based on the following fact. If G is a transitive permutation group on X, then the orbits of a subgroup $U \leq G$ on X can be bijectively mapped onto double cosets as follows: For any $x \in X$ the mapping

$$U\backslash\backslash X \to U\backslash G/G_x, \quad U(gx) \mapsto UgG_x$$

is a bijection. In the case of the (n,k)-codes we can use the fact that the general linear group $GL_n(q)$ is transitive on the set $S(n,k,q)$ of subspaces of dimension k in $GF(q)^n$, so that the isometry classes of linear (n,k)-codes turn out to be in one-one-correspondence with the set of double cosets

$$GF(q)^* \wr S_n \backslash GL_n(q)/GL_n(q)_{C_0},$$

where C_0 is any linear (n,k)-code. A computer program due to WEINRICH ([14]) allows to evaluate complete sets of representatives, and it was recently improved by using, besides of double cosets the combinatorial method of *orderly generation*. The work in this field of constructive theory is still in rapid progress, so that we cannot tell yet how far we can reach.

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Table 1: R_{nk2}

$n\backslash k$	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0
4	1	1	1	0	0	0	0	0	0	0
5	1	2	2	1	0	0	0	0	0	0
6	1	3	5	3	1	0	0	0	0	0
7	1	4	10	10	4	1	0	0	0	0
8	1	5	18	28	18	5	1	0	0	0
9	1	7	31	71	71	31	7	1	0	0
10	1	8	51	165	250	165	51	8	1	0
11	1	10	79	361	809	809	361	79	10	1
12	1	12	121	754	2484	3759	2484	754	121	12

Table 2: R_{nk3}

$n\backslash k$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	0	0	0	0	0	0
3	1	1	0	0	0	0	0
4	1	2	1	0	0	0	0
5	1	3	3	1	0	0	0
6	1	5	10	5	1	0	0
7	1	7	24	24	7	1	0
8	1	10	55	105	55	10	1
9	1	13	116	403	403	116	13
10	1	17	231	1506	3000	1506	231
11	1	21	438	5425	23579	23579	5425
12	1	27	813	19440	199473	469473	199473

Table 3: R_{nk4}

$n\backslash k$	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	0	0	0	0	0	0
3	1	1	0	0	0	0	0
4	1	2	1	0	0	0	0
5	1	4	4	1	0	0	0
6	1	6	14	6	1	0	0
7	1	9	38	38	9	1	0
8	1	13	104	238	104	13	1
9	1	18	276	1573	1573	276	18
10	1	25	711	11566	34288	11566	711
11	1	32	1793	88140	909664	909664	88140
12	1	42	4446	665736	25.020688	90.186547	25.020688

Table 4: R_{nk5}

$n\backslash k$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	0	0	0	0	0
3	1	1	0	0	0	0
4	1	2	1	0	0	0
5	1	4	4	1	0	0
6	1	8	18	8	1	0
7	1	11	62	62	11	1
8	1	18	222	659	222	18
9	1	26	800	8232	8232	800
10	1	38	2805	117351	483955	117351
11	1	51	9642	1674434	32.156437	32.156437

Table 5: R_{nk7}

$n\backslash k$	1	2	3	4	5
1	1	0	0	0	0
2	1	0	0	0	0
3	1	1	0	0	0
4	1	3	1	0	0
5	1	5	5	1	0
6	1	11	32	11	1
7	1	18	165	165	18
8	1	33	1006	4741	1006
9	1	50	6362	179586	179586
10	1	83	39417	7.058258	45.507354
11	1	123	233578	260.571116	11419.262502