ASSOCIATIVE FORMAL POWER SERIES IN TWO INDETERMINATES

HARALD FRIPERTINGER, LUDWIG REICH, JENS SCHWAIGER AND JÖRG TOMASCHEK

Abstract. Investigating the associativity equation for formal power series in two variables we show that the transcendental associative formal power series are of order one or two and that they can be represented by an invertible formal power series in one variable. We also discuss the convergence of associative formal power series.

1. Introduction

The notion of associativity of a function $F : D \times D \to D$ (or an operation $\circ$ on the set D such that $x \circ y = F(x, y)$) means in the formulation as a functional equation

$$F(F(x, y), z) = F(x, F(y, z))$$

for all $x, y, z \in D$. This functional equation has been studied for various sets D and under different regularity conditions on F, see [1]. In the present paper we consider the associativity equation for indeterminates X, Y and Z

$$F(F(X, Y), Z) = F(X, F(Y, Z)) \tag{AE}$$

for a formal power series $F(X, Y)$ in two indeterminates X and Y over a commutative field $\mathbb{K}$. A formal power series F in two variables is called associative if F fulfills the associativity equation

$$F(F(X, Y), Z) = F(X, F(Y, Z)). \tag{AE}$$

By $\mathbb{K}[\![X, Y]\!] = \left\{F \colon F(X, Y) = \sum_{p,q\geq 0} a_{p,q} X^p Y^q\right\}$ we denote the ring of formal power series in two indeterminates. For a detailed description of formal power series rings we refer the reader to [2]. For a series $F \in \mathbb{K}[\![X, Y]\!]$, $F \neq 0$, the order of F is defined by ord $F = n$ where $n \in \mathbb{N} \cup \{0\}$ is the smallest number such that $\sum_{p+q=n} a_{p,q} X^p Y^q \neq 0$, and ord $F = \infty$ if $F = 0$. In order to make unrestricted substitution possible, the order of any associative formal power series F has to be greater than zero, hence the absolute term $a_{0,0}$ of every associative formal power series is zero.

Solutions F of the classical problem of associativity of the form $F(X, Y) = X + Y + \ldots \in \mathbb{K}[\![X, Y]\!]$ are called formal group laws, see [5]. If $\mathbb{K}$ has characteristic 0, then the general solution is given by $F(X, Y) = f^{-1}(f(X) + f(Y))$ for $f \in \mathbb{K}[\![X]\!]$, see [5] pp 30.

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Recently the polynomial case was investigated, some years ago also the rational situation was considered. The article [6] provides a complete characterization of all associative polynomials in two or more variables, for the special case of three variables we refer the reader to [4]. The associative rational functions are decribed in [3].

In this paper we regard formal power series F in two indeterminates for which $F(X,Y) \neq X+Y+\ldots$. Therefore, we start with an example demonstrating that there exist associative formal power series which are not decribed in [5]. Then we show that all transcendental associative formal power series F with order 1 are given by $F(X,Y) = X+Y+\ldots$ and that all other associative formal power series F which are not polynomials are of order 2, they have to be of the form $F(X,Y) = a_{1,1}XY+\ldots$, $a_{1,1} \neq 0$, $a_{2,0} = a_{0,2} = 0$. For the series with order 2 we prove a Representation Theorem, namely Theorem 4.3, and we discuss the question of convergence of these series.

2. An example

Associative formal power series F in two indeterminates with linear part $X+Y$ are well described and there are many examples of them, see [5]. Therefore it is natural to ask wether there are any associative formal power series F with $\operatorname{ord} F \geq 2$. We start with the following example which provides us with a whole class of examples.

Example. Let $f(X) = c(X+X^2+\ldots) = cX \cdot \sum_{n=0} X^n$ where $c \neq 0$. Then f is invertible. We define $F(X,Y) := f^{-1}(f(X)f(Y))$ and we obtain

$$\begin{aligned}
F(F(X,Y),Z) &= F(f^{-1}(f(X)f(Y)),Z) \\
&= f^{-1}(f(f^{-1}(f(X)f(Y)))f(Z)) \\
&= f^{-1}(f(X)f(Y)f(Z)) \\
&= f^{-1}(f(X)(f(f^{-1}(f(Y)f(Z)))) \\
&= F(X, f^{-1}(f(Y)f(Z)))) \\
&= F(X, F(Y,Z)).
\end{aligned}$$

Hence this F is an associative formal power series. Explicitly we have

$$f(X) = \frac{cX}{1-X}, \qquad f^{-1}(X) = \frac{X}{c+X}$$

and

$$F(X,Y) = \frac{cXY}{1-X-Y+(1+c)XY}$$

or

$$\begin{aligned}
F(X,Y) = cXY + cX^2Y + cX^3Y + cXY^2 + (c-c^2)X^2Y^2 + (c-2c^2)X^3Y^2 \\
+ cXY^3 + (c-2c^2)X^2Y^3 + (c-4c^2+c^3)X^3Y^3 + \ldots.
\end{aligned}$$

Thus we know that we are not talking about an empty set when we are investigating the non-classical case of formal group laws or polynomials.

3. The structure of associative formal power series in two indeterminates

This section is devoted to the question of the order of an associative formal power series F. After showing that the order of F has to be 1 or 2, we give some conditions on the coefficients of the possible associative formal power series. Let us mention once more that we are not interesed in the case where $F(X,Y) = X + Y + \ldots$, which is the classical formal group approach. We start with the first observations.

Remark 3.1. Let $F(X,Y) = a_{1,1}XY + F_3(X,Y) + \ldots$, with $a_{1,1} \neq 0$ and $F(X,Y) \neq a_{1,1}XY$, be an associative power series. Then F is not a polynomial (i.e. there are infinitely many $k \geq 2$ such that $F_k \neq 0$).

Proof. From [6] we know that each associative polynomial $P(X,Y) = a_{1,1}XY + \ldots + F_m(X,Y)$, $a_{1,1} \neq 0$, $m \geq 3$, in two variables with $P(0,0) = 0$ is

$$P(X,Y) = a_{1,1}XY.$$

$\square$

Lemma 3.2. *Let $F \in \mathbb{K}[\![X,Y]\!]$, $F(X,Y) = a_{1,0}X + a_{0,1}Y + a_{2,0}X^2 + a_{1,1}XY + a_{0,2}Y^2 + \ldots$, be an associative formal power series. Then $a_{1,0}, a_{0,1} \in \{0,1\}$.*

Proof. To prove this, we compute the first terms of the associative equation (AE). We obtain

$$a_{1,0}(a_{1,0}X + a_{0,1}Y + \ldots) + a_{0,1}Z + \ldots = a_{1,0}X + a_{0,1}(a_{1,0}Y + a_{0,1}Z + \ldots) + \ldots.$$

Comparing the coefficients of X and Z in this equation results in $a_{1,0}^2 = a_{1,0}$ and $a_{0,1}^2 = a_{0,1}$. Hence the claim follows. $\square$

In [5] the case where $F(X,Y) = X + Y + \ldots$, F associative, is described. In this article we focus on the situation where the first coefficients $a_{1,0}$ and $a_{0,1}$ are not both one. In the next proposition we characterize those associative formal power series where only one of the starting coefficients $a_{1,0}$ or $a_{0,1}$ is different from zero.

Proposition 3.3. *Let $F \in \mathbb{K}[\![X,Y]\!]$, $F(X,Y) = \sum_{p,q\geq 0} a_{p,q}X^pY^q$, $a_{0,0} = a_{1,0} = 0$, $a_{0,1} = 1$, be an associative formal power series. Then $F(X,Y) = Y$.*

Proof. We claim that $F(X,Y) = Y$. Conversely, assuming that

$$F(X,Y) = Y + \sum_{k\geq m} F_k(X,Y)$$

with $F_m(X,Y) \neq 0$, $m \geq 2$, $F_k(X,Y)$ is the homogeneous part of F of degree k, we prove that $F_m(X,Y) = 0$, what is a contradiction. The left hand side of (AE) is computed as

$$\begin{aligned}
F(F(X,Y),Z) &= Z + \sum_{k\geq m} F_k(F(X,Y),Z) \\
&= Z + F_m(Y + \sum_{k\geq m} F_k(X,Y), Z) + \sum_{k>m} F_k(F(X,Y),Z) \\
&= Z + \sum_{p+q=m} a_{p,q}\Big(Y + \sum_{k\geq m} F_k(X,Y)\Big)^p Z^q + \sum_{k>m} F_k(F(X,Y),Z) \\
&= Z + F_m(Y,Z) + \Phi(X,Y,Z),
\end{aligned}$$

where $\operatorname{ord} \Phi(X,Y,Z) > m$. The right hand side of (AE) is

$$\begin{aligned} F(X,F(Y,Z)) &= F(Y,Z) + \sum_{k\geq m} F_k(X,F(Y,Z)) \\ &= Z + \sum_{k\geq m} F_k(Y,Z) + F_m(X, Z + \sum_{k\geq m} F_k(Y,Z)) \\ &\quad + \sum_{k>m} F_k(X,F(Y,Z)) \\ &= Z + F_m(Y,Z) + F_m(X,Z) + \Psi(X,Y,Z), \end{aligned}$$

where $\operatorname{ord} \Psi(X,Y,Z) > m$. Therefore, we obtain $F_m(X,Z) = 0$. □

Remark 3.4. If $F \in \mathbb{K}[\![X,Y]\!]$ is associative, then $F^\star$ defined by $F^\star(X,Y) := F(Y,X)$ is also associative, since

$$F^\star(F^\star(X,Y),Z) = F(Z,F(Y,X)) = F(F(Z,Y),X) = F^\star(X,F^\star(Y,Z)).$$

Thus by Proposition 3.3 any associative $F \in \mathbb{K}[\![X,Y]\!]$ of the form $F(X,Y) = X + a_{1,1}XY + a_{2,0}X^2 + a_{0,2}Y^2 + \ldots$ satisfies $F(X,Y) = X$.

Therefore, the formal power series we are interested in have order 2 or greater. Now we prove that associative formal power series with $a_{1,1} = 0$ are equal to zero.

Theorem 3.5. *Let* $F \in \mathbb{K}[\![X,Y]\!]$*,* $F(X,Y) = a_{2,0}X^2 + a_{0,2}Y^2 + a_{3,0}X^3 + a_{2,1}X^2Y + a_{1,2}XY^2 + a_{0,3}Y^3 + F_4(X,Y) + \ldots$ *be an associative formal power series. Then* $F = 0$.

Proof. We want to prove this theorem in several steps.
Step 1. First we show that $a_{n,0} = a_{0,n} = 0$ for every $n \in \mathbb{N}$.
For $n = 1$ there are no coefficients and hence the claim is fulfilled. For $n = 2$ we write

$$F(X,Y) = a_{2,0}X^2 + a_{0,2}Y^2 + \sum_{l\geq 3} F_l(X,Y).$$

The left hand side of the associativity equation (AE) results in

$$\begin{aligned} F(F(X,Y),Z) = a_{2,0}(a_{2,0}X^2 + a_{0,2}Y^2 + \sum_{l\geq 3} F_l(X,Y)) \\ + a_{0,2}Z^2 + \sum_{l\geq 3} F_l(F(X,Y),Z). \end{aligned}$$

The right hand side computes as

$$\begin{aligned} F(X,F(Y,Z)) = a_{2,0}X^2 + a_{0,2}(a_{2,0}Y^2 + a_{0,2}Z^2 + \sum_{l\geq 3} F_l(Y,Z)) \\ + \sum_{l\geq 3} F_l(X,F(Y,Z)). \end{aligned}$$

Hence we obtain $a_{0,2}Z^2 = 0$ and $a_{2,0}X^2 = 0$, therefore, $a_{0,2} = a_{2,0} = 0$. Let $n > 2$ and let the induction hypothesis be fulfilled for $1 \leq j < n$. We write

$$F(X,Y) = \sum_{l=3}^{n-1} \left(\sum_{i=1}^{l-1} a_{i,l-i} X^i Y^{l-i} \right) + \sum_{i=0}^{n} a_{i,n-i} X^i Y^{n-i} + \sum_{l\geq n} F_l(X,Y).$$

Again we compute both sides of (AE) and we obtain

$$\begin{aligned} F(F(X,Y),Z) &= \sum_{l=3}^{n-1}\left(\sum_{i=1}^{l-1} a_{i,l-i}F(X,Y)^i Y^{l-i}\right) + a_{0,n}Z^n + a_{n,0}F(X,Y)^n \\ &\quad + \sum_{l\geq n} F_l(F(X,Y),Z) \\ &= a_{0,n}Z^n + \dots \end{aligned}$$

and

$$\begin{aligned} F(X,F(Y,Z)) &= \sum_{l=3}^{n-1}\left(\sum_{i=1}^{l-1} a_{i,l-i}X^i F(Y,Z)^{l-i}\right) + a_{0,n}F(Y,Z)^n + a_{n,0}X^n \\ &\quad + \sum_{l\geq n} F_l(X,F(Y,Z)) \\ &= a_{n,0}X^n + \dots . \end{aligned}$$

Here X^n does not appear in $F(F(X,Y),Z)$ as well as Z^n does not appear in $F(X,F(Y,Z))$. Hence we get $a_{0,n} = a_{n,0} = 0$.

Step 2. In order to show that $a_{1,n} = a_{n,1} = 0$ for all $n \in \mathbb{N}$ we write

$$F(X,Y) = X\varphi_1(Y) + X^2\tilde{F}(X,Y)$$

where $\varphi_1(Y) = \sum_{n\geq 2} a_{1,n}Y^n$. We have to show that $\varphi_1 = 0$. Therefore

$$\begin{aligned} F(F(X,Y),Z) &= F(X,Y)\varphi_1(Z) + F(X,Y)^2\tilde{F}(F(X,Y),Z) \\ &= (X\varphi_1(Y) + X^2\tilde{F}(X,Y))\varphi_1(Z) + F(X,Y)^2\tilde{F}(F(X,Y),Z) \\ &= X\varphi_1(Y)\varphi_1(Z) + X^2\Phi(X,Y,Z), \end{aligned}$$

as well as

$$\begin{aligned} F(X,F(Y,Z)) &= X\varphi_1(F(Y,Z)) + X^2\tilde{F}(X,F(Y,Z)) \\ &= X\varphi_1(Y\varphi_1(Z) + Y^2\tilde{F}(Y,Z)) + X^2\tilde{F}(X,F(Y,Z)). \end{aligned}$$

Hence we have

$$\varphi_1(Y)\varphi_1(Z) = \varphi_1(Y\varphi_1(Z) + Y^2\tilde{F}(Y,Z)).$$

Assume that $\varphi_1 \neq 0$, then let $n_1 = \operatorname{ord}\varphi_1 \geq 2$. We have

$$\varphi_1(Y)\varphi_1(Z) = \sum_{n\geq n_1} a_{1,n}Y^n \sum_{n\geq n_1} a_{1,n}Z^n = a_{1,n_1}^2 Y^{n_1}Z^{n_1} + \dots$$

and

$$\begin{aligned} \varphi_1(Y\varphi_1(Z) + Y^2\tilde{F}(Y,Z)) &= a_{1,n_1}(Y\varphi_1(Z) + Y^2\tilde{F}(Y,Z))^{n_1} \\ &\quad + \sum_{n>n_1} a_{1,n}(Y\varphi_1(Z) + Y^2\tilde{F}(Y,Z))^n \\ &= a_{1,n_1}(Y^{n_1}\varphi_1(Z)^{n_1} + \dots) \\ &\quad + \sum_{n>n_1} a_{1,n}(Y\varphi_1(Z) + Y^2\tilde{F}(Y,Z))^n \\ &= a_{1,n_1}(Y^{n_1}\varphi_1(Z)^{n_1} + \dots) + \dots \end{aligned}$$

with

$$\varphi_1(Z)^{n_1} = (a_{1,n_1}Z^{n_1} + \dots)^{n_1} = a_{1,n_1}^{n_1} Z^{{n_1}^2} + \dots .$$

It follows that $n_1^2 > n_1$ since $n_1 \geq 2$, therefore there is no term of the form $Y^{n_1} Z^{n_1}$ in $\varphi_1(Y\varphi_1(Z) + Y^2 \tilde{F}(Y, Z))$ and hence $a_{1,n_1} = 0$ which is a contradiction. So we obtain that $a_{1,n} = 0$ for all $n \in \mathbb{N}$. The same argument where X and Y are interchanged shows that $a_{n,1} = 0$ for all $n \in \mathbb{N}$.

Step 3. In this step we are using an idea of [6]. Let $i = \mathrm{ord}_X F$ when F is considered as a formal power series in $\mathbb{K}[\![Y]\!][\![X]\!]$. Assume that $F(0, Y) \neq 0$, i.e. $\mathrm{ord}_X F \neq \infty$. Then $\mathrm{ord}_X F(F(X, Y), Z) = i^2$ and $\mathrm{ord}_X F(X, F(Y, Z)) = i$. Thus $i^2 = i$ and $i \in \{0, 1\}$ if F is associative. Therefore, all possible cases have been considered. $\square$

From now on we are interested in associative formal power series F

$$F(X, Y) = \sum_{\substack{i+j=n \\ n \geq 2}} a_{i,j} X^i Y^j,$$

where $a_{1,1} \neq 0$. We have the following lemma.

Lemma 3.6. *Let $F \in \mathbb{K}[\![X, Y]\!]$, $F(X, Y) = a_{2,0}X^2 + a_{1,1}XY + a_{0,2}Y^2 + \ldots$, $a_{1,1} \neq 0$, be an associative formal power series. Then $a_{k,0} = a_{0,k} = 0$ for all $k \in \mathbb{N}$, $k \geq 2$.*

Proof. We want to compare the coefficients of X^2 and Z^2 of the left- and right hand side of the associativity equation (AE). For the left hand side we obtain $0 \cdot X^2$, whereas the right hand side is given by $a_{2,0}X^2$, hence $a_{2,0} = 0$. Analogously we obtain, after comparing the coefficients of Z^2 that $a_{0,2} = 0$. Let us now assume that $a_{k,0} = a_{0,k} = 0$ for $1 \leq k \leq n$. By (AE) we get $0 \cdot X^{n+1} = a_{n+1,0}X^{n+1}$ and therefore $a_{n+1,0} = 0$. Analogously $a_{0,n+1} = 0$. $\square$

Remark 3.7. From the previous lemma it follows that an associative formal power series $F(X, Y) = a_{1,1}XY + \ldots$, $a_{1,1} \neq 0$ fulfills $F(X, 0) = F(0, Y) = 0$.

Remark 3.8. Assume that $F(X, Y) = \sum_{k,\ell \geq 1} a_{k,\ell} X^k Y^\ell$ is an associative formal series. Then

$$F(F(X, Y), Z) = \sum_{k,\ell \geq 1} a_{k,\ell} F(X, Y)^k Z^\ell.$$

We expand $F(X, Y)^k$ as $\sum_{i,j \geq 1} A_{i,j}^{(k)} X^i Y^j$. Given $i, j \geq 1$, we study for which k the coefficient $A_{i,j}^{(k)}$ is not necessarily equal to 0.

Since for all monomials $a_{r,s}X^r Y^s$ of F the minimum of r and s is greater or equal to 1, the exponent k must satisfy $1 \leq k \leq \min\{i, j\}$. For $1 \leq k \leq \min\{i, j\}$ the coefficient $A_{i,j}^{(k)}$ is computed as

$$\sum_{\substack{(i_1,\ldots,i_k) \\ (j_1,\ldots,j_k) \\ \sum_\nu i_\nu = i \\ \sum_\nu j_\nu = j}} \prod_{\nu=1}^{k} a_{i_\nu, j_\nu},$$

where i_ν and j_ν are positive integers for $1 \leq \nu \leq k$. In the sequel we will denote this sum as $\sum_{(i_\nu),(j_\nu)}^k$.

If $k > 1$, then for all sequences occurring in $\sum_{(i_\nu),(j_\nu)}^k$ we have $i_\nu < i$ and $j_\nu < j$ for all ν.

By comparison of coefficients of $X^iY^jZ^\ell$, the associativity equation yields for each triple (i,j,ℓ) of positive integers a polynomial relation of the form

$$\sum_{k=1}^{\min\{i,j\}} a_{k,\ell} \sum_{(i_\nu),(j_\nu)}^{k} \prod_{\nu=1}^{k} a_{i_\nu,j_\nu} = \sum_{k=1}^{\min\{j,\ell\}} a_{i,k} \sum_{(j_\nu),(\ell_\nu)}^{k} \prod_{\nu=1}^{k} a_{j_\nu,\ell_\nu}. \tag{3.1}$$

If $\ell = 1$, we obtain

$$\sum_{k=1}^{\min\{i,j\}} a_{k,1} \sum_{(i_\nu),(j_\nu)}^{k} \prod_{\nu=1}^{k} a_{i_\nu,j_\nu} = a_{i,1}a_{j,1},$$

from which we deduce if $a_{1,1} \neq 0$ that

$$a_{i,j} = \frac{1}{a_{1,1}}\left(a_{i,1}a_{j,1} - \sum_{k=2}^{\min\{i,j\}} a_{k,1} \sum_{(i_\nu),(j_\nu)}^{k} \prod_{\nu=1}^{k} a_{i_\nu,j_\nu}\right), \qquad i,j \geq 1. \tag{3.2}$$

Similarly, if $i = 1$ and $a_{1,1} \neq 0$ we obtain

$$a_{j,\ell} = \frac{1}{a_{1,1}}\left(a_{1,j}a_{1,\ell} - \sum_{k=2}^{\min\{j,\ell\}} a_{1,k} \sum_{(j_\nu),(\ell_\nu)}^{k} \prod_{\nu=1}^{k} a_{j_\nu,\ell_\nu}\right), \qquad j,\ell \geq 1. \tag{3.3}$$

Remark 3.9. Assume that $F(X,Y) = \sum_{i,j\geq 1} a_{i,j}X^iY^j$ is an associative formal series with $a_{1,1} \neq 0$. Then the coefficients $a_{i,j}$ with $\min\{i,j\} > 1$ are uniquely determined by the coefficients $a_{n,1}$ (respectively $a_{1,n}$) for $n \geq 1$.

Proof. Let $i = 2$ and $j \geq 2$, then $a_{i,j}$ can be expressed as in (3.2). All coefficients $a_{r,s}$ on the right hand side of (3.2) satisfy $\min\{r,s\} = 1$.

Assume that $i > 2$ and for all $r < i$ and $s \geq 1$ the coefficients $a_{r,s}$ are uniquely determined by the coefficients $a_{n,1}$, $n \geq 1$. Then in the representation (3.2) of $a_{i,j}$ all indices i_ν are smaller than i, whence all the coefficients on the right hand side of (3.2) are uniquely determined by the coefficients $a_{n,1}$, $n \geq 1$. Therefore also $a_{i,j}$ is uniquely determined by the coefficients $a_{n,1}$, $n \geq 1$. Using (3.3) we prove in a similar way that $a_{i,j}$ is uniquely determined by the coefficients $a_{1,n}$, $n \geq 1$. □

In the next proposition we show that an associative formal power series is commutative. An analougue result for the case where the formal power series is given by $F(X,Y) = X + Y + \ldots$ can be found in [5] page 38.

Proposition 3.10. *Let $F \in \mathbb{K}[\![X,Y]\!]$, $F(X,Y) = a_{1,1}XY + \ldots$, $a_{1,1} \neq 0$, be an associative formal power series. Then*

$$a_{i,j} = a_{j,i} \tag{3.4}$$

for all $i,j \in \mathbb{N}$ and hence

$$F(X,Y) = F(Y,X).$$

Proof. **Step 1.** We start by comparing the coefficients of XY^jZ for an arbitrary $j \in \mathbb{N}$. Therefore we obtain from (3.1) the relation

$$a_{1,1}a_{1,j} = a_{1,1}a_{j,1}$$

and hence $a_{1,j} = a_{j,1}$ because $a_{1,1} \neq 0$.
Step 2. Assume that $i \geq 2$, and $a_{\ell,k} = a_{k,\ell}$ for all $1 \leq \ell < i$ and for all $k \geq 1$. We prove by induction that $a_{i,j} = a_{j,i}$ for all $j \geq 1$.

For $j < i$ by assumption $a_{i,j} = a_{j,i}$. For $i = j$ there is nothing to prove. Assume that $j > i$. Then according to (3.2)

$$a_{i,j} = \frac{1}{a_{1,1}} \left(a_{i,1} a_{j,1} - \sum_{k=2}^{i} a_{k,1} \sum_{(i_\nu),(j_\nu)}^{k} \prod_{\nu=1}^{k} a_{i_\nu, j_\nu} \right).$$

On the right hand side k is greater than 1, whence, all i_ν occurring in $\sum_{(i_\nu),(j_\nu)}^{k}$ satisfy $i_\nu < i$. Thus $\min\{r, s\} < i$ for all coefficients $a_{r,s}$ occurring on the right hand side, and by the induction hypothesis they all satisfy $a_{r,s} = a_{s,r}$.

From (3.3) we obtain

$$a_{j,i} = \frac{1}{a_{1,1}} \left(a_{1,j} a_{1,i} - \sum_{k=2}^{i} a_{1,k} \sum_{(j_\nu),(i_\nu)}^{k} \prod_{\nu=1}^{k} a_{j_\nu, i_\nu} \right).$$

This expression can be obtained from the expression for $a_{i,j}$ above by interchanging the two indices of all the involved coefficients $a_{r,s}$. Therefore, $a_{i,j} = a_{j,i}$. □

Remark 3.11. In certain situations it is even possible to find simpler relations between the coefficients $a_{i,j}$ and $a_{j,i}$ of a formal series F satisfying the assumptions of Proposition 3.10. E.g. comparing the coefficients of $X^2Y^3Z^2$, we get $a_{1,2}a_{2,3} + 2a_{1,1}a_{1,2}a_{2,2} = 2a_{1,1}a_{2,1}a_{2,2} + a_{2,1}a_{3,2}$, and since $a_{1,2} = a_{2,1}$ we have $a_{1,2}a_{2,3} = a_{2,1}a_{3,2}$. If $a_{1,2} \neq 0$, then we obtain $a_{2,3} = a_{3,2}$.

4. The representation of associative formal power series of order greater than one

Remark 4.1. Assume that $F(X,Y) = \sum_{k,\ell\geq 1} a_{k,\ell} X^k Y^\ell$ is an associative formal series with $a_{1,1} \neq 0$. Then F is commutative so that the comparison of coefficients of $X^iY^jZ^\ell$ and $X^\ell Y^j Z^i$ yields the same polynomial relation. Thus it is enough to consider (3.1) only for $\ell \leq i$.

Lemma 4.2. *Let $f(X) = \sum_{n\geq 1} f_n X^n$, $f_1 \neq 0$, be an invertible formal power series. Then $F(X,Y) = f^{-1}(f(X)f(Y))$ is an associative formal power series. If we write $F(X,Y) = \sum_{k,\ell\geq 1} a_{k,\ell} X^k Y^\ell$, then $a_{1,n} = f_n = a_{n,1}$, $n \geq 1$.*

Proof. Using the method shown in the Example of section 2 we see that $F(X,Y) = f^{-1}(f(X)f(Y))$ is an associative formal power series of the form $\sum_{k,\ell\geq 1} a_{k,\ell} X^k Y^\ell$. Thus $f(F(X,Y)) = f(X)f(Y)$, which means that

$$\sum_{\nu\geq 1} f_\nu \Big(\sum_{k,\ell\geq 1} a_{k,\ell} X^k Y^\ell \Big)^\nu = \sum_{n,m\geq 1} f_n f_m X^n Y^m.$$

We compare the coefficients of X^1Y^n or X^nY^1. On the left hand side only the summand for $\nu = 1$ is possible, since otherwise the exponent 1 of X respectively Y cannot occur. Thus we get

$$f_1 a_{1,n} = f_1 f_n \text{ respectively } f_1 a_{n,1} = f_n f_1, \qquad n \geq 1,$$

which yields the assertion by cancelling f_1. □

Now we can prove the Representation Theorem.

Theorem 4.3 (Representation Theorem). (1) *The formal power series $F \in \mathbb{K}[\![X,Y]\!]$, $F(X,Y) = a_{1,1}XY + \dots$, $a_{1,1} \neq 0$, is associative if and only if there exists an invertible formal power series $f \in \mathbb{K}[\![X]\!]$ such that*

$$F(X,Y) = f^{-1}(f(X)f(Y)). \tag{4.1}$$

(2) *If $F(X,Y) = a_{1,1}XY + \dots$, $a_{1,1} \neq 0$ is associative, then f fulfilling (4.1) is uniquely determined.*

Proof. Let $F(X,Y) = \sum_{k,\ell\geq 1} a_{k,\ell}X^kY^\ell$ be an associative formal power series and let $f(X) = \sum_{n\geq 1} a_{n,1}X^n$. Then f is invertible and by Lemma 4.2 the series $\Phi(X,Y) = f^{-1}(f(X)f(Y)) = \sum_{k,\ell\geq 1} b_{k,\ell}X^kY^\ell$ is associative with $b_{n,1} = f_n = a_{n,1}$, $n \geq 1$. By Remark 3.9 the coefficients $a_{i,j}$, $\min\{i,j\} > 1$, of an associative series are uniquely determined by $a_{n,1}$, $n \geq 1$, whence $\Phi = F$. □

In this proof we did not use the commutativity of F. Actually it follows immediately from the Representation Theorem since

$$F(X,Y) = f^{-1}(f(X)f(Y)) = f^{-1}(f(Y)f(X)) = F(Y,X).$$

Another proof of Theorem 4.3 is the following. 1. Let $f \in \mathbb{K}[\![X]\!]$ be invertible. According to Lemma 4.2 the function F represented by $F(X,Y) = f^{-1}(f(X)f(Y))$ is associative.
Let F be associative and let $f(X) = \sum_{n\geq 1} a_{n,1}X^n$, then f is an invertible power series. We want to prove that $f(F(X,Y)) = f(X)f(Y)$. We have

$$\begin{aligned} F(F(X,Y),Z) &= \sum_{r,s\geq 1} a_{r,s}F(X,Y)^rZ^s \\ &= \underbrace{\sum_{r\geq 1} a_{r,1}F(X,Y)^rZ}_{=\Phi(X,Y,Z)} + \sum_{r\geq 1,\ s\geq 2} a_{r,s}F(X,Y)^rZ^s. \end{aligned}$$

The series $\Phi(X,Y,Z)$ consists of all monomials in $F(F(X,Y),Z)$ so that the exponent of Z is 1. Next we compute the subseries $\Psi(X,Y,Z)$ of $F(X,F(Y,Z))$ consisting of all monomials of the form X^iY^jZ for some $i,j \geq 1$. We get

$$\Psi(X,Y,Z) = \sum_{i\geq 1} a_{i,1}X^i \sum_{j\geq 1} a_{j,1}Y^jZ.$$

Since F is associative $\Phi(X,Y,Z) = \Psi(X,Y,Z)$. It is obvious that $\Phi(X,Y,Z) = f(F(X,Y))Z$ and $\Psi(X,Y,Z) = f(X)f(Y)Z$ which finishes the proof of the first assertion.
2. The second assertion follows from Remark 3.9. □

Remark 4.4. (1) Let $(c_n)_{n\geq 1}$ be a sequence in $\mathbb{K}$ with $c_1 \neq 0$. Then there exists exactly one associative formal series $F(X,Y) = \sum_{i,j\geq 1} a_{i,j}X^iY^j$ where $a_{1,n} = c_n = a_{n,1}$ for all $n \geq 1$. If $\min\{i,j\} > 1$, then the coefficients $a_{i,j}$ are given by (3.2).

(2) Let $F(X,Y) = \sum_{i,j\geq 1} a_{i,j}X^iY^j$ be a commutative formal series. The following assertions are equivalent:
   (a) F is associative.
   (b) For all triples (i,j,ℓ) of positive integers with $i,j \geq 2$ and $\ell \leq i$ the condition (3.1) is satisfied.

(c) For all triples $(i, j, 1)$ of positive integers with $i, j \geq 2$ the condition (3.1) is satisfied.

Remark 4.5. Let $F \in \mathbb{K}[\![X, Y]\!]$, $F(X, Y) = a_{1,1}XY + \ldots$, $a_{1,1} \neq 0$. Then F is associative if and only if for every $\pi \in \mathfrak{S}_3$

$$F(F(X_1, X_2), X_3) = F(X_{\pi(1)}, F(X_{\pi(2)}, X_{\pi(3)})). \tag{4.2}$$

Proof. Let F be associative. Then F is commutative and together with associativity F fulfills (4.2) for all permutations $\pi \in \mathfrak{S}_3$.
Let F fulfill (4.2) for all $\pi \in \mathfrak{S}_3$. Then using $\pi = id$ the formal power series F is associative. □

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ one may ask for the characterization of those associative series $F(X, Y) = cXY + \ldots$ which are convergent.

Theorem 4.6. *Let $F(X, Y) = f^{-1}(f(X)f(Y))$. Then F is convergent if and only if f is convergent.*

Proof. Let $f(X) = cX + \ldots$, $c \neq 0$, be convergent. Then f^{-1} is also convergent and thus also $f^{-1}(f(X)f(Y))$.
If, on the other hand, $F(X, Y) = a_{1,1}XY + \sum_{p+q\geq 3} a_{p,q}X^pY^q$ is convergent, we know that $f(X) = a_{1,1}X + \sum_{p\geq 2} a_{p,1}X^p$. But $\sum_{p\geq 2} a_{p,1}X^p$ is a convergent subseries of F, thus $\sum_{p\geq 2} a_{p,1}X^p y_0$ converges for some fixed y_0 and all $|X| < \delta$ for some $\delta > 0$. Accordingly

$$f(X) = \frac{1}{Y_0} \sum_{p\geq 2} a_{p,1}X^p y_0$$

is convergent. □

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Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität, Graz, Austria
E-mail address: harald.fripertinger@uni-graz.at
E-mail address: ludwig.reich@uni-graz.at
E-mail address: jens.schwaiger@uni-graz.at

Mathematics Research Unit, Université du Luxembourg, Ville de Luxembourg, Grand Duchy of Luxembourg
E-mail address: joerg.tomaschek@uni.lu