ON COVARIANT EMBEDDINGS OF A LINEAR FUNCTIONAL EQUATION WITH RESPECT TO AN ANALYTIC ITERATION GROUP

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Received December 15, 2000; Revised October 11, 2001

Let $a(x)$, $b(x)$, $p(x)$ be formal power series in the indeterminate x over $\mathbb{C}$ (i.e., elements of the ring $\mathbb{C}[\![x]\!]$ of such series) such that $\operatorname{ord} a(x) = 0$, $\operatorname{ord} p(x) = 1$ and $p(x)$ is embeddable into an analytic iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ in $\mathbb{C}[\![x]\!]$. By a covariant embedding of the linear functional equation

$$\varphi(p(x)) = a(x)\varphi(x) + b(x), \tag{L}$$

(for the unknown series $\varphi(x) \in \mathbb{C}[\![x]\!]$) with respect to $(\pi(s,x))_{s\in\mathbb{C}}$ we understand families $(\alpha(s,x))_{s\in\mathbb{C}}$ and $(\beta(s,x))_{s\in\mathbb{C}}$ with entire coefficient functions in s, such that the system of functional equations and boundary conditions

$$\begin{aligned}
&\varphi(\pi(s,x)) = \alpha(s,x)\varphi(x) + \beta(s,x) && (Ls)\\
&\alpha(t+s,x) = \alpha(s,x)\alpha(t,\pi(s,x)) && (Co1)\\
&\beta(t+s,x) = \beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x)) && (Co2)\\
&\alpha(0,x) = 1 \qquad \beta(0,x) = 0 && (B1)\\
&\alpha(1,x) = a(x) \qquad \beta(1,x) = b(x) && (B2)
\end{aligned}$$

holds for all solutions $\varphi(x)$ of (L) and for all $s,t \in \mathbb{C}$. In this paper we solve the system $((Co1),(Co2))$ (of so called cocycle equations) completely, describe when and how the boundary conditions $(B1)$ and $(B2)$ can be satisfied, and present a large class of equations (L) together with iteration groups $(\pi(s,x))_{s\in\mathbb{C}}$ for which there exist covariant embeddings of (L) with respect to $(\pi(s,x))_{s\in\mathbb{C}}$.

1. Introduction

Let $\mathbb{C}[\![x]\!]$ be the ring of formal power series in the indeterminate x with complex coefficients. Consider the linear functional equation

$$\varphi(p(x)) = a(x)\varphi(x) + b(x), \tag{L}$$

where $p(x), a(x), b(x) \in \mathbb{C}[\![x]\!]$ are given formal power series and $\varphi(x) \in \mathbb{C}[\![x]\!]$ should be determined by the functional equation. We always assume that

$$p(x) = \rho x + c_2x^2 + c_3x^3 + \cdots = \rho x + \sum_{n\geq 2} c_nx^n$$

with multiplier $\rho \neq 0$, and

$$a(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n\geq 0} a_n x^n$$

with $a_0 \neq 0$. For a foundation of the basic calculations with formal power series we refer the reader to [Henrici, 1974] and to [Cartan, 1963] or [Cartan, 1966]. If $\psi(x) \in \mathbb{C}[\![x]\!]$ is of the form $\psi(x) = \sum_{n\geq k} d_n x^n$ with $d_k \neq 0$, then k is the order of ψ, which will be indicated as $\operatorname{ord}\psi(x) = k$. Hence, $\operatorname{ord} p(x) = 1$ and $\operatorname{ord} a(x) = 0$. The set of all formal power series of order 1 is indicated by Γ, which is a group with respect to the substitution in $\mathbb{C}[\![x]\!]$. In addition to this, let Γ_0 indicate the set of all formal power series of the form $x+d_2x^2+\cdots \in \Gamma$.

Furthermore, the notion of *congruence modulo* x^r will be useful. We write $\varphi \equiv \psi \mod x^r$ for formal power series $\varphi(x), \psi(x) \in \mathbb{C}[\![x]\!]$ if x^r is a divisor of the difference $\varphi(x) - \psi(x)$. In other words $\varphi(x) - \psi(x) = 0$, or its order is greater than or equal to r.

A formal power series $\varphi(x)$ can be substituted into the series $\psi(x) = \sum_{n\geq 0} d_n x^n \in \mathbb{C}[\![x]\!]$, i.e., the series $\psi(\varphi(x)) = \sum_{n\geq 0} d_n[\varphi(x)]^n$ can be computed, if and only if $\operatorname{ord}\varphi(x) \geq 1$.

The exponential series is given as

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!},$$

and the formal logarithm is the series defined by

$$\ln(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{n}.$$

A family $\pi := (\pi(s,\cdot))_{s\in\mathbb{C}}$ in Γ is called an iteration group, (see e.g. [Scheinberg, 1970]), or a one-parameter group in Γ, if the translation equation

$$\pi(t+s,x) = \pi(t,\pi(s,x)) \tag{T}$$

holds for all $t, s \in \mathbb{C}$. Hence, $\pi(0,x) = x$ and $\pi(-1,x) = \pi^{-1}(1,x)$, the inverse of $\pi(1,x)$ with respect to substitution. If we express $\pi(s,x)$ in the form $\sum_{n\geq 1} \pi_n(s)x^n$, then π is called an analytic iteration group if all the coefficient functions $\pi_n(s)$ are entire functions.

The formal power series $p(x)$ is called (analytically) iterable, or embeddable, if there exists an (analytic) iteration group π in Γ such that $\pi(1,x) = p(x)$.

There exist only three different types of analytic iteration groups in Γ.

1. $\pi(s,x) = x$ for all $s \in \mathbb{C}$.
2. $\pi(s,x) = S^{-1}(e^{\lambda s}S(x))$ for all $s \in \mathbb{C}$, where $\lambda \in \mathbb{C}\setminus\{0\}$ and $S(x) = x+s_2x^2+\ldots$ belongs to Γ_0. These iteration groups are called iteration groups of the first type. Each iteration group of this type is simultaneously conjugate to the iteration group $(e^{\lambda s}x)_{s\in\mathbb{C}}$.
3. $\pi(s,x) = x + c_k s x^k + P_{k+1}^{(k)}(s)x^{k+1} + \ldots$ for all $s \in \mathbb{C}$, where $c_k \neq 0$, $k \geq 2$ and $P_r^{(k)}(s)$ are polynomials in s for $r > k$. These iteration groups are called iteration groups of the second type.

The formal power series $p(x) = x$ can trivially be embedded into an analytic iteration group. Assume $p(x) \neq x$ and $p(x) = \rho x + c_2x^2 + \ldots$, where $\rho \neq 0$. If ρ is not a complex root of 1, then let λ be a logarithm $\ln\rho$. In this case there exists exactly one analytic embedding $(\pi(s,x))_{s\in\mathbb{C}}$ of $p(x)$ such that $\pi(s,x) = e^{\lambda s}x + \ldots$. Let $S(x) = x + s_2x^2 + \ldots$ be the unique formal power series such that $S(\pi(1,S^{-1}(x))) = \rho x$, then $\pi(s,x) = S^{-1}(e^{\lambda s}S(x))$ for all $s \in \mathbb{C}$.

If ρ is a complex root of 1 and $\rho \neq 1$, then the series $p(x)$ need not have an analytic embedding. But if such a $p(x)$ has an analytic embedding, then it is of the first type. In this situation, however, the embedding need not be unique.

If $p(x) = x + c_kx^k + \ldots$ with $c_k \neq 0$ and $k \geq 2$, then there exists exactly one analytic embedding of $p(x)$ in an iteration group of the second type. (These facts about analytic iteration groups in $\mathbb{C}[\![x]\!]$ can also be deduced as special cases of the results in [Reich & Schwaiger, 1977].)

Assume that $a(x)$, $b(x)$, and $p(x)$ are formal power series given as above. For $n \in \mathbb{Z}$ we form the natural iterates of $p(x)$ defined by

$$p^n(x) := \begin{cases} x, & n = 0 \\ p(p^{n-1}(x)), & n > 0 \\ (p^{-1})^{-n}(x), & n < 0. \end{cases}$$

Furthermore, for $n \geq 0$ we define

$$\alpha(n,x) := \prod_{r=0}^{n-1} a(p^r(x))$$

and

$$\beta(n,x) := \alpha(n,x)\sum_{r=0}^{n-1} \frac{b(p^r(x))}{\prod_{j=0}^{r} a(p^j(x))}.$$

Then the conditions

$$\alpha(0,x) = 1 \qquad \beta(0,x) = 0 \qquad (B1)$$
$$\alpha(1,x) = a(x) \qquad \beta(1,x) = b(x) \qquad (B2)$$

are clearly satisfied.

Lemma 1.1. *The two families $(\alpha(n,x))_{n\in\mathbb{N}_0}$ and $(\beta(n,x))_{n\in\mathbb{N}_0}$ satisfy*

$$\alpha(n+m,x) = \alpha(m,x)\alpha(n,p^m(x)) \qquad (C1)$$

$$\beta(n+m,x) = \beta(m,x)\alpha(n,p^m(x)) + \beta(n,p^m(x)) \qquad (C2)$$

for all $n, m \geq 0$.

We leave the proof by induction to the reader. If for $n < 0$ we define

$$\alpha(n,x) := \frac{1}{\alpha(-n,p^n(x))} = \frac{1}{\prod_{r=0}^{-n-1} a(p^{r+n}(x))} = \frac{1}{\prod_{r=n}^{-1} a(p^r(x))}$$

and

$$\beta(n,x) := \frac{-\beta(-n,p^n(x))}{\alpha(-n,p^n(x))} = -\alpha(n,x)\beta(-n,p^n(x)),$$

then Lemma 1.1 holds for all $n, m \in \mathbb{Z}$.

Lemma 1.2. *If $\varphi(x)$ satisfies (L), then it also satisfies*

$$\varphi(p^n(x)) = \alpha(n,x)\varphi(x) + \beta(n,x) \qquad (Ln)$$

for all $n \in \mathbb{Z}$.

Proof. Obvious from Lemma 1.1 and its generalization for all $n \in \mathbb{Z}$. ■

Motivated by (Ln), $(C1)$, and $(C2)$ for natural iterates, L. Reich introduced in [Reich, 1998] the following notion.

The linear functional equation (L) has a covariant embedding with respect to the analytic iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ of $p(x)$, if there exist families $(\alpha(s,x))_{s\in\mathbb{C}}$ and $(\beta(s,x))_{s\in\mathbb{C}}$ of formal power series with entire coefficient functions $\alpha_n(s)$ and $\beta_n(s)$ for all $n \geq 0$ such that

$$\varphi(\pi(s,x)) = \alpha(s,x)\varphi(x) + \beta(s,x) \qquad (Ls)$$

holds for all $s \in \mathbb{C}$ and for all solutions $\varphi(x)$ of (L) in $\mathbb{C}[\![x]\!]$. Moreover, it is assumed that α and β satisfy both the boundary conditions $(B1)$ and $(B2)$ and the cocycle equations

$$\alpha(t+s,x) = \alpha(s,x)\alpha(t,\pi(s,x)) \qquad (Co1)$$

$$\beta(t+s,x) = \beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x)) \qquad (Co2)$$

for all $s, t \in \mathbb{C}$.

Such embeddings were studied in a much more general setting by Z. Moszner in [Moszner, 1999] and for real-valued functions by G. Guzik in [Guzik, 1999], [Guzik, 2000], and [Guzik, 2001]. For the theory of linear functional equations we refer the reader to [Kuczma *et al.*, 1990] and to [Kuczma, 1968]. In the present paper we deal with the problem of covariant embeddings in the ring of formal power series $\mathbb{C}[\![x]\!]$. In Section 2 we solve the underlying functional equations $(Co1)$ and $(Co2)$ completely. Then in Section 3 we show how to adjust these solutions to given boundary conditions. And finally, in the last section we describe how to embed the linear functional equation (L) in the generic cases.

When dealing with analytic iteration groups $(\pi(s,x))_{s\in\mathbb{C}}$ of the first type, it is enough to consider $\pi(s,x) = e^{\lambda s}x$. This is explained in the next

Theorem 1.3. *Let $\pi(s,x) = S^{-1}(e^{\lambda s}S(x))$ for $\lambda \neq 0$ and $S(x) \in \Gamma_0$ be an embedding of $p(x)$.*

1. *The formal power series $\varphi(x)$ is a solution of (L) if and only if $\tilde{\varphi} := \varphi \circ S^{-1}$ satisfies*
$$\tilde{\varphi}(e^{\lambda}y) = \tilde{a}(y)\tilde{\varphi}(y) + \tilde{b}(y) \qquad (\tilde{L})$$
where $\tilde{a} := a \circ S^{-1}$ and $\tilde{b} := b \circ S^{-1}$.

2. *The system (Ls), $(Co1)$, $(Co2)$, $(B1)$, and $(B2)$ is equivalent to the system*
$$\tilde{\varphi}(e^{\lambda s}y) = \tilde{\alpha}(s,y)\tilde{\varphi}(y) + \tilde{\beta}(s,y) \qquad (\tilde{L}s)$$
$$\tilde{\alpha}(t+s,y) = \tilde{\alpha}(s,y)\tilde{\alpha}(t,e^{\lambda s}y) \qquad (\tilde{C}o1)$$
$$\tilde{\beta}(t+s,y) = \tilde{\beta}(s,y)\tilde{\alpha}(t,e^{\lambda s}y) + \tilde{\beta}(t,e^{\lambda s}y) \qquad (\tilde{C}o2)$$
$$\tilde{\alpha}(0,y) = 1 \qquad \tilde{\beta}(0,y) = 0 \qquad (\tilde{B}1)$$
$$\tilde{\alpha}(1,y) = \tilde{a}(y) \qquad \tilde{\beta}(1,y) = \tilde{b}(y), \qquad (\tilde{B}2)$$
where $\tilde{\alpha}(s,y) = \alpha(s,S^{-1}(y))$ and $\tilde{\beta}(s,y) = \beta(s,S^{-1}(y))$.

Proof. The formal series $\varphi(x)$ satisfies (L) if and only if

$$\varphi(S^{-1}(e^{\lambda}S(x))) = a(x)\varphi(x) + b(x) \iff$$
$$(\varphi\circ S^{-1})(e^{\lambda}S(x)) =$$
$$(a\circ S^{-1})(S(x))(\varphi\circ S^{-1})(S(x)) + (b\circ S^{-1})(S(x)),$$

which is equal to $(\tilde{L})$ after replacing $S(x)$ by y.

Assuming that (Ls) holds we deduce

$$\varphi(S^{-1}(e^{\lambda s}S(x))) = \alpha(s,x)\varphi(x) + \beta(s,x) \Longrightarrow$$
$$(\varphi\circ S^{-1})(e^{\lambda s}S(x)) =$$
$$\alpha(s, S^{-1}(S(x)))(\varphi\circ S^{-1})(S(x)) + \beta(s, S^{-1}(S(x))),$$

which is equal to $(\tilde{L}s)$ after replacing $S(x)$ by y. The boundary conditions $(\tilde{B}1)$ and $(\tilde{B}2)$ are naturally equivalent to $(B1)$ and $(B2)$. Finally,

$$\tilde{\alpha}(t+s,y) = \alpha(t+s, S^{-1}(y)) =$$
$$\alpha(s, S^{-1}(y))\alpha(t, \pi(s, S^{-1}(y))) =$$
$$\tilde{\alpha}(s,y)\alpha(t, S^{-1}(e^{\lambda s}S(S^{-1}(y)))) =$$
$$\tilde{\alpha}(s,y)\alpha(t, S^{-1}(e^{\lambda s}y)) = \tilde{\alpha}(s,y)\tilde{\alpha}(t, e^{\lambda s}y),$$

hence $(\tilde{C}o1)$ is satisfied. Using similar methods, it is possible to show that $(\tilde{C}o2)$ is a consequence of $(Co2)$.

Since $(\pi(s,x))_{s\in\mathbb{C}}$ and $(e^{\lambda s}x)_{s\in\mathbb{C}}$ are conjugate via the formal power series $S(x)$, it is clear how to prove the implications into the converse direction. ∎

2. Solutions of the cocycle equations

Lemma 2.1. *Let $E(x) := e_0 + e_1x + \cdots \in \mathbb{C}[\![x]\!]$, $e_0 \neq 0$, and let $\mu\in\mathbb{C}$. Then*

$$\alpha(s,x) = e^{\mu s}\frac{E(\pi(s,x))}{E(x)}$$

is a solution of $(Co1)$.

Proof. Since π satisfies the translation equation (T), it is clear that

$$\alpha(t+s,x) = e^{\mu(t+s)}\frac{E(\pi(t+s,x))}{E(x)} =$$
$$e^{\mu t}e^{\mu s}\frac{E(\pi(t,\pi(s,x)))}{E(x)} =$$
$$e^{\mu t}\frac{E(\pi(t,\pi(s,x)))}{E(\pi(s,x))}e^{\mu s}\frac{E(\pi(s,x))}{E(x)} =$$
$$\alpha(t,\pi(s,x))\alpha(s,x).$$

∎

Lemma 2.1 also holds, when $e^{\mu s}$ is replaced by a generalized exponential function.

If we express $\alpha(s,x)$ with coefficient functions in the form

$$\alpha(s,x) = \sum_{n=0}^{\infty}\alpha_n(s)x^n,$$

then it follows from the cocycle equation $(Co1)$ that $\alpha_0(t+s) = \alpha_0(s)\alpha_0(t)$. Hence, taking into account the regularity conditions for the coefficients of α and the fact that $\alpha_0(s)\neq 0$, it is clear that $\alpha_0(s) = e^{\mu s}$ for some $\mu\in\mathbb{C}$. Consequently, $\alpha(s,x) = e^{\mu s}\hat{\alpha}(s,x)$ and $\hat{\alpha}(s,x) = 1 + \hat{\alpha}_1(s)x + \ldots$. Using the formal logarithm, there exists exactly one $\tilde{\alpha}(s,x)\in\mathbb{C}[\![x]\!]$ such that $\operatorname{ord}\tilde{\alpha}(s,x)\geq 1$ for all $s\in\mathbb{C}$, and $\hat{\alpha}(s,x) = \exp(\tilde{\alpha}(s,x))$. The coefficient functions of $\tilde{\alpha}$ are analytic if and only if the coefficient functions of $\hat{\alpha}$ are analytic, which is equivalent to the fact that the coefficient functions of α are analytic. Furthermore, $\hat{\alpha}$ is a solution of $(Co1)$ if and only if $\tilde{\alpha}$ satisfies

$$\tilde{\alpha}(t+s,x) = \tilde{\alpha}(s,x) + \tilde{\alpha}(t,\pi(s,x)) \qquad (Co1')$$

for all $s,t\in\mathbb{C}$.

Theorem 2.2. *The family $\tilde{\alpha}$ of formal power series is a solution of $(Co1')$, and $\tilde{\alpha}(0,x) = 0$, if and only if there exists a formal power series $K(y)\in\mathbb{C}[\![y]\!]$, $\operatorname{ord}K(y)\geq 1$ such that*

$$\tilde{\alpha}(s,x) = \int_0^s K(\pi(\sigma,x))d\sigma,$$

where integration is taken coefficientwise.

Proof. First assume that $\tilde{\alpha}$ is a solution of $(Co1')$ with $\tilde{\alpha}(0,x) = 0$. Coefficientwise differentiation of $(Co1')$ with respect to the variable t and the chain rule for this differentiation yields

$$\tilde{\alpha}'(t+s,x) = \tilde{\alpha}'(t,\pi(s,x)).$$

For $t = 0$ we get $\tilde{\alpha}'(s,x) = \tilde{\alpha}'(0,\pi(s,x))$. Since $\operatorname{ord}\tilde{\alpha}(s,x)\geq 1$, also $\operatorname{ord}\tilde{\alpha}'(s,x)\geq 1$. Putting

$K(y) := \tilde{\alpha}'(0,y)$, we obtain $\operatorname{ord} K(y) \geq 1$ and $\tilde{\alpha}'(s,x) = K(\pi(s,x))$. By coefficientwise integration, it follows that

$$\tilde{\alpha}(s,x) = \int_0^s K(\pi(\sigma,x))d\sigma.$$

Conversely, assume that $\tilde{\alpha}(s,x)$ is given as the integral above. We prove that $\tilde{\alpha}$ satisfies $(Co1')$:

$$\tilde{\alpha}(t+s,x) = \int_0^{t+s} K(\pi(\sigma,x))d\sigma =$$

$$\int_0^s K(\pi(\sigma,x))d\sigma + \int_s^{t+s} K(\pi(\sigma,x))d\sigma =$$

$$\tilde{\alpha}(s,x) + \int_0^t K(\pi(\tau+s,x))d\tau =$$

$$\tilde{\alpha}(s,x) + \int_0^t K(\pi(\tau,\pi(s,x)))d\tau =$$

$$\tilde{\alpha}(s,x) + \tilde{\alpha}(t,\pi(s,x)),$$

by applying (T). From the definition of $\tilde{\alpha}$ it is obvious that $\tilde{\alpha}(0,x) = 0$. ■

Corollary 2.3. *Using the notation from above, we have:*

1. *The family $(\hat{\alpha}(s,x))_{s\in\mathbb{C}}$ is a solution of $(Co1)$ if and only if there exists $K(y) \in \mathbb{C}[\![y]\!]$, $\operatorname{ord} K(y) \geq 1$ such that*

$$\hat{\alpha}(s,x) = \exp\int_0^s K(\pi(\sigma,x))d\sigma.$$

2. *The family $(\alpha(s,x))_{s\in\mathbb{C}}$ is a solution of $(Co1)$ if and only if there exist $\mu \in \mathbb{C}$ and $K(y) \in \mathbb{C}[\![y]\!]$, $\operatorname{ord} K(y) \geq 1$ such that*

$$\alpha(s,x) = e^{\mu s}\exp\int_0^s K(\pi(\sigma,x))d\sigma.$$

Now we assume that α satisfies $(Co1)$. Since $\operatorname{ord}\alpha(s,x) = 0$, it is possible to define $\gamma(s,x) \in \mathbb{C}[\![x]\!]$ by

$$\gamma(s,x) := \frac{\beta(s,x)}{\alpha(s,x)} \qquad \forall s \in \mathbb{C}.$$

The coefficient functions of γ are analytic if and only if the coefficient functions of β are analytic.

Lemma 2.4. *The families α and β satisfy the system $((Co1),(Co2))$ if and only if α satisfies $(Co1)$, and γ is a solution of*

$$\gamma(t+s,x) = \gamma(s,x) + \frac{\gamma(t,\pi(s,x))}{\alpha(s,x)}. \qquad (Co2')$$

Proof. Assume first that α is a solution of $(Co1)$, and α and β satisfy $(Co2)$. Then

$$\begin{aligned}\gamma(t+s,x) &= \frac{\beta(t+s,x)}{\alpha(t+s,x)}\\ &= \frac{\beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x))}{\alpha(s,x)\alpha(t,\pi(s,x))}\\ &= \frac{\beta(s,x)}{\alpha(s,x)} + \frac{\beta(t,\pi(s,x))}{\alpha(s,x)\alpha(t,\pi(s,x))}\\ &= \gamma(s,x) + \frac{\gamma(t,\pi(s,x))}{\alpha(s,x)}.\end{aligned}$$

Assuming conversely that α is a solution of $(Co1)$ and γ is a solution of $(Co2')$, we get

$$\begin{aligned}\beta(t+s,x) &= \alpha(t+s,x)\gamma(t+s,x)\\ &= \alpha(s,x)\alpha(t,\pi(s,x))\\ &\qquad\cdot\left[\gamma(s,x) + \frac{\gamma(t,\pi(s,x))}{\alpha(s,x)}\right]\\ &= \alpha(s,x)\gamma(s,x)\alpha(t,\pi(s,x))\\ &\qquad + \alpha(t,\pi(s,x))\gamma(t,\pi(s,x))\\ &= \beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x)).\end{aligned}$$

■

Theorem 2.5. *Assume that α satisfies the cocycle equation $(Co1)$. Then α and β form a solution of $(Co2)$ if and only if there exists a series $L(y) \in \mathbb{C}[\![y]\!]$ such that*

$$\beta(s,x) = \alpha(s,x)\int_0^s \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)}d\sigma,$$

where integration is taken coefficientwise.

Proof. First assume that α and β satisfy $(Co2)$. Then Lemma 2.4 implies that α and γ satisfy $(Co2')$. Coefficientwise differentiation of $(Co2')$ with respect to the variable t yields

$$\gamma'(t+s,x) = \frac{\gamma'(t,\pi(s,x))}{\alpha(s,x)}.$$

For $t = 0$ we get $\gamma'(s,x) = \gamma'(0,\pi(s,x))/\alpha(s,x)$. Putting $L(y) := \gamma'(0,y)$, we obtain $\gamma'(s,x) =$

$L(\pi(s,x))/\alpha(s,x)$. By coefficientwise integration it follows that

$$\gamma(s,x) = \int_0^s \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)} d\sigma$$

and

$$\beta(s,x) = \alpha(s,x) \int_0^s \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)} d\sigma.$$

Conversely, if β is given by that formula, then

$$\gamma(t+s,x) = \int_0^{t+s} \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)} d\sigma =$$

$$\int_0^{s} \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)} d\sigma + \int_s^{t+s} \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)} d\sigma =$$

$$\gamma(s,x) + \int_0^t \frac{L(\pi(\tau+s,x))}{\alpha(\tau+s,x)} d\tau =$$

$$\gamma(s,x) + \int_0^t \frac{L(\pi(\tau,\pi(s,x)))}{\alpha(s,x)\alpha(\tau,\pi(s,x))} d\tau =$$

$$\gamma(s,x) + \frac{\gamma(t,\pi(s,x))}{\alpha(s,x)}.$$

In other words, α and γ satisfy $(Co2')$, hence by Lemma 2.4 α and β satisfy $(Co2)$. ∎

Now we describe a different representation of the general solution of $(Co1)$ and of the system $((Co1),(Co2))$, involving as few integrals as possible. In Lemma 2.1 we already derived solutions α of $(Co1)$ which could be represented without integrals at all. Their form is a motivation for the representation of the general solution of $(Co1)$ we have in mind here. In the first part of Lemma 2.7 we, similarly, present a class of solutions of $((Co1),(Co2))$ which are free of integrals. This motivates the representation of the general solution of $((Co1),(Co2))$ and will be applied in the proof of the form of the general solution. In this context it is necessary and helpful to distinguish between the different types of iteration groups $(\pi(s,x))_{s\in\mathbb{C}}$, and also to consider certain special cases of μ and of (μ,λ), if iteration groups of the first type are used. In particular, we investigate under which conditions the solutions can be expressed without integrals. Theorem 2.6 summarizes our results concerning $(Co1)$, Theorem 2.8 the results concerning the system $((Co1),(Co2))$. The above mentioned form of the general solutions will be useful in solving the boundary conditions.

Theorem 2.6. *1. Let $\pi(s,x) = e^{\lambda s}x$ for $\lambda \neq 0$. Then α is a solution of $(Co1)$ if and only if there exist $\mu \in \mathbb{C}$ and a formal power series $E(x) = 1 + e_1 x + \cdots \in \mathbb{C}[\![x]\!]$ such that*

$$\alpha(s,x) = e^{\mu s} \frac{E(e^{\lambda s}x)}{E(x)}.$$

The series $E(x)$ is uniquely determined by α.

2. Let $\pi(s,x) = x + c_k s x^k + \cdots \in \mathbb{C}[\![x]\!]$ with $c_k \neq 0$ and $k \geq 2$. If $\alpha(s,x) = e^{\mu s}(1 + \hat{\alpha}_k(s)x^k + \dots) \equiv e^{\mu s} \bmod x^k$, then α is a solution of $(Co1)$ if and only if there exist $\mu \in \mathbb{C}$ and a series $E(x) = 1 + e_1 x + \cdots \in \mathbb{C}[\![x]\!]$ such that

$$\alpha(s,x) = e^{\mu s} \frac{E(\pi(s,x))}{E(x)}.$$

The series $E(x)$ is uniquely determined by α.

3. The general solution α of $(Co1)$ for iteration groups $(\pi(s,x))_{s\in\mathbb{C}}$ of the second type is

$$\alpha(s,x) =$$

$$e^{\mu s} \prod_{n=1}^{k-1} \left(\exp \int_0^s \pi(\sigma,x)^n d\sigma \right)^{\kappa_n} \frac{E(\pi(s,x))}{E(x)}$$

with $\kappa_n \in \mathbb{C}$. The series $E(x)$ and the constants κ_n are uniquely determined by α.

Proof. In Lemma 2.1 we described solutions α of $(Co1)$ which could be expressed without integrals. In Corollary 2.3 all solutions of this equation in integral form were determined. Combining these two results, we investigate when

$$e^{\mu s} \exp \int_0^s K(\pi(\sigma,x)) d\sigma = e^{\mu s} \frac{E(\pi(s,x))}{E(x)} \qquad (1)$$

holds, where $E(x) \equiv 1 \bmod x$. After applying the formal logarithm, we have to check when

$$\int_0^s K(\pi(\sigma,x)) d\sigma = \tilde{E}(\pi(s,x)) - \tilde{E}(x)$$

is true for $\tilde{E}(x) := \ln E(x)$. Coefficientwise differentiation of the last equation with respect to the variable s yields

$$K(\pi(s,x)) = \frac{d\tilde{E}}{dy}\Big|_{y=\pi(s,x)} \pi'(s,x), \qquad (2)$$

where we used the "mixed" chain rule for this derivation. **Case 1:** If $(\pi(s,x))_{s\in\mathbb{C}}$ is an iteration group of the first type this means

$$K(e^{\lambda s}x) = \frac{d\tilde{E}}{dy}\Big|_{y=e^{\lambda s}x} \lambda e^{\lambda s}x.$$

In this formula $e^{\lambda s}x$ can be replaced by the indeterminate y, hence we get $K(y) = \lambda y \frac{d\tilde{E}(y)}{dy}$. Since $\operatorname{ord} K(y) \geq 1$ and $\lambda \neq 0$, it is possible to divide by λy, and we end up with a differential equation

$$\frac{K(y)}{\lambda y} =: \tilde{K}(y) = \frac{d\tilde{E}(y)}{dy}. \tag{3}$$

Assume that $\tilde{K}(y) = \sum_{n\geq 0} \tilde{\kappa}_n y^n$, then the Ansatz $\tilde{E}(y) = \sum_{n\geq 1} \tilde{e}_n y^n$ leads to $\tilde{e}_{n+1} = \tilde{\kappa}_n/(n+1)$ for all $n \geq 0$. Hence, all the coefficients e_n of $E(x)$ for $n \geq 1$ and $n = 0$ are uniquely determined.

So far we proved that the series E is uniquely defined by K. Next we show that each solution $\tilde{E}$ of the differential equation (3) with $\tilde{E}(0) = 0$ leads to a solution E of (1) by setting $E(y) = \exp \tilde{E}(y)$. Since $\tilde{E}$ is a solution of the differential equation, it is clear that

$$K(e^{\lambda s}x) = \lambda e^{\lambda s}x \frac{d\tilde{E}}{dy}\Big|_{y=e^{\lambda s}x}.$$

The right hand side of this equation is $\frac{\partial}{\partial s}\tilde{E}(e^{\lambda s}x)$, hence

$$\int_0^s K(e^{\lambda\sigma}x)d\sigma = \tilde{E}(e^{\lambda s}x) - \tilde{E}(e^{\lambda 0}x)$$

and

$$e^{\mu s} \exp \int_0^s K(e^{\lambda\sigma}x)d\sigma =$$

$$e^{\mu s} \exp(\tilde{E}(e^{\lambda s}x) - \tilde{E}(x)) =$$

$$e^{\mu s}\frac{\exp \tilde{E}(e^{\lambda s}x)}{\exp \tilde{E}(x)} = e^{\mu s}\frac{E(e^{\lambda s}x)}{E(x)}.$$

Moreover, the coefficient e_0 of $E(x)$ is equal to 1, since $E(x) = \exp \tilde{E}(X)$ and $\operatorname{ord} \tilde{E}(x) \geq 1$, which finishes the proof for iteration groups π of the first type.

Case 2: If $(\pi(s,x))_{s\in\mathbb{C}}$ is an analytic iteration group of the second type, then from iteration theory (cf. [Scheinberg, 1970] or [Reich & Schwaiger, 1977]) it follows that $\pi'(s,x) = H(\pi(s,x))$, where $H(y) := \pi'(s,y)|_{s=0}$ is the infinitesimal generator of π. In the present situation $H(y) = c_k y^k + \ldots$, hence $\operatorname{ord} H(y) = k$, and (2) means

$$K(\pi(s,x)) = \frac{d\tilde{E}}{dy}\Big|_{y=\pi(s,x)} H(\pi(s,x)).$$

After replacing $\pi(s,x)$ by the indeterminate y, we realize that $\operatorname{ord} K(y) \geq k$, since $K(y) = H(y)\frac{d\tilde{E}(y)}{dy}$. (This, however, is equivalent to $\alpha(s,x) \equiv e^{\mu s} \bmod x^k$.) Hence, we end up with the differential equation

$$\frac{K(y)}{H(y)} =: \tilde{K}(y) = \frac{d\tilde{E}(y)}{dy}, \tag{4}$$

which, similar as in the first part of the proof, has exactly one solution $\tilde{E}(y) = \sum_{n\geq 1} \tilde{e}_n y^n$.

Finally, it remains to prove that each solution $\tilde{E}$ of this differential equation with $\tilde{E}(0) = 0$ yields a solution E of (1). Let $\tilde{E}$ be a solution of (4) with $\tilde{E}(0) = 0$, then

$$\begin{aligned} K(\pi(s,x)) &= \frac{d\tilde{E}}{dy}\Big|_{y=\pi(s,x)} H(\pi(s,x)) \\ &= \frac{d\tilde{E}}{dy}\Big|_{y=\pi(s,x)} \pi'(s,x) = \frac{\partial}{\partial s}\tilde{E}(\pi(s,x)) \end{aligned}$$

and

$$\begin{aligned} \int_0^s K(\pi(\sigma,x))d\sigma &= \tilde{E}(\pi(s,x)) - \tilde{E}(\pi(0,x)) \\ &= \tilde{E}(\pi(s,x)) - \tilde{E}(x). \end{aligned}$$

Substitution into the exponential series and multiplication by $e^{\mu s}$ yields

$$e^{\mu s} \exp \int_0^s K(\pi(\sigma,x))d\sigma =$$

$$e^{\mu s} \exp(\tilde{E}(\pi(s,x)) - \tilde{E}(x)) = e^{\mu s}\frac{E(\pi(s,x))}{E(x)}.$$

Hence, $E(x)$ satisfies (1) and $E(x) = \exp \tilde{E}(x) \equiv 1 \bmod x^0$.

Case 2.1: From Corollary 2.3 we deduce that the general solution α of $(Co1)$ is given by

$$\begin{aligned}\alpha(s,x) &= e^{\mu s}\exp\int_0^s\left(\sum_{n=1}^{k-1}\kappa_n\pi(\sigma,x)^n\right.\\ &\qquad\left.+\sum_{n\geq k}\kappa_n\pi(\sigma,x)^n\right)d\sigma\\ &= e^{\mu s}\exp\left(\sum_{n=1}^{k-1}\kappa_n\int_0^s\pi(\sigma,x)^n d\sigma\right)\\ &\qquad\cdot\exp\int_0^s\hat{K}(\pi(\sigma,x))d\sigma\end{aligned}$$

with $\hat{K}(y)=\sum_{n\geq k}\kappa_n y^n$. By Corollary 2.3

$$\exp\int_0^s\hat{K}(\pi(\sigma,x))d\sigma$$

is a solution of $(Co1)$, and it is of the form $1+\hat{\alpha}_k(s)x^k+\dots$, since $\operatorname{ord}\hat{K}(y)\geq k$. Hence, by the second part of the present theorem there exists a unique series $E(x)=1+e_1x+\dots$ such that

$$\exp\int_0^s\hat{K}(\pi(\sigma,x))d\sigma=\frac{E(\pi(s,x))}{E(x)}.$$

Summarizing, we found

$$\alpha(s,x)=$$

$$e^{\mu s}\prod_{n=1}^{k-1}\left(\exp\int_0^s\pi(\sigma,x)^n d\sigma\right)^{\kappa_n}\frac{E(\pi(s,x))}{E(x)}.$$

(We also applied the identity

$$(\exp\Phi(y))^{\kappa}=\exp(\kappa\Phi(y)),$$

holding for the formal series exp and the formal binomial series with $\operatorname{ord}\Phi(y)\geq 1$.) ■

Lemma 2.7. *Let $E(x)=e_0+e_1x+\dots\in\mathbb{C}[\![x]\!]$, $e_0\neq 0$, and assume that $F(x)\in\mathbb{C}[\![x]\!]$ and $\mu\in\mathbb{C}$.*

1. *The series*

$$\beta(s,x)=$$

$$e^{\mu s}E(\pi(s,x))[F(x)-e^{-\mu s}F(\pi(s,x))]$$

together with α given in Lemma 2.1 satisfies $(Co2)$ for any analytic iteration group π.

2. *Assume that $\pi(s,x)=x+c_ksx^k+\dots\in\mathbb{C}[\![x]\!]$ with $k\geq 2$ and $c_k\neq 0$ is an analytic iteration group of the second type, and let $P(s,x)$ denote the series*

$$P(s,x):=\prod_{n=1}^{k-1}\left(\exp\int_0^s\pi(\sigma,x)^n d\sigma\right)^{\kappa_n}.$$

Then β defined by

$$\begin{aligned}\beta(s,x)&=e^{\mu s}P(s,x)E(\pi(s,x))\\&\qquad\cdot\left[F(x)-e^{-\mu s}\frac{F(\pi(s,x))}{P(s,x)}\right]\end{aligned}$$

together with α given in the third part of Theorem 2.6 satisfies $(Co2)$.

Proof. The families α and β satisfy $(Co2)$ if and only if

$$\beta(t+s,x)-\beta(t,\pi(s,x))=\beta(s,x)\alpha(t,\pi(s,x))$$

for all $s,t\in\mathbb{C}$. If we express β and α by E, F, π, this is

$$\begin{aligned}&e^{\mu(t+s)}E(\pi(t+s,x))[F(x)-e^{-\mu(t+s)}F(\pi(t+s,x))]\\&\qquad-e^{\mu t}E(\pi(t,\pi(s,x)))\Big[F(\pi(s,x))\\&\qquad-e^{-\mu t}F(\pi(t,\pi(s,x)))\Big]=\\&e^{\mu s}E(\pi(s,x))[F(x)-e^{-\mu s}F(\pi(s,x))]\\&\qquad\cdot e^{\mu t}\frac{E(\pi(t,\pi(s,x)))}{E(\pi(s,x))}.\end{aligned}$$

Application of (T) together with simplification of both sides yields

$$\begin{aligned}&e^{\mu(t+s)}E(\pi(t+s,x))F(x)\\&\qquad-e^{\mu t}E(\pi(t,\pi(s,x)))F(\pi(s,x))=\\&e^{\mu s}F(x)e^{\mu t}E(\pi(t,\pi(s,x)))\\&\qquad-F(\pi(s,x))e^{\mu t}E(\pi(t,\pi(s,x))),\end{aligned}$$

which is always true, since π satisfies (T).

The proof of the second part is similar to the proof above; the reader only has to take into account that $(P(s,x))_{s\in\mathbb{C}}$ is a solution of $(Co1)$. ■

Theorem 2.8. *Let α be a solution of $(Co1)$.*

1. *Assume that $\pi(s,x) = e^{\lambda s}x$ for $\lambda \neq 0$ and that α is given as in the first part of Theorem 2.6.*

   If $\mu - n\lambda \neq 0$ for all $n \in \mathbb{N}_0$, then (α,β) is a solution of $(Co2)$ if and only if there exists a formal power series $F(x) \in \mathbb{C}[\![x]\!]$ such that

   $$\beta(s,x) = e^{\mu s}E(e^{\lambda s}x)[F(x) - e^{-\mu s}F(e^{\lambda s}x)].$$

   The series $F(x)$ is uniquely determined by α and β.

   If $\mu = n_0\lambda$ for $n_0 \in \mathbb{N}_0$, then (α,β) is a solution of $(Co2)$ if and only if there exist a formal power series $F(x) \in \mathbb{C}[\![x]\!]$ and $\ell_{n_0} \in \mathbb{C}$ such that

   $$\beta(s,x) = e^{\mu s}E(e^{\lambda s}x)[\ell_{n_0}sx^{n_0} + F(x) - e^{-\mu s}F(e^{\lambda s}x)].$$

2. *Let $\pi(s,x) = x + c_k s x^k + \cdots \in \mathbb{C}[\![x]\!]$ with $c_k \neq 0$ and $k \geq 2$.*

   Assume that α can be expressed as in the second part of Theorem 2.6. If $\mu \neq 0$, then (α,β) is a solution of $(Co2)$ if and only if there exists a formal power series $F(x) \in \mathbb{C}[\![x]\!]$ such that

   $$\beta(s,x) = e^{\mu s}E(\pi(s,x))[F(x) - e^{-\mu s}F(\pi(s,x))].$$

   The series $F(x)$ is uniquely determined by α and β.

   If $\mu = 0$, then (α,β) is a solution of $(Co2)$ if and only if there exist a series $F(x) \in \mathbb{C}[\![x]\!]$ and coefficients $\ell_0,\dots,\ell_{k-1} \in \mathbb{C}$ such that

   $$\beta(s,x) = E(\pi(s,x))[F(x) - F(\pi(s,x)) + Q(s,x)]$$

   where

   $$Q(s,x) = \sum_{n=0}^{k-1}\int_0^s \frac{\ell_n\pi(\sigma,x)^n}{E(\pi(\sigma,x))}d\sigma.$$

3. *Let α be the general solution of $(Co1)$ given in the third part of Theorem 2.6, and assume that*

   $$P(s,x) := \prod_{n=1}^{k-1}\left(\exp\int_0^s \pi(\sigma,x)^n d\sigma\right)^{\kappa_n}$$

   actually occurs. Moreover, let n_0 be the minimum of $\{n \in \mathbb{N} \mid 1 \leq n \leq k-1,\ \kappa_n \neq 0\}$.

   If $\mu \neq 0$, then (α,β) is a solution of $(Co2)$ if and only if there exists a formal power series $F(x) \in \mathbb{C}[\![x]\!]$ such that

   $$\beta(s,x) = e^{\mu s}P(s,x)E(\pi(s,x)) \cdot \left[F(x) - e^{-\mu s}\frac{F(\pi(s,x))}{P(s,x)}\right].$$

   The series $F(x)$ is uniquely determined by α and β.

   If $\mu = 0$ and $\big[n_0 \neq k-1$, or $\kappa_{k-1} - mc_k \neq 0$ for all $m \in \mathbb{N}\big]$, then (α,β) is a solution of $(Co2)$ if and only if there exist a formal power series $F(x) \in \mathbb{C}[\![x]\!]$ and coefficients $\ell_0,\dots,\ell_{n_0-1} \in \mathbb{C}$ such that

   $$\beta(s,x) = P(s,x)E(\pi(s,x)) \cdot \left[F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x)\right]$$

   where

   $$Q(s,x) = \sum_{n=0}^{n_0-1}\int_0^s \frac{\ell_n\pi(\sigma,x)^n}{P(\sigma,x)E(\pi(\sigma,x))}d\sigma.$$

   The series $F(x)$ and the coefficients ℓ_n are uniquely determined by α and β.

   If $\mu = 0$, $n_0 = k-1$, and $\kappa_{k-1} = n_1c_k$ for $n_1 \in \mathbb{N}$, then (α,β) is a solution of $(Co2)$ if and only if there exist a formal power series $F(x) \in \mathbb{C}[\![x]\!]$ and coefficients $\ell_0,\dots,\ell_{n_0-1}$ and $\ell''_{n_1+n_0}$ in $\mathbb{C}$ such that $\beta(s,x)$ equals

   $$E(\pi(s,x))P(s,x) \cdot \left[F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x) + \ell''_{n_1+n_0}\int_0^s \frac{\pi(\sigma,x)^{n_1+n_0}}{P(\sigma,x)}d\sigma\right].$$

Proof. We apply similar ideas and arguments as in the proof of Theorem 2.6. In Lemma 2.7 we described special solutions, and in Theorem 2.5 all solutions of $(Co2)$ in integral form were given. If α can be expressed without any integrals, then we check when

$$\alpha(s,x)\int_0^s \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)}d\sigma = e^{\mu s}E(\pi(s,x))[F(x) - e^{-\mu s}F(\pi(s,x))] \tag{5}$$

holds. This is equivalent to

$$\int_0^s \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)} d\sigma = E(x)[F(x) - e^{-\mu s}F(\pi(s,x))].$$

Coefficientwise differentiation of the last equation with respect to the variable s yields

$$\frac{L(\pi(s,x))}{\alpha(s,x)} = E(x)\Big[\mu e^{-\mu s}F(\pi(s,x)) - e^{-\mu s}\frac{dF}{dy}\Big|_{y=\pi(s,x)}\pi'(s,x)\Big].$$

Case 1: If π is an iteration group of the first type, this means

$$L(e^{\lambda s}x) = \alpha(s,x)E(x)\Big[\mu e^{-\mu s}F(e^{\lambda s}x) - e^{-\mu s}\frac{dF}{dy}\Big|_{y=e^{\lambda s}x}\lambda e^{\lambda s}x\Big].$$

Using the special form of α from the first part of Theorem 2.6 and replacing $e^{\lambda s}x$ by the indeterminate y gives

$$\tilde{L}(y) := \frac{L(y)}{E(y)} = \mu F(y) - \lambda y \frac{dF(y)}{dy}. \qquad (6)$$

Assume that $\tilde{L}(y) = \sum_{n\geq 0}\ell_n y^n$, then the Ansatz $F(y) = \sum_{n\geq 0} f_n y^n$ leads to

$$\sum_{n\geq 0}\ell_n y^n = \sum_{n\geq 0}(\mu - n\lambda)f_n y^n.$$

Case 1.1: If $\mu - n\lambda \neq 0$ for all $n \geq 0$, then $F(y)$ is uniquely given by

$$f_n = \frac{\ell_n}{\mu - n\lambda} \qquad \forall n \geq 0.$$

So far we proved that in this situation the series F is uniquely defined by L. Next we show that each solution F of the differential equation (6) is a solution of (5). Since F is a solution of (6), it is clear that

$$L(y) = E(y)\left[\mu F(y) - \lambda y \frac{dF(y)}{dy}\right].$$

After replacing y by $e^{\lambda s}x$ and using the special form of α, we derive that

$$\frac{L(e^{\lambda s}x)}{\alpha(s,x)} = E(x)\left[\mu e^{-\mu s}F(e^{\lambda s}x) - e^{-\mu s}\lambda e^{\lambda s}x\frac{dF}{dy}\Big|_{y=e^{\lambda s}x}\right] = E(x)\frac{\partial}{\partial s}(-e^{-\mu s}F(e^{\lambda s}x)).$$

Coefficientwise integration, finally, yields the desired result.

Still we are dealing with analytic iteration groups $(\pi(s,x))_{s\in\mathbb{C}}$ of the first type. But now in **case 1.2** we assume that $\mu = n_0\lambda$. In this situation, comparing the coefficients of y^{n_0} yields the condition $0 = (\mu - n_0\lambda)f_{n_0} = \ell_{n_0}$. If $\ell_{n_0} \neq 0$, then we split $L(y)$ into the form

$$\sum_{\substack{n\geq 0\\ n\neq n_0}}\ell_n y^n + \ell_{n_0}y^{n_0}.$$

From Theorem 2.6 we know that β is given as

$$\beta(s,x) = \alpha(s,x)\int_0^s \frac{L(e^{\lambda\sigma}x)}{\alpha(e^{\lambda\sigma}x)}d\sigma =$$

$$e^{\mu s}\frac{E(e^{\lambda s}x)}{E(x)}\int_0^s \frac{L(e^{\lambda\sigma}x)E(x)}{e^{\mu\sigma}E(e^{\lambda\sigma}x)}d\sigma =$$

$$e^{\mu s}E(e^{\lambda s}x)\int_0^s e^{-\mu\sigma}\sum_{n\geq 0}\ell_n e^{n\lambda\sigma}x^n d\sigma =$$

$$e^{\mu s}E(e^{\lambda s}x)\Big(\int_0^s \ell_{n_0}d\sigma\, x^{n_0} + \int_0^s \sum_{\substack{n\geq 0\\ n\neq n_0}} e^{(n\lambda-\mu)\sigma}\ell_n x^n d\sigma\Big) =$$

$$e^{\mu s}E(e^{\lambda s}x)[\ell_{n_0}sx^{n_0} + F(x) - e^{-\mu s}F(e^{\lambda s}x)].$$

For $n \neq n_0$ the coefficients f_n of $F(x)$ are uniquely given by

$$f_n = \frac{\ell_n}{\mu - n\lambda},$$

whereas f_{n_0} is not determined.

Conversely, it is left to the reader to prove that each series F with coefficients $f_n = \ell_n/(\mu - n\lambda)$ for $n \neq n_0$ and arbitrary $f_{n_0} \in \mathbb{C}$ is a solution of (5).

In **case 2** of the present proof we assume that $(\pi(s,x))_{s\in\mathbb{C}}$ is an iteration group of the second type. In **case 2.1** we investigate when (5) holds. For doing this, we assume that α is given as in the second part of Theorem 2.6. Inserting the special form of α, coefficientwise differentiation with respect to s, and expressing $\pi'(s,x)$ as $H(\pi(s,x))$, where H is the infinitesimal generator of π, yields the equation

$$L(\pi(s,x)) = E(\pi(s,x))\Big[\mu F(\pi(s,x)) - \frac{dF}{dy}\Big|_{y=\pi(s,x)} H(\pi(s,x))\Big].$$

After replacing $\pi(s,x)$ by y, we end up with the differential equation

$$\tilde{L}(y) := \frac{L(y)}{E(y)} = \mu F(y) - \frac{dF(y)}{dy}H(y). \tag{7}$$

Assume that $\tilde{L}(y) = \sum_{n\geq 0} \ell_n y^n$ and $H(y) = \sum_{n\geq k} h_n y^n$, where $h_k = c_k \neq 0$, then the Ansatz $F(y) = \sum_{n\geq 0} f_n y^n$ leads to

$$\sum_{n\geq 0} \ell_n y^n = \mu \sum_{n=0}^{k-1} f_n y^n + \sum_{n\geq k}\Big(\mu f_n - \sum_{\substack{r+s=n\\ s\geq k}} (r+1) f_{r+1} h_s\Big) y^n.$$

Comparing coefficients yields

$$\begin{aligned} \ell_n &= \mu f_n, && n<k \\ \ell_n &= \mu f_n - \sum_{r=1}^{n-k+1} r f_r h_{n-r+1}, && n \geq k. \end{aligned}$$

Case 2.1.1: If $\mu \neq 0$, then F is uniquely determined by

$$\begin{aligned} f_n &= \frac{\ell_n}{\mu}, && n<k \\ f_n &= \frac{1}{\mu}\left(\ell_n + \sum_{r=1}^{n-k+1} r f_r h_{n-r+1}\right), && n \geq k. \end{aligned}$$

Assuming conversely that F is a solution of (7), it is left to the reader to prove that F satisfies (5). (The proof is similar to that given in the first part of this proof.)

Case 2.1.2: If $\mu = 0$, then (7) reduces to

$$\tilde{L}(y) = -\frac{dF(y)}{dy}H(y) \tag{8}$$

or in more details

$$\sum_{n\geq 0} \ell_n y^n = -\sum_{n\geq k}\left(\sum_{r=1}^{n-k+1} r f_r h_{n-r+1}\right) y^n.$$

Comparing coefficients yields a necessary condition for the coefficients of $\tilde{L}$, namely

$$\ell_n = 0, \qquad n<k$$

and a formula to determine recursively the values of f_n by

$$f_n = -\frac{\ell_{k+n-1} + \sum_{r=1}^{n-1} r f_r h_{k+n-r}}{n c_k}, \qquad n \geq 1,$$

since $h_k = c_k$. The coefficient f_0 is not determined by (8). In conclusion, $\operatorname{ord}\tilde{L}(y) \geq k$, which implies that $\operatorname{ord} L(y) \geq k$, and finally $\operatorname{ord}\beta(s,x) \geq k$.

Assuming conversely that F is a solution of (8) with arbitrary $f_0 \in \mathbb{C}$, it is left to the reader to prove that F satisfies (5) for $\mu = 0$.

If $\operatorname{ord} L(x) < k$, then we get by combining the above calculations with the integral form of the general solution that

$$\beta(s,x) = E(\pi(s,x))[F(x) - F(\pi(s,x)) + Q(s,x)].$$

Case 2.2: Let α be the general solution of $(Co1)$ given in the third part of Theorem 2.6. Then

$$\beta(s,x) = e^{\mu s} P(s,x)\frac{E(\pi(s,x))}{E(x)} \int_0^s \frac{L(\pi(\sigma,x))E(x)}{e^{\mu\sigma}P(\sigma,x)E(\pi(\sigma,x))} d\sigma = e^{\mu s} P(s,x)E(\pi(s,x)) \int_0^s e^{-\mu\sigma}\frac{\tilde{L}(\pi(\sigma,x))}{P(\sigma,x)} d\sigma.$$

First we check when

$$e^{\mu s} P(s,x)E(\pi(s,x)) \int_0^s e^{-\mu\sigma}\frac{\tilde{L}(\pi(\sigma,x))}{P(\sigma,x)} d\sigma = e^{\mu s} P(s,x)E(\pi(s,x))\left[F(x) - e^{-\mu s}\frac{F(\pi(s,x))}{P(s,x)}\right] \tag{9}$$

holds. This is obviously equivalent to

$$\int_0^s e^{-\mu\sigma}\frac{\tilde{L}(\pi(\sigma,x))}{P(\sigma,x)} d\sigma = F(x) - e^{-\mu s}\frac{F(\pi(s,x))}{P(s,x)}.$$

Coefficientwise differentiation with respect to the variable s gives

$$e^{-\mu s}\frac{\tilde{L}(\pi(s,x))}{P(s,x)} = -\frac{\partial}{\partial s} e^{-\mu s}\frac{F(\pi(s,x))}{P(s,x)} = \mu e^{-\mu s}\frac{F(\pi(s,x))}{P(s,x)} - e^{-\mu s}\left(\frac{\partial}{\partial s}\frac{1}{P(s,x)}\right)F(\pi(s,x))$$

$$-e^{-\mu s}\frac{1}{P(s,x)}\frac{dF}{dy}\Big|_{y=\pi(s,x)}H(\pi(s,x)),$$

where H is the infinitesimal generator of π. Since

$$\frac{\partial}{\partial s}\frac{1}{P(s,x)}=\frac{\sum_{n=1}^{k-1}(-\kappa_n)\pi(s,x)^n}{P(s,x)},$$

we end up with the differential equation

$$\tilde{L}(y)=\left(\mu+\sum_{n=1}^{k-1}\kappa_n y^n\right)F(y)-\frac{dF(y)}{dy}H(y), \quad (10)$$

after replacing $\pi(s,x)$ by y. The usual Ansatz leads to

$$\begin{aligned}\sum_{n\geq 0}\ell_n y^n=\mu\sum_{n\geq 0}f_n y^n&\\ &+\sum_{n\geq 1}\left(\sum_{r=1}^{\min\{k-1,n\}}\kappa_r f_{n-r}\right)y^n\\ &-\sum_{n\geq k}\left(\sum_{r=1}^{n-k+1}rf_r h_{n-r+1}\right)y^n.\end{aligned}$$

Hence, the coefficients satisfy

$$\begin{aligned}\ell_0&=\mu f_0,\\ \ell_n&=\mu f_n+\sum_{r=1}^{n}\kappa_r f_{n-r}, \qquad 1\leq n<k,\end{aligned}$$

and

$$\ell_n=\mu f_n+\sum_{r=1}^{k-1}\kappa_r f_{n-r}-\sum_{r=1}^{n-k+1}rf_r h_{n-r+1}$$

for $n\geq k$. In **case 2.2.1** we assume that $\mu\neq 0$. Then F is uniquely given by

$$\begin{aligned}f_0&=\frac{\ell_0}{\mu},\\ f_n&=\frac{1}{\mu}\left(\ell_n-\sum_{r=1}^{n}\kappa_r f_{n-r}\right), \qquad 1\leq n<k,\end{aligned}$$

and

$$f_n=\frac{1}{\mu}\left(\ell_n-\sum_{r=1}^{k-1}\kappa_r f_{n-r}+\sum_{r=1}^{n-k+1}rf_r h_{n-r+1}\right)$$

for $n\geq k$.

Again it is left to the reader to prove that F satisfies (9).

What happens if $\mu=0$, which is **case 2.2.2**? If n_0 denotes $\min\{n\in\mathbb{N}\mid 1\leq n\leq k-1,\ \kappa_n\neq 0\}$, then (10) reduces to

$$\tilde{L}(y)=\left(\sum_{n=n_0}^{k-1}\kappa_n y^n\right)F(y)-\frac{dF(y)}{dy}H(y).$$

Since the right hand side is a power series of order $\geq n_0$, the coefficients of $\tilde{L}(y)$ satisfy

$$\begin{aligned}\ell_n&=0, && 0\leq n<n_0,\\ \ell_n&=\sum_{r=n_0}^{n}\kappa_r f_{n-r}, && n_0\leq n<k,\end{aligned}$$

and

$$\ell_n=\sum_{r=n_0}^{k-1}\kappa_r f_{n-r}-\sum_{r=1}^{n-k+1}rf_r h_{n-r+1}$$

for $n\geq k$.

Consequently, $\operatorname{ord}\tilde{L}(y)\geq n_0$, and $\operatorname{ord}\beta(s,x)\geq n_0$. Hence, for $n_0\leq n<k$ the coefficients f_{n-n_0} are uniquely determined by

$$f_{n-n_0}=\frac{1}{\kappa_{n_0}}\left(\ell_n-\sum_{r=n_0+1}^{n}\kappa_r f_{n-r}\right). \quad (11)$$

For $n\geq k$ we still have to consider different cases. **Case 2.2.2.1:** If $n_0<k-1$, then $n-n_0>n-k+1$ and f_{n-n_0} are uniquely given by the recursive formula

$$\begin{aligned}f_{n-n_0}=\frac{1}{\kappa_{n_0}}\Bigg(\ell_n-&\sum_{r=n_0+1}^{k-1}\kappa_r f_{n-r}\\ &+\sum_{r=1}^{n-k+1}rf_r h_{n-r+1}\Bigg).\end{aligned}$$

Case 2.2.2.2: If $n_0=k-1$, then for $n\geq k$

$$\begin{aligned}\ell_n=\kappa_{k-1}f_{n-k+1}-\sum_{r=1}^{n-k+1}rf_r h_{n-r+1}=&\\ (\kappa_{k-1}-(n-k+1)h_k)f_{n-k+1}-\sum_{r=1}^{n-k}rf_r h_{n-r+1}.&\end{aligned} \quad (12)$$

The reader should remember from case 2.1 that h_k equals c_k.

Case 2.2.2.2.1: If $\kappa_{k-1} - mc_k \neq 0$ for all $m \in \mathbb{N}$, then

$$f_{n-n_0} = f_{n-k+1} = \frac{\ell_n + \sum_{r=1}^{n-k} r f_r h_{n-r+1}}{\kappa_{k-1} - (n-k+1)c_k} \qquad (13)$$

for $n \geq k$.

Finally we have to consider in **case 2.2.2.2.2** that $\mu = 0$, $n_0 = k-1$, and that there exists $n_1 \in \mathbb{N}$ such that $\kappa_{k-1} = n_1 c_k$. If $n - k + 1 = n_1$, which is equivalent to $n = n_1 + k - 1$, we have

$$\ell_{n_1+k-1} = (\kappa_{k-1} - n_1 c_k) f_{n_1} - \sum_{r=1}^{n_1-1} r f_r h_{n_1+k-r} =$$

$$- \sum_{r=1}^{n_1-1} r f_r h_{n_1+k-r}.$$

This is a necessary condition for writing $\beta(s,x)$ as in the right hand side of (9) for $\mu = 0$. In general let

$$\ell'_{n_1+k-1} := - \sum_{r=1}^{n_1-1} r f_r h_{n_1+k-r},$$

then

$$\int_0^s \frac{\tilde{L}(\pi(\sigma,x))}{P(\sigma,x)} d\sigma = \int_0^s \frac{1}{P(\sigma,x)} \sum_{n \geq k} \ell_n \pi(\sigma,x)^n d\sigma =$$

$$\int_0^s \left(\frac{\sum_{\substack{n \geq k \\ n \neq n_1+k-1}} \ell_n \pi(\sigma,x)^n + \ell'_{n_1+k-1} \pi(\sigma,x)^{n_1+k-1}}{P(\sigma,x)} \right.$$

$$\left. + \frac{(\ell_{n_1+k-1} - \ell'_{n_1+k-1}) \pi(\sigma,x)^{n_1+k-1}}{P(\sigma,x)} \right) d\sigma =$$

$$F(x) - \frac{F(\pi(s,x))}{P(s,x)} + \ell''_{n_1+k-1} \int_0^s \frac{\pi(\sigma,x)^{n_1+k-1}}{P(\sigma,x)} d\sigma,$$

where $\ell''_{n_1+k-1} = \ell_{n_1+k-1} - \ell'_{n_1+k-1}$. For $n \neq n_1 + k - 1$ the coefficients f_{n-k+1} of the series $F(x)$ are uniquely given by the two formulae (11) and (13), and f_{n_1} is not determined.

Also in the last cases it is left to the reader to prove that each solution F of the differential equation (10) also satisfies (9) for $\mu = 0$.

In order to finish case 2.2.2 completely, if $\operatorname{ord} L(x) < n_0$, then it follows in a similar way as in case 2.1.2 that the corresponding term $Q(s,x)$ must be added to obtain the general solution. ∎

3. Solutions which satisfy the boundary conditions

In this section we assume that $(\pi(s,x))_{s\in\mathbb{C}}$ is a given iteration group. We determine solutions α and β of the cocycle equations $(Co1)$ and $(Co2)$ which also satisfy the boundary conditions $(B1)$ and $(B2)$ for given formal power series

$$a(x) = \sum_{n\geq 0} a_n x^n, \; a_0 \neq 0 \text{ and } b(x) = \sum_{n \geq 0} b_n x^n.$$

From the results of the previous section it is obvious that $(B1)$ is always satisfied. We only have to consider $(B2)$ for further investigations.

First we deal with analytic iteration groups of the first type, i.e., we consider $\pi(s,x) = e^{\lambda s} x$ for $\lambda \neq 0$. Before describing the solutions α which satisfy the boundary conditions we need a preliminary result. If $J = J(\lambda)$ denotes the set $\{n \in \mathbb{N} \mid n\lambda \in 2\pi i \mathbb{Z}\}$, then the following lemma holds.

Lemma 3.1. *Assume that J is not empty, and let j_0 be the minimum of J. Then $J = \mathbb{N} j_0$.*

Proof. Since $j_0 \in J$, there exists $z_0 \in \mathbb{Z}$ such that $j_0 \lambda = 2 z_0 \pi i$. Then $n j_0 \lambda = 2 n z_0 \pi i \in 2\pi i \mathbb{Z}$ for all $n \in \mathbb{N}$. Hence, $\mathbb{N} j_0$ is a subset of J. Conversely, assume that $n \in J$, then by division we deduce that $n = q j_0 + r$ with uniquely determined r such that $0 \leq r < j_0$. From $2\pi i \mathbb{Z} \ni n\lambda = (q j_0 + r)\lambda = q j_0 \lambda + r\lambda = 2 q z_0 \pi i + r\lambda$ it follows that $r\lambda \in 2\pi i \mathbb{Z}$, and consequently $r = 0$. Hence, $n \in \mathbb{N} j_0$, which finishes the proof. ∎

In the first part of Theorem 2.6 the general solution α of $(Co1)$ for analytic iteration groups $(\pi(s,x))_{s\in\mathbb{C}}$ of the first type was described. We want to analyze how to adjust it to the condition $\alpha(1,x) = a(x)$.

Theorem 3.2. *Assume that $a(x)$ is a given formal power series of order* 0.

If $J = \emptyset$, then there exists exactly one formal *power series $E(x) = 1 + e_1x + \dots$ such that*

$$\alpha(s,x) = e^{\mu s}\frac{E(e^{\lambda s}x)}{E(x)}$$

is a solution of $(Co1)$ which satisfies $\alpha(1,x) = a(x)$.

If $J \neq \emptyset$, then there exist formal power series $E(x) = 1 + e_1x + \dots$ such that $\alpha(s,x)$ of the above form satisfies both $(Co1)$ and the boundary condition if and only if for $n \in J$ the coefficients a_n satisfy

$$a_n = -\sum_{r=1}^{n-1} a_r e_{n-r},$$

where e_n are the coefficients of $E(x)$.

Proof. Writing α as indicated above, the assumption $\alpha(1,x) = a(x)$ is equivalent to

$$e^{\mu}\sum_{n\geq 0} e_n e^{n\lambda}x^n = \sum_{n\geq 0}\sum_{r=0}^{n} a_r e_{n-r}.$$

Comparing coefficients yields for $n = 0$ that $e^{\mu} = a_0$, since $e_0 = 1$. Hence, μ is a logarithm $\ln a_0$. For $n \geq 1$ we get

$$e^{\mu}e^{n\lambda}e_n = a_0e_n + \sum_{r=1}^{n-1} a_r e_{n-r} + a_n,$$

which implies

$$a_0(e^{n\lambda} - 1)e_n = \sum_{r=1}^{n-1} a_r e_{n-r} + a_n.$$

For $n \notin J$ the coefficient e_n is uniquely determined by

$$e_n = \frac{\sum_{r=1}^{n} a_r e_{n-r}}{a_0(e^{n\lambda} - 1)}.$$

However, for $n \in J$ the coefficient e_n is not determined and actually can be chosen arbitrarily in $\mathbb{C}$, whereas a_n must satisfy the condition above. We will not analyze these conditions further in the present paper. ∎

Before we adjust β to the condition $\beta(1,x) = b(x)$ we need another preliminary result. Let $K = K(\mu,\lambda)$ denote the set $\{n \in \mathbb{N}_0 \mid \mu - n\lambda \in 2\pi i\mathbb{Z}\}$. Then the following lemma holds.

Lemma 3.3. *Assume that the cardinality of K is greater than 1. Then J is not empty and*

$$K = \{k_0 + nj_0 \mid n \in \mathbb{N}_0\} =$$

$$\{n \in \mathbb{N}_0 \mid n \equiv k_0 \bmod j_0\},$$

where $k_0 := \min K$ and $j_0 := \min J$. If $|K| = 1$, then $J = \emptyset$.

Proof. First we prove that when n_1 and n_2 are two different elements of K such that $n_1 > n_2$, then $n_1 - n_2$ belongs to J. Since $n_1, n_2 \in K$, there exist $z_1, z_2 \in \mathbb{Z}$ such that $\mu - n_1\lambda = 2z_1\pi i$ and $\mu - n_2\lambda = 2z_2\pi i$. Then $(n_1 - n_2)\lambda = (\mu - n_2\lambda) - (\mu - n_1\lambda) = 2(z_2 - z_1)\pi i \in 2\pi i\mathbb{Z}$ and $n_1 - n_2 \in \mathbb{N}$. Hence $n_1 - n_2 \in J$.

Since $k_0 \in K$ and $j_0 \in J$, there exist $z_0, z_1 \in \mathbb{Z}$ such that $\mu - k_0\lambda = 2z_0\pi i$ and $j_0\lambda = 2z_1\pi i$. Let $n \in \mathbb{N}_0$, then $\mu - (k_0 + nj_0)\lambda = \mu - k_0\lambda - nj_0\lambda = 2z_0\pi i - 2nz_1\pi i = 2(z_0 - nz_1)\pi i \in 2\pi i\mathbb{Z}$ and consequently $k_0 + nj_0 \in K$. Thus, $k_0 + \mathbb{N}_0j_0 \subseteq K$.

In the next step we prove that $k_0 < j_0$. (Then it is clear that $k_0 + \mathbb{N}_0j_0$ is the set of all positive integers congruent k_0 modulo j_0.) If we assume that $k_0 - j_0 \geq 0$, then $k_0 - j_0 \in K$ since $\mu - (k_0 - j_0)\lambda = 2(z_0 + z_1)\pi i \in 2\pi i\mathbb{Z}$. Moreover, $k_0 - j_0 < k_0$, which is a contradiction to the construction of k_0.

Finally, we have to prove that $K \subseteq k_0 + \mathbb{N}_0j_0$. Let $n \in K$. If $n \neq k_0$, then $n > k_0$ and then there exists $z \in \mathbb{Z}$, such that $\mu - n\lambda = 2z\pi i$. Moreover, $(n - k_0)\lambda = 2(z_0 - z)\pi i \in 2\pi i\mathbb{Z}$, thus $n - k_0 \in J = \mathbb{N}j_0$ by Lemma 3.1. Hence, $n \in k_0 + \mathbb{N}j_0$.

If $|K| = 1$, then necessarily $J = \emptyset$. Because if we assume that $J \neq \emptyset$, then $J = \mathbb{N}j_0$. Hence, $k_0 + J \subset K$, which is a contradiction to $|K| = 1$. ∎

Let α be a solution of $(Co1)$ where π is an analytic iteration group of the first type. The general form of β, which satisfies together with α the cocycle equation $(Co2)$, was given in the first part of Theorem 2.8.

Theorem 3.4. *Let π be an analytic iteration group of the first type. Assume that $b(x)$ is a given formal power series and α is a solution of $(Co1)$ given by*

$$\alpha(s,x) = e^{\mu s}\frac{E(e^{\lambda s}x)}{E(x)}$$

with $E(x) = 1 + e_1x + \dots$.

1. *If $K=\emptyset$, then there exists exactly one formal power series $F(x)$ such that*
$$\beta(s,x)=e^{\mu s}E(e^{\lambda s}x)[F(x)-e^{-\mu s}F(e^{\lambda s}x)]$$
together with α is a solution of $(Co2)$ satisfying $\beta(1,x)=b(x)$.

2. *If $K\neq\emptyset$ and $\mu-n\lambda\neq 0$ for all $n\in\mathbb{N}_0$, then there exist formal power series $F(x)$ such that*
$$\beta(s,x)=e^{\mu s}E(e^{\lambda s}x)[F(x)-e^{-\mu s}F(e^{\lambda s}x)]$$
together with α is a solution of $(Co2)$ satisfying the boundary condition if and only if for $n\in K$ the coefficients b_n satisfy
$$b_n=\sum_{r=0}^{n-1}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda}), \qquad (14)$$
where e_n and f_n are the coefficients of $E(x)$ and $F(x)$.

3. *If $K\neq\emptyset$ and $\mu-n_0\lambda=0$, then there exist formal power series $F(x)$ and $\ell_{n_0}\in\mathbb{C}$ such that*
$$\beta(s,x)=e^{\mu s}E(e^{\lambda s}x)\big[\ell_{n_0}sx^{n_0} + F(x)-e^{-\mu s}F(e^{\lambda s}x)\big]$$
together with α is a solution of $(Co2)$ satisfying the boundary condition if and only if for $n\in K\setminus\{n_0\}$ the coefficients b_n satisfy (14) for $n<n_0$, and b_n equals
$$\sum_{\substack{r=0\\ r\neq n_0}}^{n-1}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda})+\ell_{n_0}e_{n-n_0}e^{n\lambda}$$
for $n\in K$ and $n>n_0$.

Proof. Writing β in the form
$$\beta(s,x)=e^{\mu s}E(e^{\lambda s}x)[F(x)-e^{-\mu s}F(e^{\lambda s}x)],$$
and assuming that there is no $n_0\in\mathbb{N}$ such that $\mu=n_0\lambda$, the assumption $b(x)=\beta(1,x)$ is equivalent to
$$\sum_{n\geq 0}b_nx^n=e^{\mu}\sum_{n\geq 0}\left(\sum_{r=0}^{n}e_{n-r}f_re^{(n-r)\lambda}\right)x^n - \sum_{n\geq 0}\left(\sum_{r=0}^{n}e_{n-r}f_r\right)e^{n\lambda}x^n,$$
which yields for all $n\geq 0$ that b_n equals
$$\sum_{r=0}^{n-1}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda})+e_0f_ne^{n\lambda}(e^{\mu-n\lambda}-1).$$
If $n\notin K$, then f_n is uniquely given by
$$f_n=\frac{b_n-\sum_{r=0}^{n-1}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda})}{e^{n\lambda}(e^{\mu-n\lambda}-1)}.$$
For $n\in K$ the coefficient f_n can be chosen arbitrarily in $\mathbb{C}$, and b_n must satisfy
$$b_n=\sum_{r=0}^{n-1}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda}).$$

If there is $n_0\in\mathbb{N}$ such that $\mu=n_0\lambda$, then $n_0\in K$. Since in this situation
$$\beta(s,x)=e^{\mu s}E(e^{\lambda s}x)[\ell_{n_0}sx^{n_0}+F(x)-e^{-\mu s}F(e^{\lambda s}x)],$$
the formulae above are only correct for $n<n_0$. Comparing coefficients for $n\geq n_0$ yields
$$b_n=\sum_{\substack{r=0\\ r\neq n_0}}^{n}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda})+\ell_{n_0}e_{n-n_0}e^{n\lambda}.$$
For $n=n_0$ the coefficient ℓ_{n_0} is uniquely determined by
$$\ell_{n_0}=e^{-\mu}\left(b_{n_0}-\sum_{r=0}^{n_0-1}e_{n_0-r}f_re^{(n_0-r)\lambda}(e^{\mu}-e^{r\lambda})\right).$$
Furthermore, for $n>n_0$ and $n\notin K$, the coefficient f_n is given by the fraction
$$\frac{b_n-\sum\limits_{\substack{r=0\\ r\neq n_0}}^{n-1}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda})-\ell_{n_0}e_{n-n_0}e^{n\lambda}}{e^{n\lambda}(e^{\mu-n\lambda}-1)}$$
and for $n>n_0$, $n\in K$ the coefficients b_n must satisfy the condition
$$b_n=\sum_{\substack{r=0\\ r\neq n_0}}^{n-1}e_{n-r}f_re^{(n-r)\lambda}(e^{\mu}-e^{r\lambda})+\ell_{n_0}e_{n-n_0}e^{n\lambda}.$$
We will not analyze these conditions further. ∎

The importance of the next theorem will be clear in connection with Theorem 4.1.

Theorem 3.5. *Let $\pi(s,x)=e^{\lambda s}x$, where $\rho=e^{\lambda}$ is not a complex root of* 1. *Assume that $a(x)$ and $b(x)$ are given power series, where $\operatorname{ord} a(x)=0$. Then there exists a suitable α, which satisfies $(Co1)$ and the two boundary conditions, such that there is exactly one β, which satisfies the boundary conditions $(B1)$ and $(B2)$ and together with α also the cocycle equation $(Co2)$.*

Proof. Since ρ is not a complex root of 1, it is obvious that $J=J(\lambda)$ is empty. Hence, according to Lemma 3.3 the set $K=K(\mu,\lambda)$ is empty or K has cardinality 1. In the first case everything is clear from Theorem 3.2 and Theorem 3.4. If $K=\{k_0\}$, let μ be a logarithm of a_0 and $\alpha(s,x)$ be given as in Theorem 3.2. Hence, there exists some $z\in\mathbb{Z}$ such that $\mu-k_0\lambda=2z\pi i$. Then $\mu-2z\pi i=k_0\lambda$. If we replace μ by $\mu':=\mu-2z\pi i$, then $a_0=e^{\mu}=e^{\mu'}$ and $\mu'=k_0\lambda$, which means that $K(\mu',\lambda)=\{k_0\}$ and $\mu'-k_0\lambda=0$, hence $k_0=n_0$ from the second part of Theorem 3.4. Moreover, $\alpha'(s,x):=e^{(\mu'-\mu)s}\alpha(s,x)$ is also a solution of $(Co1)$, which satisfies the boundary conditions. As was described in the proof of Theorem 3.4, the coefficients f_n (for $n\neq k_0$) and ℓ_{k_0} are uniquely determined. Just f_{k_0} can be chosen arbitrarily in $\mathbb{C}$. Moreover, the series $b(x)$ need not satisfy any necessary conditions, so also in this situation there always exists a β satisfying $(Co2)$ and $(B2)$. According to Theorem 2.8, for computing $\beta(s,x)$ it is necessary to determine $F(x)-e^{-\mu's}F(e^{\lambda s}x)$. Because of the special choice of μ' and λ, this difference reads as

$$\sum_{n\geq 0} f_n x^n - e^{-\mu's}\sum_{n\geq 0} f_n e^{n\lambda s}x^n =$$

$$\sum_{n\geq 0}(1-e^{(-\mu'+n\lambda)s})f_n x^n = \sum_{\substack{n\geq 0\\ n\neq k_0}}(1-e^{(-\mu'+n\lambda)s})f_n x^n,$$

consequently it does not depend on the coefficient f_{k_0}, which still could be chosen arbitrarily. Hence, β is uniquely determined in this situation. ■

Now we come back to the analytic iteration groups of the second type, i.e., $\pi(s,x)=x+c_k s x^k+\dots$ with $k\geq 2$ and $c_k\neq 0$. The embedding for those α, which are of the form $\alpha(s,x)=e^{\mu s}+\alpha_k(s)x^k+\dots$, is described in

Theorem 3.6. *Assume that $a(x)$ is a given formal power series of order* 0, *and π is an analytic iteration group of the second type. There exists a formal power series $E(x)=1+e_1x+\dots$ such that*

$$\alpha(s,x)=e^{\mu s}\frac{E(\pi(s,x))}{E(x)}$$

is a solution of $(Co1)$ satisfying the boundary condition if and only if $a_n=0$ for $1\leq n<k$.

If $E(x)$ exists, then it is uniquely determined.

Proof. In this situation again the boundary condition $\alpha(1,x)=a(x)$ is equivalent to $a(x)E(x)=e^{\mu}E(p(x))$. First we compute $E(p(x))$ which is equal to

$$\sum_{n\geq 0} e_n[p(x)]^n = \sum_{n\geq 0} e_n[x+c_kx^k+\dots]^n =$$

$$\sum_{n\geq 0} e_n(x^n+nc_kx^{n-1+k}+\dots) =$$

$$\sum_{n=0}^{k-1} e_n x^n + \sum_{n\geq k}\Big(e_n+(n-k+1)e_{n-k+1}c_k$$

$$+R_{n-k+1}(e_1,\dots,e_{n-k})\Big)x^n,$$

where $R_{n-k+1}=R_{n-k+1}(e_1,\dots,e_{n-k})$ are universal polynomials in $e_1,\dots,e_{n-k}$, and $R_1=0$. Hence, α satisfies the boundary condition if and only if

$$\sum_{n\geq 0}\left(\sum_{r=0}^{n} a_{n-r}e_r\right)x^n = e^{\mu}\sum_{n=0}^{k-1} e_n x^n$$

$$+e^{\mu}\sum_{n\geq k}\left(e_n+(n-k+1)e_{n-k+1}c_k+R_{n-k+1}\right)x^n.$$

Comparing coefficients on both sides, we derive for $n=0$ that $a_0=e^{\mu}$, since $e_0=1$, hence $\mu=\ln a_0$. Then for $1\leq n<k$ the coefficient $a_n=0$, since

$$\sum_{r=0}^{n-1} a_{n-r}e_r + a_0e_n = e^{\mu}e_n$$

is equivalent to

$$a_n+\sum_{r=1}^{n-1} a_{n-r}e_r = 0.$$

Hence, recursively we get

$$a_n = -\sum_{r=1}^{n-1} a_{n-r}e_r = 0.$$

Finally, for $n \geq k$, write n as $k+j$ for $j \geq 0$. Comparison of coefficients of x^{k+j} yields

$$a_0 e_{k+j} + \sum_{s=k}^{k+j} a_s e_{k+j-s} =$$

$$e^{\mu}(e_{k+j} + (j+1)e_{j+1}c_k + R_{j+1}),$$

which reduces to

$$\sum_{r=0}^{j} a_{k+j-r}e_r = e^{\mu}((j+1)e_{j+1}c_k + R_{j+1}).$$

From this formula the coefficients e_{j+1} can uniquely be determined by

$$e_{j+1} = \frac{e^{-\mu}\sum_{r=0}^{j} a_{k+j-r}e_r - R_{j+1}}{(j+1)c_k}.$$

■

In order to deal with the general form of α, let $P_{n,\kappa}(s,x)$ be given by

$$P_{n,\kappa}(s,x) := \left(\exp\int_0^s \pi(\sigma,x)^n d\sigma\right)^{\kappa} =$$

$$\exp\left(\kappa\int_0^s \pi(\sigma,x)^n d\sigma\right)$$

for $1 \leq n < k$. Then $P(s,x) = \prod_{n=1}^{k-1} P_{n,\kappa_n}(s,x)$.

Lemma 3.7. *Let $(\pi(s,x))_{s\in\mathbb{C}}$ be an analytic iteration group of the second type, $\kappa \in \mathbb{C}$, and assume that $1 \leq n < k$. Then $P_{n,\kappa}(s,x) = 1 + \kappa s x^n + \dots$.*

Proof. Computing the first coefficients, we get

$$\pi(\sigma,x)^n = x^n + nc_k\sigma x^{n-1+k} + \dots,$$

hence

$$\kappa\int_0^s \pi(\sigma,x)^n d\sigma = \kappa s x^n + \dots,$$

and consequently $P_{n,\kappa}(s,x)$ is of the given form. ■

Standard computations can be used in order to prove

Lemma 3.8. *Writing the series $P(s,x)$, which is the product $\prod_{n=1}^{k-1} P_{n,\kappa_n}(s,x)$, in the form*

$$P(s,x) = \sum_{n\geq 0} p_n(s)x^n,$$

then

$$p_n(s) = \begin{cases} 1, & n = 0 \\ \kappa_1 s, & n = 1 \\ \kappa_n s + q_n(\kappa_1,\dots,\kappa_{n-1},s), & 2 \leq n < k \end{cases}$$

where $q_n(\kappa_1,\dots,\kappa_{n-1},s)$ is a polynomial in $\kappa_1,\dots,\kappa_{n-1}$ and s. From this explicit form of $p_n(s)$ for $1 \leq n < k$ it is possible to determine the vector of parameters $(\kappa_1,\dots,\kappa_{k-1})$ of a given polynomial $P(s,x)$ in a unique way.

Already at the very beginning of this article we realized that $\alpha_0(s) = e^{\mu s}$, hence $\alpha_0(1) = e^{\mu} = a_0$. Consequently, it is enough and also easier to adjust $\hat{\alpha}(s,x) := e^{-\mu s}\alpha(s,x)$ to the boundary condition $\hat{\alpha}(1,x) = \hat{a}(x) := e^{-\mu}a(x)$. The main idea is formulated in the next

Lemma 3.9. *Let $\hat{a}(x) = 1 + \sum_{n\geq n_0} \hat{a}_n x^n$ for $1 \leq n_0 < k$. Then there exists exactly one $P_{\nu,\kappa}(s,x)$ such that*

$$\frac{\hat{a}(x)}{P_{\nu,\kappa}(1,x)} \equiv 1 \bmod x^{n_0+1}.$$

Proof. When we choose $\nu = n_0$ and $\kappa = \hat{a}_{n_0}$, then it is clear from Lemma 3.7 that $\hat{a}(x) \equiv P_{\nu,\kappa}(1,x) \bmod x^{n_0+1}$. In order to prove that $P_{\nu,\kappa}(s,x)$ is uniquely defined, assume that there exists a series $P_{\nu',\kappa'}(s,x)$ such that $\hat{a}(x) \equiv P_{\nu',\kappa'}(1,x) \bmod x^{n_0+1}$, then $\mathrm{ord}(\hat{a}(x) - P_{\nu',\kappa'}(1,x)) \geq n_0 + 1$. Hence, $P_{\nu',\kappa'}(1,x)$ starts with $1 + \hat{a}_{n_0}x^{n_0}$. Consequently, $\nu' = n_0 = \nu$ and $\kappa' = \hat{a}_{n_0} = \kappa$ by Lemma 3.7. Hence, $P_{\nu',\kappa'}(s,x) = P_{\nu,\kappa}(s,x)$. ■

From this lemma it is obvious that

$$\hat{a}(x) \equiv P_{n_0,\hat{a}_{n_0}}(1,x) \bmod x^{n_0+1}.$$

Now we can adjust the general solution α given in the last part of Theorem 2.6 to the boundary condition.

Theorem 3.10. *Let $\hat{a}(x) = 1 + \sum_{n\geq 1} \hat{a}_n x^n$ be a given formal power series of order 0, and assume that π is an analytic iteration group of the second type. Then there exists exactly one solution*

$$\hat{\alpha}(s,x) = P(s,x)\frac{E(\pi(s,x))}{E(x)}$$

of $(Co1)$ with $E(x) = 1 + e_1 x + \dots$ which also satisfies the boundary condition $\hat{\alpha}(1,x) = \hat{a}(x)$.

Proof. According to Lemma 3.9, there exists exactly one $P_{1,\kappa_1}(s,x)$ such that

$$\frac{\hat{a}(x)}{P_{1,\kappa_1}(1,x)} \equiv 1 \bmod x^2.$$

Assume that recursively for $1 \leq n < k-1$ we found uniquely defined $P_{n,\kappa_n}(s,x)$ such that

$$\frac{\hat{a}(x)}{P_{1,\kappa_1}(1,x)\cdots P_{n,\kappa_n}(1,x)} \equiv 1 \bmod x^{n+1},$$

then by Lemma 3.9 there exists exactly one series $P_{n+1,\kappa_{n+1}}(s,x)$ such that

$$\frac{\hat{a}(x)}{P_{1,\kappa_1}(1,x)\cdots P_{n+1,\kappa_{n+1}}(1,x)} \equiv 1 \bmod x^{n+2}$$

holds. Hence, we end up with

$$\frac{\hat{a}(x)}{P_{1,\kappa_1}(1,x)\cdots P_{k-1,\kappa_{k-1}}(1,x)} \equiv 1 \bmod x^{k},$$

where $P_{1,\kappa_1}(s,x), \dots, P_{k-1,\kappa_{k-1}}(s,x)$ are uniquely determined. From Theorem 3.6 we deduce the existence of a uniquely determined formal power series $E(x) = 1 + e_1 x + \dots$ such that

$$\frac{\hat{a}(x)}{P_{1,\kappa_1}(1,x)\cdots P_{k-1,\kappa_{k-1}}(1,x)} = \frac{E(p(x))}{E(x)}.$$

Thus, $\hat{a}(x)$ can be written as

$$\hat{a}(x) = \prod_{n=1}^{k-1} P_{n,\kappa_n}(1,x)\frac{E(p(x))}{E(x)},$$

where $E(x)$ and $P_{n,\kappa_n}(s,x)$ are uniquely determined for $1 \leq n < k$. From Lemma 3.8 it follows that there is exactly one vector of parameters of $P(s,x) := \prod_{n=1}^{k-1} P_{n,\kappa_n}(s,x)$, namely $(\kappa_1,\dots,\kappa_{k-1})$, hence

$$\hat{\alpha}(s,x) := P(s,x)\frac{E(\pi(s,x))}{E(x)}$$

is also uniquely determined by $\hat{a}(x)$. It is a solution of $(Co1)$, and it satisfies the boundary condition, which finishes the proof. ∎

Summarizing, we found the following result: To any given formal power series $a(x)$ of order 0 and any analytic iteration group π of the second type, there exist solutions

$$\alpha(s,x) = e^{\mu s}P(s,x)\frac{E(\pi(s,x))}{E(x)} \tag{15}$$

of $(Co1)$ with $E(x) = 1 + e_1 x + \dots$ which also satisfy the boundary condition $\alpha(1,x) = a(x)$.

Theorem 3.11. *Assume that π is an analytic iteration group of the second type. Let $a(x)$ and $b(x)$ be given formal power series,* $\operatorname{ord} a(x) = 0$*, and let α be a solution of $(Co1)$ of the form (15) which satisfies the boundary condition $(B2)$.*

1. *If $a_0 \neq 1$, then there exists exactly one*

$$\beta(s,x) = e^{\mu s}P(s,x)E(\pi(s,x)) \cdot \left[F(x) - e^{-\mu s}\frac{F(\pi(s,x))}{P(s,x)}\right]$$

which satisfies together with α the cocycle equation $(Co2)$ and the boundary condition $\beta(1,x) = b(x)$.

2. *Assume that $a_0 = 1$. If $a(x) = 1$, let $m_0 = k$, otherwise let m_0 be the smallest element in $\{n \in \mathbb{N} \mid a_n \neq 0\}$, and let $n_0 := \min\{m_0, k\}$.*

a) *First we assume that α is a solution of $(Co1)$ with $\mu \neq 0$.*

If $n_0 = k$, then there exist families β of the above form with $P(s,x) = 1$ which satisfy together with α the cocycle equation $(Co2)$ and the boundary condition $\beta(1,x) = b(x)$ if and only if $b_n = 0$ for all $0 \leq n < k$. However, β is not uniquely determined.

If $n_0 < k-1$, or $\left[n_0 = k-1 \text{ and } \kappa_{k-1} - nc_k \neq 0 \text{ for all } n \in \mathbb{N}\right]$, then there exists a family β of the above form, which satisfies together with α the cocycle equation $(Co2)$ and the boundary condition $\beta(1,x) = b(x)$, if and only if $b_n = 0$ for all $0 \leq n < n_0$. If β exists, then it is uniquely determined.

If $n_0 = k-1$ and $\kappa_{k-1} = n_1 c_k$ for $n_1 \in \mathbb{N}$, then there exists a family β of the above form which satisfies together with α the cocycle equation $(Co2)$ and the boundary condition $\beta(1,x) = b(x)$, if and only if $b_n = 0$ for all

$0 \le n < n_0$ and b_{n_1+k-1} satisfies a condition which is implicitly given in the proof. However, β is not uniquely determined.

b) *Finally we assume that α is a solution of $(Co1)$ with $\mu = 0$.*

If $n_0 = k$, or $n_0 < k-1$ or $\left[n_0 = k-1 \text{ and } \kappa_{k-1} - nc_k \neq 0 \text{ for all } n \in \mathbb{N}\right]$, then there exists exactly one family β of the form

$$\beta(s,x) = P(s,x)E(\pi(s,x)) \cdot \left[F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x)\right]$$

where

$$Q(s,x) = \sum_{n=0}^{n_0-1} \int_0^s \frac{\ell_n \pi(\sigma,x)^n}{P(\sigma,x)E(\pi(\sigma,x))} d\sigma$$

which satisfies together with α the cocycle equation $(Co2)$ and the boundary condition $\beta(1,x) = b(x)$.

If $n_0 = k-1$ and $\kappa_{k-1} = n_1 c_k$ for $n_1 \in \mathbb{N}$, then there exists a family $\beta(s,x)$ of the form

$$E(\pi(s,x))P(s,x)\left[F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x) + \ell''_{n_1+n_0} \int_0^s \frac{\pi(\sigma,x)^{n_1+k-1}}{P(\sigma,x)} d\sigma\right]$$

which satisfies together with α the cocycle equation $(Co2)$ and the boundary condition $\beta(1,x) = b(x)$. However, β is not uniquely determined.

Proof. Writing β as indicated in the first part of this theorem, the condition $b(x) = \beta(1,x)$ is equivalent to

$$\frac{b(x)}{E(p(x))} = e^{\mu} P(1,x)F(x) - F(p(x)). \tag{16}$$

From the proof of Theorem 3.6, we know that

$$E(p(x)) = \sum_{n\ge 0} e_n x^n + \sum_{n \ge k}\Big((n-k+1)e_{n-k+1}c_k + R_{n-k+1}\Big)x^n \equiv \sum_{n\ge 0} e_n x^n \bmod x^k.$$

If we denote $b(x)/E(p(x))$ by $\sum_{n\ge 0} \tilde{b}_n x^n$, then

$$\left(\sum_{n\ge 0} \tilde{b}_n x^n\right) E(p(x)) = \sum_{n\ge 0} b_n x^n.$$

Hence,

$$\sum_{n\ge 0} b_n x^n \equiv \sum_{n\ge 0}\left(\sum_{r=0}^{n} \tilde{b}_r e_{n-r}\right) x^n \bmod x^k,$$

and the coefficients b_n are uniquely determined by the $\tilde{b}_n$ for $0 \le n < k$.

Using the notation of Lemma 3.8 for the coefficients of $P(s,x)$, condition (16) can be written as

$$\sum_{n\ge 0} \tilde{b}_n x^n = e^{\mu}\left(\sum_{n\ge 0} p_n(1)x^n\right)\left(\sum_{n\ge 0} f_n x^n\right) - \sum_{n\ge 0} f_n [p(x)]^n =$$
$$e^{\mu} \sum_{n\ge 0}\left(\sum_{r=0}^{n} p_r(1) f_{n-r}\right) x^n - \sum_{n\ge 0} f_n x^n - \sum_{n\ge k}\left((n-k+1)f_{n-k+1}c_k + S_{n-k+1}\right)x^n,$$

where $S_{n-k+1}(f_0,\dots,f_{n-k})$ are universal polynomials in $f_0,\dots,f_{n-k}$. Comparing coefficients yields

$$\tilde{b}_n = e^{\mu}\left(f_n + \sum_{r=1}^{n} p_r(1) f_{n-r}\right) - f_n \tag{17}$$

for $0 \le n < k$, and

$$\tilde{b}_n = e^{\mu}\left(f_n + \sum_{r=1}^{n} p_r(1) f_{n-r}\right) - f_n - (n-k+1)f_{n-k+1}c_k - S_{n-k+1} \tag{18}$$

for $n \ge k$.

In **case 1** we assume that $a_0 = e^{\mu} \neq 1$. Then $\mu \neq 0$, and f_n is uniquely determined by

$$f_n = \frac{\tilde{b}_n - e^{\mu}\sum_{r=1}^{n} p_r(1) f_{n-r}}{e^{\mu} - 1}$$

for $0 \le n < k$, and

$$f_n = (e^{\mu}-1)^{-1}\left(\tilde{b}_n - e^{\mu}\sum_{r=1}^{n} p_r(1) f_{n-r} + (n-k+1)f_{n-k+1}c_k + S_{n-k+1}\right)$$

for $n \ge k$.

In **case 2** we assume that $a_0 = e^\mu = 1$ and $\mu \neq 0$. From the definition of n_0 we deduce that $\kappa_n = 0$ for $1 \leq n < n_0$, and by Lemma 3.8 also $p_n(s) = 0$ for $1 \leq n < n_0$. If moreover $n_0 < k$, then $p_{n_0}(s) = \kappa_{n_0} s$ and $\kappa_{n_0} \neq 0$. For $n = 0$ we deduce that $\tilde{b}_0 = 0$ and recursively $\tilde{b}_n = 0$ for all $1 \leq n < n_0$. Hence, $b_n = 0$ for $0 \leq n < n_0$.

Case 2.1: If $n_0 = k$, then $a_1 = \cdots = a_{k-1} = 0$ and according to Theorem 3.6 there exists a family α of the form (15) satisfying $(Co1)$ and $(B2)$ with $P(s,x) = 1$. Thus, (16) reduces to

$$\frac{b(x)}{E(p(x))} = F(x) - F(p(x)).$$

Introducing coefficients, this is

$$\sum_{n\geq 0} \tilde{b}_n x^n = \sum_{n\geq 0} f_n x^n - \sum_{n\geq 0} f_n x^n - \sum_{n\geq k} \Big((n-k+1) f_{n-k+1} c_k + S_{n-k+1} \Big) x^n.$$

Hence,

$$\sum_{n\geq 0} \tilde{b}_n x^n = -\sum_{n\geq k} \Big((n-k+1) f_{n-k+1} c_k + S_{n-k+1} \Big) x^n.$$

By comparing coefficients we get $\tilde{b}_n = 0$ for $0 \leq n < k$ and

$$\tilde{b}_n = -\Big((n-k+1) f_{n-k+1} c_k + S_{n-k+1} \Big)$$

for $n \geq k$. This formula allows to compute f_{n-k+1} recursively for $n \geq k$, whence f_n are uniquely determined for $n \geq 1$, whereas f_0 is not determined and can be chosen arbitrarily in $\mathbb{C}$.

Case 2.2: If $n_0 < k$, then (17) means

$$\tilde{b}_n = \sum_{r=n_0}^{n} p_r(1) f_{n-r} = \kappa_{n_0} f_{n-n_0} + \sum_{r=n_0+1}^{n} p_r(1) f_{n-r}$$

for $n_0 \leq n < k$. Hence, f_{n-n_0} is uniquely determined by

$$f_{n-n_0} = \frac{\tilde{b}_n - \sum_{r=n_0+1}^{n} p_r(1) f_{n-r}}{\kappa_{n_0}}$$

for $n_0 \leq n < k$.

Finally, assume that $n \geq k$. **Case 2.2.1:** If $n_0 < k-1$ (which is equivalent to $n - n_0 > n-k+1$), then from (18) we deduce that $\tilde{b}_n$ equals

$$\sum_{r=n_0}^{n} p_r(1) f_{n-r} - (n-k+1) f_{n-k+1} c_k - S_{n-k+1},$$

whence

$$\kappa_{n_0} f_{n-n_0} = \tilde{b}_n - \sum_{r=n_0+1}^{n} p_r(1) f_{n-r} + (n-k+1) f_{n-k+1} c_k + S_{n-k+1},$$

which allows to determine f_{n-n_0} for $n \geq k$.

Case 2.2.2: If $n_0 = k-1$, then

$$\begin{aligned}\tilde{b}_n &= \kappa_{k-1} f_{n-k+1} + \sum_{r=k}^{n} p_r(1) f_{n-r} \\ &\quad - (n-k+1) f_{n-k+1} c_k - S_{n-k+1} \\ &= (\kappa_{k-1} - (n-k+1) c_k) f_{n-k+1} \\ &\quad + \sum_{r=k}^{n} p_r(1) f_{n-r} - S_{n-k+1}.\end{aligned}$$

Case 2.2.2.1: If $\kappa_{k-1} \neq m c_k$ for all $m \in \mathbb{N}$, then f_{n-k+1} is uniquely determined by

$$f_{n-k+1} = \frac{\tilde{b}_n - \sum_{r=k}^{n} p_r(1) f_{n-r} + S_{n-k+1}}{\kappa_{k-1} - (n-k+1) c_k} \tag{19}$$

for all $n \geq k$.

Finally in the **case 2.2.2.2** we deal with the computation of f_n for $n \geq k$, where $a_0 = 1$, $\mu \neq 0$, $n_0 = k-1$, and $\kappa_{k-1} = n_1 c_k$ for $n_1 \in \mathbb{N}$. Then for $n = n_1 + k - 1$, which is equivalent to $n - k + 1 = n_1$, we get

$$\tilde{b}_{n_1+k-1} = \sum_{r=k}^{n_1+k-1} p_r(1) f_{n_1+k-1-r} - S_{n_1}. \tag{20}$$

This is a necessary condition for $\tilde{b}_{n_1+k-1}$ in order to guarantee that $\beta(s,x)$ can be adjusted to $(B2)$. Hence, b_{n_1+k-1} must satisfy a corresponding condition. If it is satisfied, then the coefficient f_{n_1} of $F(x)$ can be chosen arbitrarily in $\mathbb{C}$, and the coefficients f_{n-k+1} for $n \geq k$ and $n \neq n_1 + k - 1$ are uniquely determined by (19).

Case 3: Now we assume that $a_0 = 1$ and $\mu = 0$. The general solution $\beta(s,x)$ of $(Co2)$ is given at the beginning of 2.b) in the present theorem. Let $\tilde{Q}(s,x)$ denote the product $P(s,x)E(\pi(s,x))Q(s,x)$. It is easy to prove that

$$\tilde{Q}(s,x) \equiv \sum_{n=0}^{n_0-1} \tilde{\ell}_n s x^n \bmod x^{n_0}$$

where $\tilde{\ell}_n = \ell_n + Z_n(\ell_0, \ldots, \ell_{n-1})$ with polynomials Z_n in ℓ_j. From Theorem 2.8 it follows that $\tilde{Q}(s,x)$

is a solution of $(Co2)$. From $b(x) = \beta(1,x)$, we deduce that $\tilde{\ell}_n = b_n$ for $0 \le n < n_0$. If we put $\delta(s,x) = \beta(s,x) - \tilde{Q}(s,x)$, then $\delta(s,x)$ is also a solution of $(Co2)$ with $\operatorname{ord}\delta(s,x) \ge n_0$. It can be expressed as

$$P(s,x)E(\pi(s,x))\left[F(x) - \frac{F(\pi(s,x))}{P(s,x)}\right].$$

The coefficients ℓ_n of $Q(s,x)$ are uniquely determined by $b(x)$. In order to satisfy $(B2)$ we have to find a series $F(x)$, determining $\delta(s,x)$, such that

$$\delta(1,x) = b(x) - \sum_{n=0}^{n_0-1} b_n x^n.$$

Similar as in case 2 it follows that for $n_0 < k-1$ or $\left[n_0 = k-1 \text{ and } \kappa_{k-1} \ne nc_k \text{ for all } n \in \mathbb{N}\right]$ there exists exactly one family δ which satisfies together with α the cocycle equation $(Co2)$ and the boundary condition from above. Thus, $\beta(s,x) = \delta(s,x) + \tilde{Q}(s,x)$ is uniquely determined.

If $n_0 = k$, then similarly to case 2 only the coefficient f_0 of the series $F(x)$ is not uniquely determined. But since $\mu = 0$, this coefficient does not influence δ. Hence also in this situation δ and consequently β are uniquely determined.

Finally, if $n_0 = k-1$ and $\kappa_{k-1} = n_1 c_k$ for $n_1 \in \mathbb{N}$, then according to the last part of Theorem 2.8 we are allowed to add

$$E(\pi(s,x))P(s,x)\ell''_{n_1+n_0}\int_0^s \frac{\pi(\sigma,x)^{n_1+n_0}}{P(\sigma,x)}d\sigma$$

to the general solution $\beta(s,x)$, where $\ell''_{n_1+n_0}$ is an arbitrary complex number. By doing this it is possible to skip the additional condition (20) for $\tilde{b}_{n_1+k-1}$ which occurred in case 2.2.2.2. In order to compute the integral from above we derive

$$\pi(\sigma,x)^{n_1+n_0} =$$
$$x^{n_1+k-1} + (n_1+k-1)c_k\sigma x^{n_1+2(k-1)} + \dots$$

and

$$\frac{\pi(\sigma,x)^{n_1+n_0}}{P(\sigma,x)} =$$
$$\frac{x^{n_1+k-1} + (n_1+k-1)c_k\sigma x^{n_1+2(k-1)} + \dots}{1 + \kappa_{k-1}\sigma x^{k-1} + \dots} =$$
$$x^{n_1+k-1} + ((n_1+k-1)c_k - \kappa_{k-1})\sigma x^{n_1+2(k-1)} + \dots,$$

hence

$$\int_0^s \frac{\pi(\sigma,x)^{n_1+n_0}}{P(\sigma,x)}d\sigma =$$
$$sx^{n_1+k-1} + \frac{n_0 c_k}{2}s^2 x^{n_1+2(k-1)} + \dots.$$

Consequently, for $s = 1$ we get

$$P(1,x)\ell''_{n_1+n_0}\int_0^1 \frac{\pi(\sigma,x)^{n_1+n_0}}{P(\sigma,x)}d\sigma \equiv$$
$$\ell''_{n_1+k-1}x^{n_1+k-1} \bmod x^{n_1+k}.$$

We indicate this formal power series by

$$\sum_{n \ge n_1+k-1} q_n x^n,$$

thus $q_{n_1+k-1} = \ell''_{n_1+k-1}$.

For $k \le n < n_1+k-1$ the coefficient f_{n-k+1} is uniquely determined by (19). For $n = n_1+k-1$ we determine ℓ''_{n_1+k-1} by

$$\ell''_{n_1+k-1} = \tilde{b}_{n_1+k-1} - \sum_{r=k}^{n_1+k-1} p_r(1)f_{n_1+k-1-r} + S_{n_1}$$

and f_{n_1} can be chosen arbitrarily in $\mathbb{C}$. Finally, for $n > n_1+k-1$ the coefficients f_{n-k+1} is uniquely given by

$$f_{n-k+1} = \frac{\tilde{b}_n - \sum_{r=k}^n p_r(1)f_{n-r} + S_{n-k+1} - q_n}{\kappa_{k-1} - (n-k+1)c_k}.$$

∎

In the case $a_0 = 1$, the condition $b_n = 0$ for all $0 \le n < n_0$ is also a necessary condition for the existence of a solution $\varphi(x)$ of (L). This fact is shown in the next

Lemma 3.12. *Let $p(x) = x + c_k x^k + \dots$ for $k \ge 2$ and $c_k \ne 0$, and assume that $a(x) = 1$ (then set $m_0 := k$) or $a(x) = 1 + a_{m_0}x^{m_0} + \dots$ for $1 \le m_0$ and $a_{m_0} \ne 0$. If $\varphi(x) \in \mathbb{C}[\![x]\!]$ is a solution of (L), then $b_n = 0$ for $0 \le n < n_0 := \min\{m_0, k\}$.*

Proof. Elementary computations yield

$$\varphi(p(x)) = \sum_{n \ge 0}\varphi_n[p(x)]^n =$$
$$\sum_{n \ge 0}\varphi_n(x^n + nc_k x^{n-1+k} + \dots) \equiv$$

$$\sum_{n\geq 0} \varphi_n x^n \bmod x^k,$$

and

$$a(x)\varphi(x)+b(x)=\sum_{n\geq 0}\left(\sum_{r=0}^{n} a_r\varphi_{n-r}+b_n\right)x^n=$$

$$\sum_{n\geq 0}\left(\varphi_n+\sum_{r=m_0}^{n} a_r\varphi_{n-r}+b_n\right)x^n.$$

Hence, comparing coefficients of x^n on the left and on the right side of (L) yields

$$\varphi_n=\varphi_n+b_n$$

for $0\leq n<n_0$, thus $b_n=0$. ∎

4. Solution of the problem of covariant embeddings in certain special cases

In this section we give in Theorem 4.1 a necessary condition that a given linear functional equation (L) (with a non-empty set of solutions) has an embedding with respect to a given analytic iteration group of $p(x)$. In Corollary 4.2 we present, as a consequence, a rather large class of such embeddings. However, there remain some special cases of solutions (α,β) of the system $((Co1),(Co2))$ and the boundary conditions corresponding to (L) where the existence of an embedding is still open.

Theorem 4.1. *Assume that the linear functional equation* (L) *has a solution* $\varphi(x)\in\mathbb{C}[\![x]\!]$, *and let* $(\pi(s,x))_{s\in\mathbb{C}}$ *be an analytic iteration group of* $p(x)$. *Furthermore, assume that* α *satisfies* $(Co1)$ *and the two boundary conditions* $(B1)$ *and* $(B2)$. *If there exists exactly one* β *which also satisfies* $(B1)$ *and* $(B2)$, *such that* (α,β) *is a solution of* $(Co2)$, *then there exists an embedding of* (L) *with respect to the iteration group* $(\pi(s,x))_{s\in\mathbb{C}}$.

Proof. Let φ be a solution of (L). Then $\Phi_\varphi(s,x)$ defined by

$$\Phi_\varphi(s,x):=\varphi(\pi(s,x))-\alpha(s,x)\varphi(x)$$

satisfies both $\Phi_\varphi(0,x)=\varphi(x)-1\varphi(x)=0$ and $\Phi_\varphi(1,x)=\varphi(p(x))-a(x)\varphi(x)=b(x)$. Furthermore, the pair (α,Φ_φ) satisfies $(Co2)$, since

$$\Phi_\varphi(t+s,x)=\varphi(\pi(t+s,x))-\alpha(t+s,x)\varphi(x)=$$

$$\varphi(\pi(t,\pi(s,x)))-\alpha(s,x)\alpha(t,\pi(s,x))\varphi(x)=$$

$$\Phi_\varphi(t,\pi(s,x))+\alpha(t,\pi(s,x))\varphi(\pi(s,x))$$

$$-\alpha(s,x)\alpha(t,\pi(s,x))\varphi(x)=$$

$$\Phi_\varphi(t,\pi(s,x))+\alpha(t,\pi(s,x))$$

$$\cdot[\Phi_\varphi(s,x)+\alpha(s,x)\varphi(x)-\alpha(s,x)\varphi(x)]=$$

$$\Phi_\varphi(t,\pi(s,x))+\alpha(t,\pi(s,x))\Phi_\varphi(s,x).$$

In other words, Φ_φ satisfies the same conditions as β, i.e., $(B1)$, $(B2)$ and together with α the cocycle equation $(Co2)$. Hence, since there exists exactly one β with these properties, $\Phi_\varphi(s,x)=\beta(s,x)$ for all $s\in\mathbb{C}$ and all solutions $\varphi(x)$ of (L). ∎

Combining this result with Theorem 3.5 and Theorem 3.11, we get

Corollary 4.2. *If* $\pi(s,x)=e^{\lambda s}x$ *is an analytic iteration group of the first type, and* e^λ *is not a complex root of* 1, *then there exists an embedding of* (L) *with respect to the iteration group* π.

Assume that $\pi(s,x)=x+c_ksx^k+\dots$ *with* $k\geq 2$ *and* $c_k\neq 0$ *is an analytic iteration group of the second type. If* $a_0\neq 1$, *then there exists an embedding of* (L) *with respect to the iteration group* π. *Assume that* $a_0=1$. *If* $n_0<k-1$, *or* $n_0=k$, *or* $\left[n_0=k-1\right.$ *and* $\left.a_{k-1}\neq nc_k \text{ for all } n\in\mathbb{N}\right]$, *then there exists an embedding of* (L) *with respect to the iteration group* π.

If $p(x)=\rho x+c_2x^2+\dots$, where $\rho\neq 1$ is a complex root of 1, and $p(x)$ does not have an embedding in an analytic iteration group, then there exists no covariant embedding of the linear functional equation. If $p(x)$ has an embedding, then it is still open whether there exists a covariant embedding of (L). In addition to this the embedding problem is also still open for analytic iteration groups π of the second type, when $\pi(s,x)=x+c_ksx^k+\dots$ with $k\geq 2$ and $c_k\neq 0$ and $a(x)=1+a_{k-1}x^{k-1}+\dots$ with $a_{k-1}=n_1c_k$ for $n_1\in\mathbb{N}$.

5. Acknowledgment

This paper has been supported by the "Fonds zur Förderung der Wissenschaftlichen Forschung P14342-MAT".

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