On covariant embeddings of a linear functional equation with respect to an analytic iteration group in some non-generic cases

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Abstract

In the paper *On covariant embeddings of a linear functional equation with respect to an analytic iteration group* (accepted for publication in the *International Journal of Bifurcation and Chaos*) the authors described the problem of covariant embeddings of a linear functional equation with respect to analytic iteration groups and solved it in the generic cases. However, some cases remained unsolved. In this paper we present the solution for these open cases.

1 Introduction

Let $\mathbb{C}\llbracket x \rrbracket$ be the ring of formal power series in the indeterminate x with complex coefficients. Consider the linear functional equation

$$\varphi(p(x)) = a(x)\varphi(x) + b(x), \tag{L}$$

where $p(x), a(x), b(x) \in \mathbb{C}\llbracket x \rrbracket$ are given formal power series, and $\varphi(x) \in \mathbb{C}\llbracket x \rrbracket$ should be determined by the functional equation. We always assume that

$$p(x) = \rho x + c_2x^2 + c_3x^3 + \ldots = \rho x + \sum_{n\geq 2} c_n x^n$$

with multiplier $\rho \neq 0$, and

$$a(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{n\geq 0} a_n x^n$$

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with $a_0 \neq 0$. For a foundation of the basic calculations with formal power series we refer the reader to [9] and to [1] or [2].

L. Reich introduced in [11] the notion of a covariant embedding of (L) with respect to an analytic iteration group.

The linear functional equation (L) *has a covariant embedding with respect to the analytic iteration group* $(\pi(s,x))_{s\in\mathbb{C}}$ *of* $p(x)$*, if there exist families* $(\alpha(s,x))_{s\in\mathbb{C}}$ *and* $(\beta(s,x))_{s\in\mathbb{C}}$ *of formal power series with entire coefficient functions* α_n *and* β_n *for all* $n \geq 0$

$$\alpha(s,x) = \sum_{n\geq 0} \alpha_n(s)x^n, \qquad \beta(s,x) = \sum_{n\geq 0} \beta_n(s)x^n$$

such that

$$\varphi(\pi(s,x)) = \alpha(s,x)\varphi(x) + \beta(s,x) \tag{Ls}$$

holds for all $s \in \mathbb{C}$ *and for all solutions* $\varphi(x)$ *of* (L) *in* $\mathbb{C}[\![x]\!]$*. Moreover, it is assumed that* α *and* β *satisfy both the boundary conditions*

$$\alpha(0,x) = 1 \qquad \beta(0,x) = 0 \tag{B1}$$

$$\alpha(1,x) = a(x) \qquad \beta(1,x) = b(x) \tag{B2}$$

and the cocycle equations

$$\alpha(t+s,x) = \alpha(s,x)\alpha(t,\pi(s,x)) \tag{Co1}$$

$$\beta(t+s,x) = \beta(s,x)\alpha(t,\pi(s,x)) + \beta(t,\pi(s,x)) \tag{Co2}$$

for all $s,t \in \mathbb{C}$.

In [3] the general solutions of (Co1) and of the system ((Co1),(Co2)) are derived; then these functional equations are solved under the additional boundary conditions (B1) and (B2). Finally, the next theorem is proved as Theorem 4.1 in [3], which describes a sufficient condition for the existence of a covariant embedding of (L) with respect to an analytic iteration group.

Theorem 1. *Assume that the linear functional equation* (L) *has a solution* $\varphi(x) \in \mathbb{C}[\![x]\!]$*, and let* $(\pi(s,x))_{s\in\mathbb{C}}$ *be an analytic iteration group of* $p(x)$*. Furthermore, assume that* α *satisfies* (Co1) *and the two boundary conditions* (B1) *and* (B2)*. If there exists exactly one* β*, which also satisfies* (B1) *and* (B2)*, such that* (α,β) *is a solution of* (Co2)*, then there exists a covariant embedding of* (L) *with respect to the iteration group* $(\pi(s,x))_{s\in\mathbb{C}}$*.*

Covariant embeddings of a linear functional equation were studied in a much more general setting by Z. Moszner in [10] and for functions defined on a real interval by G. Guzik in [5] and [7]. The first cocycle equation is also studied in [6] and [8].

In [3] we called an analytic iteration group a *group of first type,* if $\pi(s,x)$ equals $S^{-1}(e^{\lambda s}S(x))$ for all $s \in \mathbb{C}$, where $\lambda \in \mathbb{C}\setminus\{0\}$ and $S(x) = x + s_2x^2 + \ldots$.

Each iteration group of this type is simultaneously conjugate to the iteration group $(e^{\lambda s}x)_{s\in\mathbb{C}}$. Iteration groups of the form $\pi(s,x) = x + c_k s x^k + P_{k+1}^{(k)}(s)x^{k+1} + \ldots$ for all $s \in \mathbb{C}$, where $c_k \neq 0$, $k \geq 2$, and $P_r^{(k)}(s)$ are polynomials in s for $r > k$, were called *iteration groups of second type.* Each non-trivial analytic iteration group is either of first or of second type.

In [3] the problem of the existence of a covariant embedding was solved in the generic cases. There we proved in Theorem 3.5, Theorem 3.11, and Corollary 4.2 the following facts.

1. Assume that $(\pi(s,x))_{s\in\mathbb{C}}$ is an analytic iteration group of the first type which is simultaneously conjugate to $e^{\lambda s}x$, where e^{λ} is not a complex root of 1. Then there exist suitable solutions (α,β) of the system consisting of (Co1), (Co2), (B1), and (B2) which yield covariant embeddings of (L) with respect to π.

2. Assume that $(\pi(s,x))_{s\in\mathbb{C}}$ is an analytic iteration group of the second type. If the coefficient a_0 of $a(x)$ is different from 1, then there exist covariant embeddings of (L) with respect to the iteration group π. Assume that $a_0 = 1$. Depending on the coefficient function α_0 of $\alpha(s,x)$, a solution of (Co1), we have: If $a(x) = 1$ then there exists a covariant embedding (α,β) with $\alpha_0 = 1$. Finally assume that $a(x) = 1 + \sum_{n\geq n_0} a_n x^n$ with $a_{n_0} \neq 0$. If $\alpha_0(s) = e^{\mu s}$ for some $\mu \in 2\pi i\mathbb{Z}\setminus\{0\}$ and $\Big[n_0 < k-1$, or $\big[n_0 = k-1$ and $a_{k-1} \neq nc_k$ for all $n \in \mathbb{N}\big]\Big]$[1], then there exist covariant embeddings (α,β) of (L) with respect to the iteration group π, if and only if $b_n = 0$ for all $0 \leq n < n_0$. If $\alpha_0 = 1$ and $\Big[n_0 \neq k-1$, or $\big[n_0 = k-1$ and $a_{k-1} \neq nc_k$ for all $n \in \mathbb{N}\big]\Big]$, then there exists a covariant embedding (α,β) of (L) with respect to the iteration group π.

The following cases were not investigated so far.

The iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ of the first type is given as

$$\pi(s,x) = S^{-1}(e^{\lambda s}S(x)), \qquad s \in \mathbb{C},$$

where $\lambda \neq 0$ and $\rho := e^{\lambda}$ is a complex root of 1, primitive of order $j_0 > 1$. As it was shown in [3] this situation can always be simplified to $\pi(s,x) = e^{\lambda s}x$, which yields $p(x) = \rho x$. (1)

[1]We use square brackets $[\ldots]$ in order to indicate the logical structure.

The iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ of the first type is given as

$$\pi(s,x)=S^{-1}(e^{\lambda s}S(x)), \qquad s\in\mathbb{C},$$

where $\rho:=e^{\lambda}$ is not a complex root of 1. As it was shown in [3] this situation can always be simplified to $\pi(s,x)=e^{\lambda s}x$, which yields $p(x)=\rho x$.
Disregarding the covariant embedding (α,β) of (L) described in Theorem 3.5 of [3], do there exist further solutions (α,β) of (Co1), (Co2), (B1), and (B2) which are covariant embeddings of (L)? (2)

The series $p(x)$, $a(x)$ and $b(x)$ are of the form

$$p(x)=x+c_kx^k+P_{k+1}^{(k)}(1)x^{k+1}+\dots, \qquad k\geq 2,\ c_k\neq 0,$$

$$a(x)=1+\sum_{n\geq k-1}a_nx^n, \qquad a_{k-1}=n_1c_k,\ n_1\in\mathbb{N},$$

$$b(x)=\sum_{n\geq 0}b_nx^n,$$

where $p(x)$ is embedded into an analytic iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ with

$$\pi(s,x)=x+c_ksx+P_{k+1}^{(k)}(s)x^{k+1}+\dots,$$

an analytic iteration group of the second type. (3)

The series $p(x)$, $a(x)$ and $b(x)$ are of the form

$$p(x)=x+c_kx^k+P_{k+1}^{(k)}(1)x^{k+1}+\dots, \qquad k\geq 2,\ c_k\neq 0,$$

$$a(x)=1 \quad\text{or}\quad a(x)=1+\sum_{n\geq k}a_nx^n,$$

$$b(x)=\sum_{n\geq 0}b_nx^n,$$

where $p(x)$ is embedded into an analytic iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ with

$$\pi(s,x)=x+c_ksx+P_{k+1}^{(k)}(s)x^{k+1}+\dots,$$

an analytic iteration group of the second type. Disregarding the covariant embeddings (α,β) described in [3] with $\alpha_0=1$, what about other covariant embeddings (α,β) with $\alpha_0(s)=e^{\mu s}$ for some $\mu\in 2\pi i\mathbb{Z}\setminus\{0\}$? (4)

Whereas in [3] the structure of the set of solutions of (L) did not play an explicit role, here it will be of importance. The set of solutions of (L) in the special case (1) was investigated in [4], in a few other situations it will be dealt with in the present paper. We will list the covariant embeddings of (L) in the various cases at the very end of this paper.

2 The non-generic situations for iteration groups of type 1

In Theorem 2.6 of [3], we proved that for analytic iteration groups $\pi(s,x) = e^{\lambda s}x$ of the first type the general solution α of (Co1) is given by

$$\alpha(s,x) = e^{\mu s}\frac{E(e^{\lambda s}x)}{E(x)}, \tag{5}$$

where $e^{\mu} = a_0$ and $E(x) = 1 + e_1x + \ldots \in \mathbb{C}\,[\![x]\!]$. If $\rho = e^{\lambda}$ is not a complex root of 1, then according to Theorem 3.2 of [3], for each $a(x)$ there exist solutions α of (Co1) which satisfy (B2). (The series $E(x)$ is uniquely determined, whereas μ can be any logarithm of a_0.) If ρ is a complex root of 1, then we gave in the same theorem a necessary and sufficient condition for the existence of solutions α of (Co1) also satisfying the boundary condition (B2).

2.A ρ a complex root of 1 primitive of order $j_0 > 1$

Let us first deal with the problem described in (1), hence ρ is a complex root of 1 primitive of order $j_0 > 1$. Here we present another characterization for the existence of solutions α of (Co1) which also satisfy (B2). Using the explicit form of α given above, the boundary condition (B2) for α is

$$a_0\frac{E(\rho x)}{E(x)} = a(x),$$

which is equivalent to

$$E(\rho x) = \hat{a}(x)E(x), \tag{6}$$

where

$$\hat{a}(x) := \frac{a(x)}{a_0}. \tag{7}$$

This is a homogeneous linear functional equation for the unknown series $E(x) = 1 + e_1x + \ldots$. Consequently, there exists a solution α of (Co1) and (B2) if and only if (6) has a non-trivial solution. According to Theorem 5 of [4], there exist non-trivial solutions $E(x)$ of (6) if and only if

$$\prod_{\ell=0}^{j_0-1}\hat{a}(\rho^{\ell}x) = 1. \tag{8}$$

Moreover, from Lemma 2 and Theorem 12 or Theorem 24 of [4] we know that the homogeneous linear functional equation

$$\varphi(\rho x) = a(x)\varphi(x) \tag{$\mathrm{L_h}$}$$

has non-trivial solutions if and only if $a_0^{j_0} = 1$ and (8) is satisfied.

Case 1: If there is no α which satisfies (Co1) and (B2), then there does not exist a covariant embedding of (L). For that reason, we assume now in **case 2** that α is a solution of (Co1) satisfying (B2), whence (8) is also satisfied. In order to determine all β satisfying (B2) such that (α, β) is a solution of (Co2), in [3] we introduced the set $K = K(\lambda, \mu) := \{n \in \mathbb{N}_0 \mid \mu - n\lambda \in 2\pi i\mathbb{Z}\}$. Actually, this set does not depend on the particular choice of the values $\lambda = \ln \rho$ and $\mu = \ln a_0$, consequently $K = \{n \in \mathbb{N}_0 \mid \rho^n = a_0\}$. **Case 2.1:** If $a_0^{j_0} \neq 1$, then $K = \emptyset$, and it follows from Theorem 3.4 of [3] that there exists exactly one β of the form

$$\beta(s,x) = e^{\mu s} E(e^{\lambda s}x)[F(x) - e^{-\mu s} F(e^{\lambda s}x)] \tag{9}$$

with $F[x] \in \mathbb{C}[\![x]\!]$ which satisfies (B2) and together with α also the cocycle equation (Co1). Hence, we still have to deal with the **case 2.2** where $a_0^{j_0} = 1$, which means that $(\mathrm{L_h})$ has non-trivial solutions. It also implies that $K \neq \emptyset$, whence $K = k_0 + \mathbb{N}_0 j_0$ with $k_0 = \min K = \min \{n \in \mathbb{N}_0 \mid \rho^n = a_0\}$. In Theorem 2.8 of [3] the general solution (α, β) of (Co2) in the present situation was given by (5) and

$$\beta(s,x) = e^{\mu s} E(e^{\lambda s}x)[\ell_{n_0} s x^{n_0} + F(x) - e^{-\mu s} F(e^{\lambda s}x)], \tag{10}$$

where $n_0 \in \mathbb{N}_0$, $\ell_{n_0} \in \mathbb{C}$ and $F(x) \in \mathbb{C}[\![x]\!]$. Moreover, the summand $\ell_{n_0} x^{n_0}$ occurs only in the particular situation when $\mu = n_0\lambda$ for some $n_0 \in \mathbb{N}_0$.

Since, given ρ and a_0, we are still free to choose the values of μ and λ modulo integer multiples of $2\pi i$, the integer n_0 can be an arbitrary element of K. (For $k \in K$ there exists an integer z such that $\mu - k\lambda = 2z\pi i$, which is equivalent to $\mu' := \mu - 2z\pi i = k\lambda$. We used this choice of μ in order to construct covariant embeddings in [3].) On the other hand, there exist choices of μ and λ such that $\mu - n\lambda \neq 0$ for all $n \in \mathbb{N}_0$.

The boundary condition (B2) for the special form of β is

$$b(x) = a_0 E(\rho x)[\ell_{n_0} x^{n_0} + F(x) - \frac{1}{a_0} F(\rho x)]. \tag{11}$$

This can be rewritten as a linear functional equation for the power series $F(x)$, namely

$$F(\rho x) = a_0 F(x) + a_0 \ell_{n_0} x^{n_0} - \frac{b(x)}{E(\rho x)}. \tag{12}$$

Depending on ℓ_{n_0} we have to consider two cases. **Case 2.2.1:** Assume that $\ell_{n_0} = 0$ or $\mu \neq n\lambda$ for all $n \in \mathbb{N}_0$, thus $\ell_{n_0} x^{n_0}$ does not occur. Then it is convenient to introduce $G(x) := -E(x)F(x)$. By an application of (6), we derive from (11) that

$$G(\rho x) = a(x)G(x) + b(x). \tag{13}$$

Our assumptions on $a(x)$ already guarantee that the homogeneous linear equation $G(\rho x) = a(x)G(x)$ has non-trivial solutions. The necessary and sufficient condition on $b(x)$ for the existence of solutions of (L) (or (13)) are given in Theorem 15 of [4] as

$$\sum_{k=0}^{j_0-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} = 0. \tag{14}$$

Consequently, (L) has solutions if and only if (13) has solutions, which is equivalent to the existence of solutions (α, β) of (Co1), (Co2), and (B2).

Case 2.2.1.1: If (14) is not satisfied, then for any choice of α satisfying (Co1) and (B2) there does not exist a β such that (Co2) and (B2) are satisfied, thus there is no covariant embedding of (L). Hence, we assume in **case 2.2.1.2** that also (14) is satisfied, consequently (13) has solutions $G(x)$, which allow to compute the solutions $F(x)$ of (12) and the solutions (α, β) of (Co2) and (B2) via (10). Then according to Theorem 15 of [4], the set of solutions of (L) is given by the series

$$\varphi(x) = \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x)\right]^{-1} \left(j_0 \sum_{t\geq 0} \varphi_{tj_0} x^{tj_0} - \sum_{n=1}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \sum_{k=0}^{n-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} \right),$$

for any choice of $(\varphi_{tj_0})_{t\geq 0}$ in $\mathbb{C}$. We claim that in the present situation, i.e. in case 2.2.1.2, there does not exist a covariant embedding of (L). Assuming that there were a covariant embedding of (L), each solution of (L) also satisfies (Ls) for all $s \in \mathbb{C}$.

Choosing $s_0 \in \mathbb{C}$ such that $\tau := e^{\lambda s_0}$ is not a complex root of 1, we derive from (Ls)

$$\varphi(\tau x) = \alpha(s_0, x)\varphi(x) + \beta(s_0, x) \tag{Lτ}$$

The next lemma proves that the set of solutions of (Lτ) is much smaller than the set of solutions of (L) which gives a contradiction to the assumption that there exists a covariant embedding of (L) with respect to π. Smaller means here that for (L) we can choose the whole family $(\varphi_{tj_0})_{t\geq 0}$ of coefficients arbitrarily, whereas for (Lτ) at most one coefficient can be chosen arbitrarily.

Lemma 2. *If τ is not a complex root of 1, then the set of solutions of (Lτ) is either empty, or it consists of exactly one formal power series, or it is given by*

$$\left\{ \sum_{n\geq 0} \varphi_n x^n \mid \varphi_n = \frac{\sum_{r=1}^{n} \alpha_r(s_0)\varphi_{n-r} + \beta_n(s_0)}{\tau^n - \alpha_0(s_0)} \text{ for } n \neq n_1,\ \varphi_{n_1} \in \mathbb{C} \right\},$$

where $\tau^{n_1} = \alpha_0(s_0)$ for a uniquely determined $n_1 \in \mathbb{N}_0$.

Proof. Introducing coefficients of the series in (Lτ), we derive that $\varphi(x)$ satisfies (Lτ) if and only if

$$\varphi_n(\tau^n - \alpha_0(s_0)) = \sum_{r=1}^{n} \alpha_r(s_0)\varphi_{n-r} + \beta_n(s_0)$$

for all $n \in \mathbb{N}_0$. If $\tau^n \neq \alpha_0(s_0)$ for all $n \in \mathbb{N}_0$, then the coefficients of $\varphi(x)$ are uniquely determined, and there exists exactly one solution of (Lτ). Otherwise, there exists exactly one $n_1 \in \mathbb{N}_0$ such that $\tau^{n_1} = \alpha_0(s_0)$. In this situation there exist solutions $\varphi(x)$ of $(\mathrm{L}\tau)$ if and only if

$$\sum_{r=1}^{n_1} \alpha_r(s_0)\varphi_{n_1-r} + \beta_{n_1}(s_0) = 0.$$

In this case the coefficients φ_n of $\varphi(x)$ are uniquely determined for $n \neq n_1$, whereas φ_{n_1} is not determined by the functional equation and can be arbitrarily chosen in $\mathbb{C}$. □

Finally, we have to consider the **case 2.2.2**, where $\mu = n_0\lambda$ for some $n_0 \in \mathbb{N}_0$ and $\ell_{n_0} \neq 0$. In this situation we study the linear functional equation (12) for the unknown series $F(x)$. Denoting $a_0\ell_{n_0}x^{n_0} - [E(\rho x)]^{-1}b(x)$ by $\hat{b}(x)$, we derive from Theorem 15 of [4] that there exist solutions $F(x)$ of (12) if and only if

$$\sum_{k=0}^{j_0-1} \frac{\hat{b}(\rho^k x)}{\prod_{j=0}^{k} a_0} = 0. \tag{15}$$

Using the explicit form of $\hat{b}(x)$ we determine the left hand side of (15) as

$$\sum_{k=0}^{j_0-1} \frac{\hat{b}(\rho^k x)}{a_0^{k+1}} = \sum_{k=0}^{j_0-1} \frac{1}{a_0^{k+1}} \left(a_0\ell_{n_0}\rho^{kn_0}x^{n_0} - \frac{b(\rho^k x)}{E(\rho^{k+1}x)} \right).$$

According to the assumptions on ρ and μ, we have $\rho^{kn_0} = e^{n_0\lambda k} = e^{\mu k} = a_0^k$. By application of (7) and by iteration of (6), the right hand side of the last expression yields

$$\sum_{k=0}^{j_0-1} \left(\ell_{n_0}x^{n_0} - \frac{b(\rho^k x)}{a_0^{k+1}\left(\prod_{j=0}^{k} \hat{a}(\rho^j x)\right) E(x)} \right) =$$

$$= j_0\ell_{n_0}x^{n_0} - \sum_{k=0}^{j_0-1} \frac{b(\rho^k x)}{\left(\prod_{j=0}^{k} a(\rho^j x)\right) E(x)}.$$

Introducing $B(x)$ as

$$B(x) := \sum_{k=0}^{j_0-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)}, \tag{16}$$

we derive that there exist solutions $F(x)$ of (12) if and only if

$$j_0\ell_{n_0}x^{n_0}E(x) = B(x).$$

From Theorem 12 of [4] we know that $B(x)$ is a solution of $(\mathrm{L_h})$. It is easy to check that also $j_0\ell_{n_0}x^{n_0}E(x)$ satisfies $(\mathrm{L_h})$ since $E(x)$ is a solution of (6) and $n_0 \in K$, i.e. $n_0 = k_0 + r_0j_0$ for some $r_0 \in \mathbb{N}_0$, whence $\rho^{n_0} = \rho^{k_0} = a_0$.

The next remark describes that in this final situation, i.e. in case 2.2.2 with $B(x) \neq 0$, it is possible to find a covariant embedding of (L).

Remark 3. *Assume that $a_0^{j_0} = 1$, that $\hat{a}(x)$ satisfies* (8), *and that $B(x)$ is given by* (16). *If $B(x) = 0$, then* (12) *is satisfied only for $\ell_{n_0} = 0$, but then we are in the previous case, where $\ell_{n_0} x^{n_0}$ did not occur, and there is no covariant embedding of* (L). *If $B(x) \neq 0$, then let n_0 be the order of $B(x)$. Since $B(x)$ is a solution of* ($\mathrm{L_h}$), *it follows from Theorem 10 of* [4] *that $n_0 \equiv k_0 \bmod j_0$, where k_0 is the smallest non-negative integer n such that $\rho^n = a_0$. Defining ℓ_{n_0} as B_{n_0}/j_0, where B_n are the coefficients of $B(x)$, we get a series*

$$E(x) := \frac{B(x)}{j_0 \ell_{n_0} x^{n_0}}$$

satisfying (6) *with $E(x) \equiv 1 \bmod x$. Moreover, it is possible to determine μ such that $\mu = n_0\lambda$ and $e^\mu = a_0$. Using* (5), *we determine a solution α of* (Co1) *satisfying* (B2). *By construction,* (15) *is satisfied, whence we can find a solution $F(x)$ of* (12) *which allows to determine β by* (10) *such that (α, β) satisfies* (Co2) *and* (B2). *Since $B(x) \neq 0$, the linear equation* (L) *does not have any solution, hence the set of solutions of* (L), *being empty, is trivially a subset of the set of solutions of* (Ls) *for each s, and consequently there exists a covariant embedding of* (L).

Summarizing, we proved for the case described in (1)

Theorem 4. *Assume that $\pi(s, x) = e^{\lambda s}x$ is an analytic iteration group of $p(x)$, thus $\lambda \neq 0$ is a logarithm of ρ, a complex root of* 1 *primitive of order $j_0 > 1$.*

If $\hat{a}(x)$ given by (7) *does not satisfy* (8), *then there is no covariant embedding of* (L).

Now we assume that (8) *is satisfied. If $a_0^{j_0} \neq 1$, let α be given by* (5) *with an arbitrary logarithm μ of a_0 and any solution $E(x)$ of* (6). *Then there exists exactly one β given by* (9) *such that (α, β) is a covariant embedding of* (L) *with respect to π.*

If $a_0^{j_0} = 1$ let $B(x)$ be given by (16). *If $B(x) = 0$, then* (L) *has solutions, but there is no covariant embedding of* (L). *If $B(x) \neq 0$ let n_0 be the order of $B(x)$. The covariant embeddings in this situation are given by all pairs (α, β) where α is given by* (5) *with $\mu = n_0\lambda$, $E(x) = B(x)/(B_{n_0} x^{n_0})$ and where β is given by* (10) *with $\ell_{n_0} = B_{n_0}/j_0$ and $F(x)$ is an arbitrary solution of* (12).

Remark 5. *Formal power series $p(x) = \rho x + c_2 x^2 + \dots$ where ρ is a complex root of* 1, *primitive of order $j_0 > 0$, which possess an embedding into an analytic iteration group $(\pi(s, x))_{s \in \mathbb{C}}$ have such embeddings into different iteration groups. It is well known from iteration theory that a series $p(x)$ as above has an embedding if and only if there exists $S(x) = x + s_2 x^2 + \dots$ such that*

$$p(x) = S^{-1}(\rho S(x)). \tag{17}$$

For each solution S of this functional equation, $(S^{-1}(e^{\lambda s} S(x))_{s \in \mathbb{C}}$ with $e^\lambda = \rho$ is an analytic iteration group of $p(x)$, and all analytic iteration groups of $p(x)$ are obtained in this way. Moreover, different solutions of (17) *yield different iteration groups.*

We are now considering the question how the existence of a covariant embedding of (L) *in this situation depends on the iteration group* $(S^{-1}(e^{\lambda s}S(x))_{s\in\mathbb{C}}$ *of* $p(x)$*. As it was shown in [3] we may assume that* $p(x) = \rho x$*. Then the iteration groups of* $p(x)$ *are given by* $(S^{-1}(e^{\lambda s}S(x))_{s\in\mathbb{C}}$ *where* $S(x) = x + s_2x^2 + \dots$ *is any solution of*

$$\rho S(x) = S(\rho x). \tag{18}$$

The set of these S*, which forms a group with respect to substitution, consists of the series*

$$S(x) = x + \sum_{k\geq 1} s_{kj_0+1}x^{kj_0+1}$$

with arbitrary $s_{kj_0+1} \in \mathbb{C}$*.*

Let λ *with* $e^{\lambda} = \rho$ *be fixed. We claim that* (L) *has a covariant embedding with respect to the iteration group* $(e^{\lambda s}x)_{s\in\mathbb{C}}$ *of* $p(x)$ *if and only if it has a covariant embedding with respect to all iteration groups* $(S^{-1}(e^{\lambda s}S(x))_{s\in\mathbb{C}}$*.*

Proof. We consider the general form $(S^{-1}(e^{\lambda s}S(x))_{s\in\mathbb{C}}$ of an analytic iteration group of $p(x) = \rho x$. As it was shown in Theorem 1.3 of [3] we reduce this case equivalently by the transformation $y = S(x)$ to the situation of an iteration group $(e^{\lambda s}x)_{s\in\mathbb{C}}$. This yields for (L)

$$\tilde{\varphi}(e^{\lambda}y) = \tilde{a}(y)\tilde{\varphi}(y) + \tilde{b}(y) \tag{$\tilde{\mathrm{L}}$}$$

where $\tilde{a} := a \circ S^{-1}$ and $\tilde{b} := b \circ S^{-1}$. The system (Ls), (Co1), (Co2), (B1), and (B2) is equivalent to the system

$$\tilde{\varphi}(e^{\lambda s}y) = \tilde{\alpha}(s,y)\tilde{\varphi}(y) + \tilde{\beta}(s,y) \tag{$\tilde{\mathrm{L}}$s}$$

$$\tilde{\alpha}(t+s,y) = \tilde{\alpha}(s,y)\tilde{\alpha}(t,e^{\lambda s}y) \tag{$\tilde{\mathrm{C}}$o1}$$

$$\tilde{\beta}(t+s,y) = \tilde{\beta}(s,y)\tilde{\alpha}(t,e^{\lambda s}y) + \tilde{\beta}(t,e^{\lambda s}y) \tag{$\tilde{\mathrm{C}}$o2}$$

$$\tilde{\alpha}(0,y) = 1 \qquad \tilde{\beta}(0,y) = 0 \tag{$\tilde{\mathrm{B}}$1}$$

$$\tilde{\alpha}(1,y) = \tilde{a}(y) \qquad \tilde{\beta}(1,y) = \tilde{b}(y) \tag{$\tilde{\mathrm{B}}$2}$$

where $\tilde{\alpha}(s,y) = \alpha(s,S^{-1}(y))$ and $\tilde{\beta}(s,y) = \beta(s,S^{-1}(y))$. The necessary and sufficient conditions for the existence of covariant embeddings of (L) with respect to $(e^{\lambda s})_{s\in\mathbb{C}}$, given in Theorem 4 are satisfied if and only if the corresponding conditions for the existence of covariant embeddings of ($\tilde{\mathrm{L}}$) with respect to $(e^{\lambda s})_{s\in\mathbb{C}}$ are satisfied, since S is a solution of (18). □

2.B ρ not a complex root of 1

For the rest of this section we are dealing with the problem described in (2). We have already repeated that in the present situation there exist α, given by (5), which satisfy both (Co1) and (B2). Let α be a solution of (Co1) and (B2). We also introduced the set $K = K(\mu,\lambda)$. **Case 1:** If $K = \emptyset$, then from Theorem 3.4 of [3] it follows that there exists exactly one β which is together with α a solution of (Co2) and (B2). If $K \neq \emptyset$, then K is a set of cardinality one, since ρ is not a complex root of 1. **Case 2:** Assume that $K = \{k_0\}$ for some $k_0 \in \mathbb{N}_0$. **Case 2.1:** If k_0, μ, and λ satisfy the equation $\mu = k_0\lambda$, then it is shown in the proof of Theorem 3.5 of [3] that there exists exactly one β of the form (10) such that (α, β) is a solution of (Co2) and (B2). So far, it is possible to apply Theorem 1 in order to derive that (α, β) yields a covariant embedding of (L). **Case 2.2:** Assume that $\mu - n\lambda \neq 0$ for all $n \in \mathbb{N}_0$. In Theorem 2.8 of [3], the general solution (α, β) of (Co2) is determined, where β is given by (9) with an arbitrary formal power series $F(x)$. In Theorem 3.4 of [3] we describe under which conditions β also satisfies (B2). Here we present another characterization. The boundary condition (B2) written for β, given by (9), is

$$b(x) = e^{\mu} E(\rho x)[F(x) - e^{-\mu} F(\rho x)].$$

Taking into account that $E(\rho x)$ satisfies (6) and introducing a formal series $G(x) := -E(x)F(x)$ we derive $b(x) = -a(x)G(x) + G(\rho x)$, which can be written as a linear functional equation for $G(x)$, namely

$$G(\rho x) = a(x)G(x) + b(x). \tag{19}$$

Hence, there exist solutions (α, β) of (Co2) satisfying (B2) if and only if (19) has a solution, thus if and only if (L) can be solved. For that reason, in the next lemma we describe the set of solutions of (L) in the situation described in (2).

Lemma 6. *Assume that $\varphi(x)$ is a solution of* (L) *under the assumptions of* (2). *If $K = \emptyset$, then the coefficients φ_n of $\varphi(x)$ are uniquely determined by*

$$\varphi_n = \frac{\sum_{r=1}^{n} a_r \varphi_{n-r} + b_n}{\rho^n - a_0} \tag{20}$$

for all $n \geq 0$.

If $K = \{k_0\}$, then the coefficients φ_n are uniquely determined by (20) *for all $n \neq k_0$, and φ_{k_0} remains undetermined.*

Conversely, if for $K = \{k_0\}$ the coefficient b_{k_0} satisfies the condition

$$b_{k_0} = -\sum_{r=1}^{k_0} a_r \varphi_{k_0 - r},$$

then for each choice of φ_{k_0} in $\mathbb{C}$ we get a solution $\varphi(x) = \sum_{n \geq 0} \varphi_n x^n$ of (L), *where φ_n are given by* (20) *for $n \neq k_0$.*

The simple proof by comparison of coefficients is left to the reader. Here in case 2 we are only interested in $K = \{k_0\}$.

Case 2.2.1: If there are no solutions of (L), then it is impossible to find β such that (α, β) is a solution of (Co2) and (B2), consequently there is no covariant embedding of (L).

Case 2.2.2: If (L) has solutions, then according to Lemma 6 for each coefficient φ_{k_0} there exists a solution $\varphi(x)$ of (L). Moreover, (19) has also solutions, whence there exist solutions (α, β) of (Co2) and (B2) where β is of the form (9). We claim that there does not exist a covariant embedding (α, β) of (L) with respect to π. If we assume that there exists a covariant embedding of (L), then each solution $\varphi(x)$ of (L) satisfies for all $s \in \mathbb{C}$

$$\begin{aligned}\varphi(e^{\lambda s}x) &= e^{\mu s}\frac{E(e^{\lambda s}x)}{E(x)}\varphi(x) + e^{\mu s}E(e^{\lambda s}x)\left[F(x) - e^{-\mu s}F(e^{\lambda s}x)\right] \\ &= e^{\mu s}E(e^{\lambda s}x)\left[\frac{\varphi(x)}{E(x)} + F(x)\right] - E(e^{\lambda s}x)F(e^{\lambda s}x).\end{aligned}$$

Hence,

$$\frac{\varphi(e^{\lambda s}x) + E(e^{\lambda s}x)F(e^{\lambda s}x)}{E(e^{\lambda s}x)} = e^{\mu s}\left[\frac{\varphi(x)}{E(x)} + F(x)\right]. \tag{21}$$

The left hand side of this equation can be written as a formal power series

$$\sum_{n\geq 0} P_n(e^{\lambda s})x^n$$

where the coefficient functions P_n are polynomials in $e^{\lambda s}$. The right hand side is of the form

$$e^{\mu s}\sum_{n\geq 0}\zeta_n x^n.$$

Lemma 7. *Let $P_n(y)$ be polynomials for $n \geq 0$. If $\lambda \neq 0$ and $\mu - n\lambda \neq 0$ for all $n \in \mathbb{N}_0$, then*

$$\sum_{n\geq 0} P_n(e^{\lambda s})x^n = e^{\mu s}\sum_{n\geq 0}\zeta_n x^n, \qquad \forall s \in \mathbb{C} \tag{22}$$

implies that $\zeta_n = 0$ for all $n \geq 0$, whence also $P_n(y) = 0$ for all $n \geq 0$.

Proof. Comparison of coefficients in (22) leads to

$$P_n(e^{\lambda s}) = e^{\mu s}\zeta_n, \qquad \forall s \in \mathbb{C},\ \forall n \geq 0. \tag{23}$$

Derivation with respect to s and (23) yield

$$\lambda e^{\lambda s}\frac{dP_n}{dy}(e^{\lambda s}) = \mu e^{\mu s}\zeta_n = \mu P_n(e^{\lambda s}) \qquad \forall s \in \mathbb{C}.$$

Since $\lambda \neq 0$, $e^{\lambda s}$ takes infinitely many values, thus for each $n \geq 0$ the last equality can be written as an identity in the indeterminate y, namely

$$\lambda y\frac{dP_n(y)}{dy} = \mu P_n(y) \qquad \forall n \geq 0.$$

Introducing coefficients for the polynomials P_n of the form $P_n(y) = \sum_{j=0}^{r(n)} p_j^{(n)}y^j$, we derive

$$\lambda\sum_{j=1}^{r(n)} jp_j^{(n)}y^j = \mu\sum_{j=0}^{r(n)} p_j^{(n)}y^j \qquad \forall n \geq 0,$$

consequently

$$(\mu - j\lambda)p_j^{(n)} = 0 \quad \text{for all } 0 \leq j \leq r(n) \qquad \forall n \geq 0.$$

For that reason $p_j^{(n)} = 0$ for all j and for all all $n \geq 0$, thus $P_n(y) = 0$ for all $n \geq 0$ and the proof is finished. □

The assumptions on λ and μ from case 2.2 in situation (2) guarantee that the assumptions of Lemma 7 are satisfied. Hence, the equality in (21) holds only in the case when both sides are equal to zero for all $s \in \mathbb{C}$ and all solutions $\varphi(x)$ of (L). (This result also could have been obtained from a more general result published in [12] as Theorem 6.) Especially for $s = 0$ we get from the right hand side of (21) that all solutions $\varphi(x)$ of (L) satisfy

$$\varphi(x) = -F(x)E(x),$$

which is impossible since different solutions φ of (L) have different coefficients φ_{k_0}, whereas the right hand side of the last equation is just one formal power series. So there is no covariant embedding of (L).

Summarizing, we proved for the case described in (2)

Theorem 8. *Assume that $\pi(s, x) = e^{\lambda s}x$ is an analytic iteration group of $p(x)$, thus λ is a logarithm of ρ which is not a complex root of* 1. *Let α, given by* (5), *be a solution of* (Co1) *and* (B2) *where $E(x) = 1 + e_1 x + \dots$ is uniquely determined and μ is a logarithm of a_0.*

If $\mu - n\lambda \notin 2\pi i\mathbb{Z}$ for all $n \in \mathbb{N}_0$, then there exists exactly one β of the form (9) *such that (α, β) is a covariant embedding of* (L) *with respect to π.*

If $\mu = n_0\lambda$ for some $n_0 \in \mathbb{N}_0$, then there exists exactly one β of the form (10) *with uniquely determined ℓ_{n_0} but not uniquely determined $F(x)$, such that (α, β) is a covariant embedding of* (L) *with respect to π.*

Otherwise, i.e. if there exists $n_0 \in \mathbb{N}_0$ such that $\mu - n_0\lambda \in 2\pi i\mathbb{Z} \setminus \{0\}$, there is no covariant embedding (α, β) of (L) *with respect to π.*

3 The non-generic situations for iteration groups of type 2

3.A $a_{k-1} = n_1 c_k$ for some $n_1 \in \mathbb{N}$

According to Theorem 2.6 of [3] the general solution α of (Co1) in case (3) is

$$\alpha(s, x) = e^{\mu s} P_{k-1,\kappa_{k-1}}(s, x)\frac{E(\pi(s, x))}{E(x)},$$

where $e^{\mu} = 1$, $E(x) = 1 + e_1 x + \dots$, and

$$P_{k-1,\kappa_{k-1}}(s, x) = \exp\left(\kappa_{k-1}\int_0^s \pi(\sigma, x)^{k-1} d\sigma\right)$$

with $\kappa_{k-1} \in \mathbb{C}$. From Theorem 3.10 and (15) of [3] we derive that for any $a(x)$ it is possible to find solutions α of (Co1) and (B2). They are given by

$$\alpha(s,x) = e^{\mu s} P_{k-1,a_{k-1}}(s,x) \frac{E(\pi(s,x))}{E(x)}, \tag{24}$$

with a uniquely determined series $E(x) = 1 + e_1 x + \ldots$.

Now we assume that α is a solution of (Co1) and (B2). The general solution (α, β) of (Co2) depends on the special choice of μ. If $\mu \neq 0$, then it follows from Theorem 2.8 of [3] that

$$\beta(s,x) = e^{\mu s} P_{k-1,a_{k-1}}(s,x) E(\pi(s,x)) \left[F(x) - e^{-\mu s} \frac{F(\pi(s,x))}{P_{k-1,a_{k-1}}(s,x)} \right], \tag{25}$$

where $F(x)$ is an arbitrary formal power series. According to Theorem 3.11 of [3] the boundary condition $\beta(1,x) = b(x)$ is satisfied if and only if $b_n = 0$ for all $n < k-1$ and b_{n_1+k-1} satisfies an implicitly given condition, which we do not explain in more details. However, after Lemma 9 we present another characterization when β satisfies this boundary condition.

If $\mu = 0$, then it follows from the same theorems that the general solution $\beta(s,x)$ is given by

$$\begin{aligned} &E(\pi(s,x)) P_{k-1,a_{k-1}}(s,x) \cdot \\ &\quad \cdot \left[F(x) - \frac{F(\pi(s,x))}{P_{k-1,a_{k-1}}(s,x)} + Q(s,x) + \ell \int_0^s \frac{\pi(\sigma,x)^{n_1+k-1}}{P_{k-1,a_{k-1}}(\sigma,x)} d\sigma \right], \end{aligned} \tag{26}$$

with

$$Q(s,x) = \sum_{n=0}^{k-2} \int_0^s \frac{\ell_n \pi(\sigma,x)^n}{P_{k-1,a_{k-1}}(\sigma,x) E(\pi(\sigma,x))} d\sigma$$

where $F(x)$ is an arbitrary formal power series and ℓ and ℓ_n are arbitrary complex numbers. In this situation, the boundary condition $\beta(1,x) = b(x)$ can always be satisfied. However, β is not uniquely determined as a solution of (Co2) and (B2).

First we determine all solutions of (L). In Lemma 3.12 of [3] it was shown that (L) has solutions only in the case when $b_n = 0$ for all $0 \leq n < k-1$.

Lemma 9. *Let $b_n = 0$ for all $0 \leq n < k-1$, and let $a_{k-1} = n_1 c_k$ for some $n_1 \in \mathbb{N}$.*

If $\varphi(x)$ is a solution of (L) *under the assumptions of* (3)*, then the coefficients φ_n of $\varphi(x)$ are uniquely determined by*

$$\varphi_n = \frac{\sum_{r=k}^{n+k-1} a_r \varphi_{n+k-1-r} + b_{n+k-1} - Q_n(\varphi_1, \ldots, \varphi_{n-1})}{n c_k - a_{k-1}} \tag{27}$$

for $n \neq n_1$, with universal polynomials $Q_n(\varphi_1, \ldots, \varphi_{n-1})$. The coefficient φ_{n_1} remains undetermined. Conversely, if φ_n are given by (27) *for $n \neq n_1$ and if* *the coefficient b_{n_1+k-1} satisfies the condition*

$$b_{n_1+k-1} = Q_{n_1}(\varphi_1, \ldots, \varphi_{n_1-1}) - \sum_{r=k}^{n_1+k-1} a_r \varphi_{n_1+k-1-r}, \tag{28}$$

then for each choice of φ_{n_1} in $\mathbb{C}$ we obtain a solution $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$ of (L)*.*

Proof. The series $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$ is a solution of (L) if and only if

$$\sum_{n\geq 0} \varphi_n [p(x)]^n = \left(1 + \sum_{n\geq k-1} a_n x^n\right)\left(\sum_{n\geq 0} \varphi_n x^n\right) + \sum_{n\geq k-1} b_n x^n.$$

The left hand side of this equation yields

$$\varphi_0 + \sum_{n\geq 1} \varphi_n [x + c_k x^k + P^{(k)}_{k+1}(1) x^{k+1} + \ldots]^n =$$

$$\varphi_0 + \sum_{n\geq 1} \varphi_n (x^n + n c_k x^{n-1+k} + n P^{(k)}_{k+1}(1) x^{n+k} + \ldots).$$

Hence, $\varphi(x)$ is a solution of (L) if and only if

$$\varphi_0 + \sum_{n\geq 1} \varphi_n (x^n + n c_k x^{n-1+k} + n P^{(k)}_{k+1}(1) x^{n+k} + \ldots) =$$

$$\sum_{n\geq 0} \varphi_n x^n + \sum_{n\geq k-1} \left(\sum_{r=k-1}^{n} a_r \varphi_{n-r} + b_n\right) x^n,$$

which is equivalent to

$$\sum_{n\geq 1} \varphi_n (n c_k x^{n-1+k} + n P^{(k)}_{k+1}(1) x^{n+k} + \ldots) = \sum_{n\geq k-1} \left(\sum_{r=k-1}^{n} a_r \varphi_{n-r} + b_n\right) x^n.$$

Comparing the coefficients of x^{j+k-1} for $j \geq 0$, we get by induction necessary and sufficient conditions on the coefficients for the existence of a solution of (L):

$$\begin{aligned} -a_{k-1}\varphi_0 &= b_{k-1} & j &= 0 \\ (jc_k - a_{k-1})\varphi_j &= \sum_{r=k}^{j+k-1} a_r \varphi_{j+k-1-r} + b_{j+k-1} - Q_j(\varphi_1, \ldots, \varphi_{j-1}), & j &> 0 \end{aligned}$$

where $Q_j(\varphi_1, \ldots, \varphi_{j-1})$ are universal polynomials in the coefficients $\varphi_1, \ldots,$ φ_{j-1}. This yields for $j \neq n_1$ a unique way to determine φ_j by (27).

In order to satisfy the necessary and sufficient condition for the existence of a solution of (L) in the case $j = n_1$, the coefficient φ_{n_1} can be chosen arbitrarily in $\mathbb{C}$ if and only if (28) is satisfied. □

Now we come back to the boundary condition $\beta(1,x) = b(x)$ for $\mu \neq 0$. Since $e^\mu = 1$ and since α satisfies (Co1) and (B2) we have

$$a(x) = P_{k-1,a_{k-1}}(1,x)\frac{E(p(x))}{E(x)}. \tag{29}$$

The boundary condition for β of the form (25) is

$$\begin{aligned} b(x) &= P_{k-1,a_{k-1}}(1,x)E(p(x))\left[F(x) - \frac{F(p(x))}{P_{k-1,a_{k-1}}(1,x)}\right] \\ &= P_{k-1,a_{k-1}}(1,x)E(p(x))F(x) - E(p(x))F(p(x)). \end{aligned}$$

Introducing the formal power series $G(x) := -E(x)F(x)$ and applying (29) yields

$$G(p(x)) = a(x)G(x) + b(x),$$

which is a linear functional equation for $G(x)$. Consequently, we derive that β satisfies the boundary condition (B2) if and only if the linear functional equation (L) has a solution.

The next lemma will be applied in the proof of Lemma 11. It could also be deduced from a more general result published in [12] as Theorem 6.

Lemma 10. *For $n \geq 0$ let $Q_n(y)$ be universal polynomials. If $\mu \neq 0$, then*

$$\sum_{n\geq 0} Q_n(s)x^n = e^{\mu s}\sum_{n\geq 0}\zeta_n x^n, \qquad \forall s \in \mathbb{C} \tag{30}$$

implies that $\zeta_n = 0$ for all $n \geq 0$, whence also $Q_n(y) = 0$ for all $n \geq 0$.

Proof. Comparison of coefficients in (30) leads to

$$Q_n(s) = e^{\mu s}\zeta_n, \qquad \forall s \in \mathbb{C},\ \forall n \geq 0. \tag{31}$$

Derivation with respect to s and (31) yield

$$Q_n'(s) = \mu e^{\mu s}\zeta_n = \mu Q_n(s) \qquad \forall s \in \mathbb{C}, \forall n \geq 0.$$

Thus, for each $n \geq 0$ the last equality can be written as an identity in the indeterminate y, namely

$$Q_n'(y) = \mu Q_n(y) \qquad n \geq 0.$$

This is a homogeneous linear differential equation which has the general solution $Q_n(y) = ce^{\mu y}$. Under the additional assumption that Q_n is a polynomial we get $c = 0$, whence $Q_n = 0$ for all $n \geq 0$ and the proof is finished. □

In the next lemma we prove that there is no covariant embedding of (L) involving a solution α of (Co1) and (B2) with $\mu \neq 0$.

Lemma 11. *If α is a solution of* (Co1) *and* (B2) *with $\mu \neq 0$, then* (L) *does not have a covariant embedding (α, β) under the hypotheses of* (3).

Proof. Assuming that (α, β) is a covariant embedding of (L) with $\mu \neq 0$, then β is a solution of (Co2) and (B2), whence necessarily (L) has solutions, thus $b_n = 0$ for all $0 \leq n < k-1$. Hence the general assumption of Lemma 9 is satisfied. Moreover, we derive that there exist formal power series $E(x) = 1 + e_1 x + \ldots$ and $F(x)$ such that

$$\begin{aligned}\varphi(\pi(s,x)) &= e^{\mu s} P_{k-1,a_{k-1}}(s,x) \frac{E(\pi(s,x))}{E(x)} \varphi(x) + \\ &\quad e^{\mu s} P_{k-1,a_{k-1}}(s,x) E(\pi(s,x)) \left[F(x) - e^{-\mu s} \frac{F(\pi(s,x))}{P_{k-1,a_{k-1}}(s,x)} \right] \\ &= e^{\mu s} P_{k-1,a_{k-1}}(s,x) E(\pi(s,x)) \left[\frac{\varphi(x)}{E(x)} + F(x) \right] - \\ &\quad E(\pi(s,x)) F(\pi(s,x))\end{aligned}$$

holds for all solutions $\varphi(x)$ of (L) and all $s \in \mathbb{C}$. Hence,

$$\frac{\varphi(\pi(s,x)) + E(\pi(s,x))F(\pi(s,x))}{P_{k-1,a_{k-1}}(s,x) E(\pi(s,x))} = e^{\mu s} \left[\frac{\varphi(x) + E(x)F(x)}{E(x)} \right] \tag{32}$$

holds for all solutions $\varphi(x)$ of (L) and all $s \in \mathbb{C}$. The left hand side of this equation can be written as a formal power series

$$\sum_{n \geq 0} Q_n(s) x^n$$

where the coefficient functions Q_n are polynomials in s. The right hand side is of the form

$$e^{\mu s} \sum_{n \geq 0} \zeta_n x^n.$$

Since $\mu \neq 0$, it follows from Lemma 10 that both sides of (32) vanish, and consequently $\varphi(x) = -E(x)F(x)$. In other words, for all solutions $\varphi(x) = \sum_{n \geq 0} \varphi_n x^n$ of (L)

$$\sum_{n \geq 0} \varphi_n x^n = -\sum_{n \geq 0} \left(\sum_{r=0}^{n} e_r f_{n-r} \right) x^n$$

is satisfied. Comparing the coefficients for x^{n_1}, we see that this is impossible since according to Lemma 9 two different solutions $\varphi(x)$ and $\tilde{\varphi}(x)$ of (L) have different coefficients $\varphi_{n_1} \neq \tilde{\varphi}_{n_1}$. Thus we end up with a contradiction. $\square$

Hence, we only have to investigate the situation where α is a solution of (Co1) with $\mu = 0$. As **case 1** we assume that (L) has solutions, whence $b_n = 0$ for all $0 \leq n < k-1$ and b_{n_1+k-1} satisfies (28). If $\varphi(x) = \sum_{n \geq 0} \varphi_n x^n$ and $\tilde{\varphi}(x) = \sum_{n \geq 0} \tilde{\varphi}_n x^n$ are two different solutions of (L), then according to Lemma 9 we have $\varphi_n = \tilde{\varphi}_n$ for $n < n_1$, $\varphi_{n_1} \neq \tilde{\varphi}_{n_1}$, and $\psi(x) := \varphi(x) - \tilde{\varphi}(x)$ is a formal power series of order n_1, which is a solution of the homogeneous linear equation

$$\psi(p(x)) = a(x)\psi(x). \tag{$\mathrm{L_h}$}$$

Since α is a solution of (Co1) and $\pi(m,x) = p^m(x)$, the m-th iterate of $p(x)$ for $m \in \mathbb{N}$, we also derive that

$$\psi(p^m(x)) = \psi(\pi(m,x)) = \alpha(m,x)\psi(x) \tag{$\mathrm{L_h, m}$}$$

for all $m \in \mathbb{N}$.

In the proof of Theorem 1 (cf. [3]) we showed that for a solution $\varphi(x)$ of (L) and a solution α the family $(\Phi_\varphi(s,x))_{s\in\mathbb{C}}$ of formal power series

$$\Phi_\varphi(s,x) := \varphi(\pi(s,x)) - \alpha(s,x)\varphi(x)$$

satisfies the boundary conditions (B1), (B2), and together with α the cocycle equation (Co2). In other words, $(\Phi_\varphi(s,x))_{s\in\mathbb{C}}$ is a solution β of (Co2) under the additional boundary conditions. In order to guarantee in the present situation the existence of a covariant embedding, we only have to check whether $(\Phi_\varphi(s,x))_{s\in\mathbb{C}}$ determines for all solutions $\varphi(x)$ of (L) the same family β. If it is so, then there exists a covariant embedding of (L) with respect to the analytic iteration group $(\pi(s,x))_{s\in\mathbb{C}}$, since then $\varphi(\pi(s,x)) = \alpha(s,x)\varphi(x) + \Phi_\varphi(s,x) = \alpha(s,x)\varphi(x) + \beta(s,x)$, for all solutions $\varphi(x)$ of (L). Otherwise there is no covariant embedding.

If $\varphi(x)$ and $\tilde{\varphi}(x)$ denote two solutions of (L), and if $\psi(x)$ is the difference $\varphi(x) - \tilde{\varphi}(x)$, then

$$\Phi_\varphi(s,x) - \Phi_{\tilde{\varphi}}(s,x) = \varphi(\pi(s,x)) - \alpha(s,x)\varphi(x) - \tilde{\varphi}(\pi(s,x)) + \alpha(s,x)\tilde{\varphi}(x) =$$

$$\psi(\pi(s,x)) - \alpha(s,x)\psi(x).$$

Hence, $\Phi_\varphi(s,x) = \Phi_{\tilde{\varphi}}(s,x)$ for all $s \in \mathbb{C}$ if and only if $\psi(\pi(s,x)) = \alpha(s,x)\psi(x)$ for all $s \in \mathbb{C}$. Inserting the explicit form of α and using the fact that $\mu = 0$, there exists a covariant embedding of (L) with respect to π if and only if

$$\psi(\pi(s,x)) = P_{k-1,a_{k-1}}(s,x)\frac{E(\pi(s,x))}{E(x)}\psi(x) \tag{33}$$

for all solutions $\psi(x)$ of ($\mathrm{L_h}$). This is equivalent to

$$\frac{\psi(\pi(s,x))}{E(\pi(s,x))P_{k-1,a_{k-1}}(s,x)} = \frac{\psi(x)}{E(x)}$$

for all solutions ψ of ($\mathrm{L_h}$). Now we introduce coefficients for

$$\frac{\psi(\pi(s,x))}{E(\pi(s,x))P_{k-1,a_{k-1}}(s,x)} =: \Psi^\psi(s,x) = \sum_{n\geq 0} \Psi_n^\psi(s)x^n$$

and for

$$\frac{\psi(x)}{E(x)} =: \Theta^{\psi}(x) = \sum_{n\geq 0} \Theta_n^{\psi} x^n.$$

The coefficient functions Ψ_n^{ψ} are polynomials in s. As a consequence of $(\mathrm{L_h}, \mathrm{m})$ we derive that that $\Psi_n^{\psi}(m) = \Theta_n^{\psi}$ for all $m \in \mathbb{N}$, all $n \geq 0$, and all solutions ψ of $(\mathrm{L_h})$. Since the last relation is a polynomial relation, it holds for all $s \in \mathbb{C}$, and (33) is satisfied for all $s \in \mathbb{C}$ and all solutions ψ of $(\mathrm{L_h})$. This means that there exists a covariant embedding of (L).

In **case 2** we assume that (L) does not have any solutions. In this situation however each solution (α, β) of the system (Co1), (Co2), (B1), and (B2) is a covariant embedding of (L) with respect to π. According to Theorem 3.11 of [3] this system of functional equations can always be solved, whence there always exists a covariant embedding of (L).

Summarizing we proved for the situation described in (3)

Theorem 12. *Assume that $\pi(s, x) = x + c_k s x^k + \ldots$, with $k \geq 2$ and $c_k \neq 0$, is an analytic iteration group of $p(x)$.*

Let α, given by (24)*, be a solution of* (Co1) *and* (B2) *where μ is a logarithm of* 1*.*

If $\mu \neq 0$, then there is no covariant embedding (α, β) of (L) *with respect to π.*

If $\mu = 0$ and $b_n = 0$ for $0 \leq n < k-1$ and b_{n_1+k-1} satisfies (28)*, then there exists exactly one β given by*

$$\beta(s, x) = \varphi(\pi(s, x)) - \alpha(s, x)\varphi(x)$$

for an arbitrary solution $\varphi(x)$ of (L)*, such that (α, β) is a covariant embedding of* (L) *with respect to π.*

If $\mu = 0$ and $\big[$there exists at least one $n \in \{0, \ldots, k-2\}$ such that $b_n \neq 0$ or b_{n_1+k-1} does not satisfy (28)$\big]$*, then* (L) *does not have any solutions, but any solution (α, β) of* (Co2) *and* (B2) *is a covariant embedding of* (L) *with respect to π. The families β which occur in these covariant embeddings are given by* (26) *with uniquely determined $Q(s, x)$ and ℓ, but not uniquely determined $F(x) \in \mathbb{C}[\![x]\!]$.*

3.B Covariant embeddings in certain cases with $\mu \neq 0$

Finally we are dealing with the problem presented in (4). According to Theorem 2.6 of [3] the general solution α of (Co1) is

$$\alpha(s, x) = e^{\mu s} \frac{E(\pi(s, x))}{E(x)}, \tag{34}$$

where $e^{\mu} = 1$ and $E(x) = 1 + e_1 x + \ldots \in \mathbb{C}[\![x]\!]$. According to Theorem 3.6 of [3] for any $a(x)$ satisfying the assumptions of (4) it is possible to find solutions α of (Co1) and (B2). They are given by (34) with a uniquely determined series

$E(x) = 1 + e_1 x + \ldots$. In order to deal with the problem described in (4) we can restrict to solutions α with $\mu \neq 0$.

Assuming that α with $\mu \neq 0$ is a solution of (Co1) and (B2), the general solution (α, β) of (Co2) is given in Theorem 2.8 of [3] by

$$\beta(s,x) = e^{\mu s} E(\pi(s,x)) \left[F(x) - e^{-\mu s} F(\pi(s,x)) \right],$$

where $F(x)$ is an arbitrary formal power series. According to Theorem 3.11 of [3] the boundary condition $\beta(1,x) = b(x)$ is satisfied if and only if $b_n = 0$ for all $n < k$. However β is not uniquely determined, since the coefficient f_0 of $F(x)$ is not determined by (B2). Hence, there may exist covariant embeddings of (L) in the situation (4) only if $b_n = 0$ for all $n < k$.

First we determine all solutions of (L) for the situation described in (4) with $b_n = 0$ for all $0 \leq n < k$.

Lemma 13. *Assume that $a(x) = 1 + \sum_{n \geq k} a_n x^n$ and $b_n = 0$ for all $0 \leq n < k$. If* (L) *has a solution, then the coefficient φ_0 of a solution φ of* (L) *is not determined, however, φ_n for $n \geq 1$ are uniquely determined (depending on φ_0).*

Conversely, if $b_n = 0$ for all $0 \leq n < k$, then for each $\varphi_0 \in \mathbb{C}$ there exists exactly one solution φ of (L) *such that $\varphi \equiv \varphi_0 \mod x$.*

More generally, it can be proved that (L) *has a solution if and only if $b_n = 0$ for $0 \leq n < k$.*

Proof. Using the same methods as in the proof of Lemma 9 we derive that

$$\varphi(p(x)) = \varphi_0 + \sum_{n \geq 1} \varphi_n (x^n + n c_k x^{n-1+k} + n P_{k+1}^{(k)}(1) x^{n+k} + \ldots).$$

Hence, each solution φ of (L) satisfies

$$\sum_{n \geq 1} \varphi_n (n c_k x^{n-1+k} + n P_{k+1}^{(k)}(1) x^{n+k} + \ldots) = \sum_{n \geq k} \left(\sum_{r=k}^{n} a_r \varphi_{n-r} + b_n \right) x^n. \quad (35)$$

Comparing the coefficients of x^{k+j} for $j \geq 0$, we get

$$(j+1)\varphi_{j+1} c_k = \sum_{r=k}^{k+j} a_r \varphi_{k+j-r} + b_{k+j} - Q_j(\varphi_1, \ldots, \varphi_{j-1})$$

where $Q_j(\varphi_1, \ldots, \varphi_{j-1})$ are universal polynomials in the coefficients $\varphi_1, \ldots, \varphi_{j-1}$, which are already determined. This yields for $n = j+1 \neq 0$ a unique way to determine φ_n by

$$\varphi_{j+1} = \frac{1}{(j+1)c_k} \left(\sum_{r=k}^{k+j} a_r \varphi_{k+j-r} + b_{k+j} - Q_j(\varphi_1, \ldots, \varphi_{j-1}) \right). \quad (36)$$

Conversely, if φ_0 is an arbitrary complex number and φ_{j+1} given by (36) for $j \geq 0$, then (35) is satisfied, thus $\varphi(x) = \sum_{n \geq 0} \varphi_n x^n$ is a solution of (L).

If the series φ is a solution of (L), then

$$\sum_{n\geq 0}\varphi_n x^n+\sum_{n\geq 1}\varphi_n(nc_kx^{n-1+k}+nP_{k+1}^{(k)}(1)x^{n+k}+\ldots)=$$

$$\sum_{n\geq 0}\varphi_n x^n+\sum_{n\geq k}\left(\sum_{r=k}^{n}a_r\varphi_{n-r}+b_n\right)x^n+\sum_{n=0}^{k-1}b_nx^n.$$

Comparison of coefficients yields, $\sum_{n=0}^{k-1}b_nx^n=0$ and (35) is satisfied. □

From the last lemma we deduce the following characterization: It is possible to find a solution (α,β) of (Co2) and (B2) with $\mu\neq 0$ if and only if (L) can be solved.

Hence, if (L) has no solutions, then there does not exist a covariant embedding of (L) with respect to π. From now on we assume that (L) has solutions, thus for each $\varphi_0\in\mathbb{C}$ there exists exactly one solution φ of (L) with $\varphi\equiv\varphi_0 \bmod x$.

Using exactly the same method as in the proof of Lemma 11 we can prove

Lemma 14. *If α is a solution of* (Co1) *with $\mu\neq 0$, then* (L) *does not have a covariant embedding (α,β) under the hypotheses of* (4)*.*

Proof. Assuming that (α,β) is a covariant embedding of (L) with $\mu\neq 0$, then β is a solution of (Co2) and (B2), whence necessarily (L) has solutions, thus $b_n=0$ for all $0\leq n<k$. Moreover, we derive that there exist formal power series $E(x)=1+e_1x+\ldots$ and $F(x)$ such that

$$\frac{\varphi(\pi(s,x))+E(\pi(s,x))F(\pi(s,x))}{E(\pi(s,x))}=e^{\mu s}\left[\frac{\varphi(x)+E(x)F(x)}{E(x)}\right]$$

holds for all solutions $\varphi(x)$ of (L) and all $s\in\mathbb{C}$. Since $\mu\neq 0$, it follows from Lemma 10 that both sides of (32) vanish, and consequently $\varphi(x)=-E(x)F(x)$ for all solutions φ of (L). This leads to a contradiction, since according to Lemma 13 there exist different solutions φ of (L). □

Summarizing we proved for the situation described in (4)

Theorem 15. *Let $\pi(s,x)=x+c_ksx^k+\ldots$, with $k\geq 2$ and $c_k\neq 0$, be an analytic iteration group of $p(x)$ and let α, given by* (34)*, with $\mu\neq 0$ be a solution of* (Co1) *and* (B2)*. Then there is no covariant embedding (α,β) of* (L) *with respect to the analytic iteration group π.*

4 The situation of the improper functional equation

If $p(x)=x$, then (L) also makes sense, but then it is not a functional equation in the usual sense. Nevertheless it is possible to study the covariant embeddings of (L) in this situation as well, i.e. the embedding problem for

$$\varphi(x) = a(x)\varphi(x) + b(x).$$

All the analytic iteration groups of $p(x)$ are of the form $(\pi(s,x))_{s\in\mathbb{C}}$, where $\pi(s,x) = S^{-1}(e^{2\pi izs}S(x))$ with an arbitrary integer z and an arbitrary series S of the form $S(x) = x + s_2x^2 + \ldots$. Depending on z we have to consider two cases, namely $z \neq 0$ and $z = 0$.

4.A Embeddings with respect to iteration groups of type 1

Assume that π is an analytic iteration group of the first type, whence it is of the form

$$\pi(s,x) = S^{-1}(e^{\lambda_0 s}S(x))$$

where $2\pi iz =: \lambda_0 \neq 0$ and $e^{\lambda_0} = 1$. Hence (cf. Theorem 1.3 of [3]) we may assume by an appropriate transformation of the indeterminate x that $\pi(s,x) = e^{\lambda_0 s}x$ for all $s \in \mathbb{C}$.

From Corollary 2.3 of [3] we get that the general solution α of (Co1) is given by

$$\alpha(s,x) = e^{\mu s}\frac{E(e^{\lambda_0 s}x)}{E(x)} \tag{37}$$

where $E(x) = 1 + e_1x + \ldots$ and $e^\mu = a_0$. In order to determine all α which also satisfy the boundary condition (B2), we introduced the set $J = J(\lambda_0)$ which in the present situation equals the set of all natural numbers. As a consequence of Theorem 3.2 of [3] it is possible to adapt the solution α to the boundary condition (B2) if and only if $a(x) = a_0$. In this situation any formal power series $E(x)$ with constant term equal to 1 can be used to determine a family α.

Assume that α of the form (37) is a solution of (Co1) and (B2). If $\mu - n\lambda_0 \neq 0$ for all $n \in \mathbb{N}_0$, then according to Theorem 2.8 of [3] the general solution (α, β) of (Co2) is given by

$$\beta(s,x) = e^{\mu s}E(e^{\lambda_0 s}s)[F(x) - e^{-\mu s}F(e^{\lambda_0 s}x)] \tag{38}$$

where $F(x)$ is an arbitrary formal power series. If $\mu = n_0\lambda_0$ for some $n_0 \in \mathbb{N}_0$, then

$$\beta(s,x) = e^{\mu s}E(e^{\lambda_0 s}s)[\ell_{n_0}sx^{n_0} + F(x) - e^{-\mu s}F(e^{\lambda_0 s}x)] \tag{39}$$

with $\ell_{n_0} \in \mathbb{C}$ and $F(x) \in \mathbb{C}\,[\![x]\!]$. In order to determine all the solutions (α, β) of (Co2) and (B2), we introduced the set $K = K(\mu, \lambda_0)$. In the present situation $K = \emptyset$ if $\mu \notin 2\pi i\mathbb{Z}$, and $K = \mathbb{N}_0$ if $\mu \in 2\pi i\mathbb{Z}$. From Theorem 3.4 of [3] it follows that if $K = \emptyset$, then there exists exactly one series $F(x)$ such that β of the form (38) satisfies both (Co2) and (B2). If $K = \mathbb{N}_0$ and $\mu \neq n\lambda_0$ for all $n \in \mathbb{N}_0$, then β of the form (38) satisfies (B2) if and only if $b(x) = 0$. Finally, if $K = \mathbb{N}_0$ and $\mu = n_0\lambda_0$ for some $n_0 \in \mathbb{N}_0$, then β of the form (39) satisfies (B2), if and only if $b(x) = \ell_{n_0}E(x)x^{n_0}$ for some $\ell_{n_0} \in \mathbb{C}$. If in the last two situations $b(x)$ satisfies the necessary condition, then any $F(x)$ can be used to determine β.

Theorem 16. *Assume that* $p(x) = x$ *is embedded into the analytic iteration group* $(\pi(s,x))_{s\in\mathbb{C}}$ *with* $\pi(s,x) = e^{\lambda_0 s}x$ *for* $\lambda_0 \in 2\pi i\mathbb{Z} \setminus \{0\}$.

If $a(x) \in \mathbb{C}[\![x]\!] \setminus \mathbb{C}$, *then there is no covariant embedding of* (L).

Assume that $a(x) = a_0$ *(and moreover* $a_0 \neq 0$ *as a general assumption), and let* α *be a solution of* (Co1) *and* (B2) *given by* (37) *where* μ *is a logarithm of* a_0 *and* $E(x) = 1 + e_1x + \dots$ *an arbitrary series in* $\mathbb{C}[\![x]\!]$.

If $a_0 \neq 1$, *then there is exactly one* β *of the form* (38) *such that* (α, β) *is a covariant embedding of* (L) *with respect to* π.

If $a_0 = 1$ *and* $\mu - n\lambda_0 \neq 0$ *for all* $n \in \mathbb{N}_0$, *then there is no covariant embedding* (α, β) *of* (L) *with respect to* π.

If $a_0 = 1$ *and* $\mu = n_0\lambda_0$ *for some* $n_0 \in \mathbb{N}_0$, *then the pairs* (α, β) *with* β *given by* (39), *where* $\ell_{n_0} \neq 0$ *and* $F(x)$ *is an arbitrary series, is a covariant embedding of* (L) *with respect to* π *if and only if* $b(x) = \ell_{n_0}E(x)x^{n_0}$.

Proof. The existence of a solution α of (Co1) which satisfies (B2) is a necessary condition for the existence of a covariant embedding of (L). Hence, only in the case $a(x) = a_0$ it may be possible to find covariant embeddings of (L). For that reason we assume that $a(x) = a_0$ and that α is a solution of (Co1) and (B2).

Case 1: If $a_0 \neq 1$, then $\mu \notin 2\pi i\mathbb{Z}$, whence $K = \emptyset$. Consequently the series $F(x)$ defining β is uniquely determined by (B2), and from Theorem 1 we deduce that there exist covariant embeddings of (L) with respect to π.

Case 2: If $a_0 = 1$, then $K = \mathbb{N}_0$. **Case 2.1:** If $\mu \neq n\lambda_0$ for all $n \in \mathbb{N}_0$, then a necessary condition for adapting a solution (α, β) of (Co2) to (B2) is $b(x) = 0$. Consequently, (L) becomes $\varphi(x) = \varphi(x)$ which is satisfied by any formal series $\varphi(x)$. If there were a covariant embedding (α, β) of (L), then for each $s \in \mathbb{C}$ and for any $\varphi(x) \in \mathbb{C}[\![x]\!]$ the equation

$$\varphi(e^{\lambda_0 s}x) = \alpha(s,x)\varphi(x) + \beta(s,x)$$

holds. If we choose $s_0 \in \mathbb{C}$ such that $\tau := e^{\lambda_0 s_0}$ is not a complex root of 1, then the equation above yields $\varphi(\tau x) = \alpha(s_0,x)\varphi(x) + \beta(s_0,x)$, and according to Lemma 2 the set of solutions of this equation is smaller than $\mathbb{C}[\![x]\!]$, which is a contradiction to the assumption that there is a covariant embedding of (L).

Case 2.2: If $\mu = n_0\lambda_0$ for some $n_0 \in \mathbb{N}_0$, then there exist solutions (α, β) of (Co2) and (B2) if and only if $b(x) = \ell_{n_0}E(x)x^{n_0}$. If $\ell_{n_0} = 0$ the same method as above proves that there is no covariant embedding (α, β) of (L). If, however, $\ell_{n_0} \neq 0$ then (L) has no solutions, whence all pairs (α, β) which are solutions of (Co1), (Co2), and (B2) are covariant embeddings of (L) with respect to π. □

4.B Embeddings with respect to $\pi(s,x) = x$

Now we consider the situation $z = 0$, whence $\pi(s,x) = x$ for all $s \in \mathbb{C}$. In order to derive the form of the general solution α of (Co1), we have to go back to Corollary 2.3 of [3]. We get

$$\alpha(s,x) = e^{\mu s} \exp \int_0^s K(x)d\sigma$$

where $K(x)$ is a formal power series of order ≥ 1. Hence

$$\alpha(s,x) = e^{\mu s}\exp(sK(x)). \tag{40}$$

Moreover, α satisfies (B2) if and only if $e^{\mu} = a_0$ and $K(x) = \ln(a(x)/a_0)$.

Let α be a solution of (Co1) and (B2). The general solution β of (Co2) is given in Theorem 2.5 of [3] as

$$\beta(s,x) = \alpha(s,x)\int_0^s \frac{L(x)}{\alpha(\sigma,x)}d\sigma. \tag{41}$$

with $L(x) \in \mathbb{C}[\![x]\!]$.

If $\mu \neq 0$ or $K(x) \neq 0$, then

$$\beta(s,x) = e^{\mu s}\exp(sK(x))L(x)\int_0^s e^{-\mu\sigma}\exp(-\sigma K(x))d\sigma =$$

$$e^{\mu s}\exp(sK(x))L(x)\left(\frac{1 - e^{-\mu\sigma}\exp(-\sigma K(x))}{K(x)+\mu}\Big|_0^s\right) =$$

$$e^{\mu s}\exp(sK(x))L(x)\frac{1 - e^{-\mu s}\exp(-sK(x))}{K(x)+\mu} = \frac{L(x)(\alpha(s,x)-1)}{K(x)+\mu}.$$

In this situation $\beta(s,x)$ is indeed a formal power series. (If $\mu \neq 0$, then $\operatorname{ord}(K(x)+\mu) = 0$, and the reciprocal of $K(x)+\mu$ exists in $\mathbb{C}[\![x]\!]$. If $\mu = 0$ then $K(x) \neq 0$, and $K(x)$ is a divisor of $\alpha(s,x) - 1 = \exp(sK(x)) - 1$ by (40).)

If both $\mu = 0$ and $K(x) = 0$, then $\alpha(s,x) = 1$ and from (41) it follows that

$$\beta(s,x) = sL(x)$$

with an arbitrary formal power series $L(x)$.

How to adapt these solutions β to the boundary condition (B2)? **Case 1:** If $a_0 \neq 1$, then there exists exactly one $L(x)$ such that $\beta(1,x) = b(x)$, namely

$$L(x) = [a(x) - 1]^{-1}(K(x)+\mu)b(x). \tag{42}$$

In **case 2** we assume that $a_0 = 1$. **Case 2.1:** If $a(x) = 1$, then $K(x) = 0$ and $\mu = \ln 1$. **Case 2.1.1:** If $\mu = 0$, then $\beta(s,x) = sL(x)$, and $\beta(1,x)$ equals $b(x)$ if and only if $L(x) = b(x)$. Thus, there exists exactly one β satisfying the boundary condition. In **case 2.1.2** we assume that $\mu \neq 0$. Then $\beta(s,x) = L(x)(\alpha(s,x)-1)/\mu$ satisfies (B2) if and only if $b(x) = 0$, and then any series $L(x)$ can be used to determine β.

Case 2.2: Let $a(x) = 1 + a_t x^t + \ldots$, $t \geq 1$, and $a_t \neq 0$. Hence $\operatorname{ord}(K(x)) = t$. Introducing coefficients in an obvious way, (B2), which can be written as

$$L(x)(a(x)-1) = (K(x)+\mu)b(x),$$

is equivalent to

$$\left(\sum_{n\geq 0}\ell_n x^n\right)\left(\sum_{n\geq t}a_n x^n\right) = \left(\mu + \sum_{n\geq t}k_n x^n\right)\left(\sum_{n\geq 0}b_n x^n\right).$$

If $\mu = 0$, which is **case 2.2.1**, then there exists exactly one $L(x)$ with coefficients

$$\ell_n = \frac{\sum_{r=t}^{t+n} k_r b_{t+n-r} - \sum_{r=t+1}^{t+n} a_r \ell_{t+n-r}}{a_t} \tag{43}$$

for $n \geq 0$. Finally in **case 2.2.2** we assume that $\mu \neq 0$. Then necessarily $b_n = 0$ for $0 \leq n < t$, if (B2) is satisfied. If this condition is fulfilled, then the series $L(x)$ is uniquely determined with coefficients

$$\ell_n = \frac{\sum_{r=t}^{t+n} k_r b_{t+n-r} + \mu b_{t+n} - \sum_{r=t+1}^{t+n} a_r \ell_{t+n-r}}{a_t} \tag{44}$$

for $n \geq 0$.

The solutions of the improper functional equation (L) for $p(x) = x$ are described in the next

Lemma 17. *If $a_0 \neq 0$, then there exists exactly one solution φ of* (L)*.*

If $a(x) = 1$, then there exist solutions of (L) *if and only if $b(x) = 0$. If $b(x) = 0$, then any formal power series $\varphi(x)$ satisfies* (L)*.*

If $a(x) = 1 + a_t x^t + \dots$, $t \geq 1$ and $a_t \neq 0$. The necessary and sufficient conditions for the existence of solutions of (L) *are $b_n = 0$ for $0 \leq n < t$. If they are satisfied, then the solution of* (L) *is uniquely determined.*

Theorem 18. *Assume that $p(x) = x$ is embedded into the analytic iteration group $(\pi(s, x))_{s\in\mathbb{C}}$ with $\pi(s, x) = x$.*

Assume that the coefficient a_0 of $a(x)$ is different from 1. Let α be a solution of (Co1) *and* (B2) *given by* (40) *with a uniquely determined series $K(x)$ and with an arbitrary logarithm μ of a_0. Then there exists exactly one β given by* (41) *and* (42) *such that (α, β) is a covariant embedding of* (L) *with respect to π.*

Assume that $a(x) = 1$. Let α be a solution of (Co1) *and* (B2) *given by* (40) *with $K(x) = 0$ and with an arbitrary logarithm μ of 1. If $\mu = 0$ then there exists exactly one β given by $\beta(s, x) = sb(x)$ such that (α, β) is a covariant embedding of* (L) *with respect to π. If $\mu \neq 0$ then there is no covariant embedding (α, β) of* (L) *with respect to π.*

Assume that $a(x) = 1 + a_t x^t + \dots$ with $t \geq 1$ and $a_t \neq 0$. Let α be a solution of (Co1) *and* (B2) *given by* (40) *with a uniquely determined series $K(x)$ and with an arbitrary logarithm μ of 1. If $\mu = 0$, then there exists exactly one β given by* (41)*, where the coefficients of $L(x)$ are given by* (43)*, such that (α, β) is a covariant embedding of* (L) *with respect to π. If $\mu \neq 0$ and $b_n = 0$ for all $0 \leq n < t$, then there exists exactly one β given by* (41)*, where the coefficients of $L(x)$ are given by* (44)*, such that (α, β) is a covariant embedding of* (L) *with respect to π. If $\mu \neq 0$ and there exists at least one $b_j \neq 0$ for $0 \leq j < t$, then there is no covariant embedding (α, β) of* (L) *with respect to π.*

Proof. **Case 1:** If $a_0 \neq 1$, then for each α satisfying (Co1) and (B2) there exists exactly one β such that (α, β) is a solution of (Co2) and (B2), whence according to Theorem 1 there exists a covariant embedding of (L).

Assume that α is a solution of (Co1) and (B2). For the **case 2.1.1**, which was $a(x) = 1$ and $\mu = 0$, the same method can be applied. In **case 2.1.2** we had assumed that $a(x) = 1$ and $\mu \neq 0$. If $b(x) \neq 0$, then there is no covariant embedding, since we cannot find a family β such that (α, β) is a solution of (Co2) and (B2). If $b(x) = 0$, then (L) becomes $\varphi(x) = \varphi(x)$ which is satisfied by any formal series. If there were a covariant embedding of (L), then for each $s \in \mathbb{C}$ and for any $\varphi(x) \in \mathbb{C}\llbracket x \rrbracket$ the equation

$$\varphi(x) = e^{\mu s}\varphi(x) + \frac{L(x)(e^{\mu s} - 1)}{\mu}$$

is satisfied, since in this case $\alpha(s, x) = e^{\mu s}$ and $\beta(s, x) = \mu^{-1}L(x)(e^{\mu s} - 1)$. If we choose $s_0 \in \mathbb{C}$ such that $e^{\mu s_0} \neq 1$, then this equation has a unique solution

$$\varphi(x) = \frac{L(x)(e^{\mu s_0} - 1)}{(1 - e^{\mu s_0})\mu} = -\frac{L(x)}{\mu}$$

which is a contradiction to the assumption that there is a covariant embedding of (L).

Finally, in **case 2.2** we assume that $a(x) = 1 + a_t x^t + \ldots$, $t \geq 1$ and $a_t \neq 0$. **Case 2.2.1:** If $\mu = 0$, then there exists exactly one β such that (α, β) is a solution of (Co2) and (B2), whence by Theorem 1, no matter whether (L) can be solved or not, (L) has a covariant embedding with respect to π. In **case 2.2.2** we assume that $\mu \neq 0$. If $b(x)$ satisfies the necessary condition $b_n = 0$ for all $0 \leq n < t$ for finding a solution (α, β) of (Co2) and (B2), then β is uniquely determined and the assertion follows from Theorem 1. If $b(x)$ does not satisfy this condition, i.e. if there exists a coefficient $b_j \neq 0$ for $0 \leq j < t$, then there is no solution (α, β) of (Co2) and (B2), whence there is no covariant embedding of (L). □

5 Survey of results

In order to present all the covariant embeddings of (L) with respect to an analytic iteration group π we add a detailed description of those covariant embeddings with respect to iteration groups of second type which were already found in [3].

Theorem 19. *Assume that $\pi(s, x) = x + c_k s x^k + \ldots$, with $k \geq 2$ and $c_k \neq 0$, is an analytic iteration group of $p(x)$.*

If the coefficient a_0 of $a(x)$ is different from 1, let α be a solution of (Co1) *and* (B2) *given by*

$$\alpha(s, x) = e^{\mu s} P(s, x) \frac{E(\pi(s, x))}{E(x)} \tag{45}$$

where μ is an arbitrary logarithm of a_0, $E(x) = 1 + e_1 x + \ldots$ a uniquely determined series, and

$$P(s, x) := \prod_{n=1}^{k-1} P_{n,\kappa_n}(s, x) = \prod_{n=1}^{k-1} \exp\left(\kappa_n \int_0^s \pi(\sigma, x)^n d\sigma\right)$$

with uniquely determined $\kappa_n \in \mathbb{C}$. Then there exists exactly one β given by

$$\beta(s,x) = e^{\mu s} P(s,x) E(\pi(s,x)) \left[F(x) - e^{-\mu s} \frac{F(\pi(s,x))}{P(s,x)} \right] \tag{46}$$

with $F(x) \in \mathbb{C}[\![x]\!]$ such that (α, β) is a covariant embedding of (L) *with respect to π.*

Assume that $a_0 = 1$ and that α given by (45) *with $\mu \neq 0$ is a solution of* (Co1) *and* (B2)*. If $a(x) = 1 + a_{n_0} x^{n_0} + \dots$ with $n_0 \geq 1$, $a_{n_0} \neq 0$, and $\Big[n_0 < k-1$ or $\big[n_0 = k-1$ and $a_{k-1} - nc_k \neq 0$ for all $n \in \mathbb{N}\big]\Big]$, then there exist solutions (α, β) of* (Co2) *and* (B2) *if and only if $b_n = 0$ for all $0 \leq n < n_0$. If these conditions are satisfied then β of the form* (46) *is uniquely determined by* (B2) *and (α, β) is a covariant embedding of* (L) *with respect to π.*

Assume that $a_0 = 1$ and that α given by (45) *with $\mu = 0$ is a solution of* (Co1) *and* (B2)*. If $a(x) = 1$, let $m_0 = k$, otherwise let m_0 be the smallest element in $\{n \in \mathbb{N} \mid a_n \neq 0\}$, and let $n_0 := \min\{m_0, k\}$. There exists exactly one β of the form*

$$\beta(s,x) = P(s,x) E(\pi(s,x)) \left[F(x) - \frac{F(\pi(s,x))}{P(s,x)} + Q(s,x) \right]$$

where

$$Q(s,x) = \sum_{n=0}^{n_0-1} \int_0^s \frac{\ell_n \pi(\sigma,x)^n}{P(\sigma,x) E(\pi(\sigma,x))} d\sigma$$

with $\ell_n \in \mathbb{C}$ such that (α, β) is a covariant embedding of (L) *with respect to π.*

Furthermore, we have shown in [3] that we may consider without loss of generality only analytic iteration groups of type 1 of the special form $(e^{\lambda s} x)_{s \in \mathbb{C}}$ (and hence $p(x) = \rho x$ with $\rho = e^{\lambda}$). However, according to Theorem 1.3 of [3] our results are valid for any chosen iteration group $(S^{-1}(e^{\lambda s} S(x)))_{s \in \mathbb{C}}$ of type 1 with $S(x) = x + s_2 x^2 + \dots$.

In Remark 5 we proofed: If $p(x) = \rho x + c_2 x^2 + \dots$ has several embeddings into analytic iteration groups with a fixed λ satisfying $e^{\lambda} = \rho$, then (L) has a covariant embedding with respect to a particular analytic iteration group $(S_0^{-1}(e^{\lambda s} S_0(x)))_{s \in \mathbb{C}}$ if and only if it is embeddable with respect to any other analytic iteration group of $p(x)$.

The existence or non-existence of a covariant embedding of (L) with respect to an analytic iteration group π is now completely solved and described for all the different kinds of analytic iteration groups. The theorems dealing with the different situations are collected in the following table:

Situation	Solution
$p(x) = x$ and π of type 1	Theorem 16
$p(x) = x$ and $\pi(s, x) = x$	Theorem 18
$p(x) = e^\lambda x$, $\lambda \neq 0$, e^λ a root of 1	Theorem 4
$p(x) = e^\lambda x$, e^λ not a root of 1	Corollary 4.2 of [3], Theorem 8
$\pi(s, x) = S^{-1}(e^{\lambda s} S(x))$, $\lambda \neq 0$	Theorem 1.3 of [3]
$p(x) = x + c_k x^k + \ldots$, $a_0 = 1$	Corollary 4.2 of [3], Theorem 12, Theorem 15, Theorem 19
$p(x) = x + c_k x^k + \ldots$, $a_0 \neq 1$	Corollary 4.2 of [3], Theorem 19

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