Enumeration in Musical Theory

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Abstract

Being a mathematician and a musician (I play the flute) I found it very interesting to deal with Pólya's counting theory in my Master's thesis. When reading about Pólya's theory I came across an article, called "Enumeration in Music Theory" by D. L. Reiner [11]. I took up his ideas and tried to enumerate some other "musical objects".

At first I would like to generalize certain aspects of 12-tone music to n-tone music, where n is a positive integer. Then I will explain how to interpret intervals, chords, tone-rows, all-interval-rows, rhythms, motifs and tropes in n-tone music. Transposing, inversion and retrogradation are defined to be permutations on the sets of "musical objects". These permutations generate permutation groups, and these groups induce equivalence relations on the sets of "musical objects". The aim of this article is to determine the number of equivalence classes (I will call them patterns) of "musical objects". Pólya's enumeration theory is the right tool to solve this problem.

In the first chapter I will present a short survey of parts of Pólya's counting theory. In the second chapter I will investigate several "musical objects".

Abstract

In dieser Arbeit wird der Begriff von 12-Ton Musik auf n-Ton Musik, wobei n eine natürliche Zahl ist, erweitert. Objekte der Musiktheorie wie Intervall, Akkord, Takt, Motiv, Tonreihe, Allintervallreihe und Trope werden mathematisch gedeutet. Transponieren, Inversion (Umkehrung) und Krebs werden als Permutationen auf geeigneten Mengen interpretiert. Zwei "musikalische Objekte" heißen wesentlich verschieden, falls man sie nicht durch solche Permutationen ineinander überführen kann. In die Sprache der Mathematik übersetzt, bedeutet dies: Abzählen von Äquivalenzklassen (von Funktionen), wobei die Äquivalenz durch eine Permutationsgruppe induziert wird. Dieses Problem wird von der Abzähltheorie von Pólya und von Sätzen, die in Anschluß an diese Theorie entstanden sind, gelöst. Zu diesen Sätzen gehören Theoreme von N.G. de Bruijn und das Power Group Enumeration Theorem von F. Harary.

Im ersten Kapitel stelle ich alle grundlegenden Definitionen zusammen. Dann folgen oben erwähnte Sätze, welche hier in dieser Arbeit nicht bewiesen sind. Das daran anschließende Kapitel beschäftigt sich mit den Anzahlbestimmungen "musikalischer Objekte". Diese Sätze sind nun vollständig bewiesen.

Die Grundidee zu dieser Arbeit habe ich [11] entnommen. Daraufhin habe ich versucht diese Gedanken weiter auszubauen und die Anzahlbestimmung anderer "musikalischer Objekte" durchzuführen. Bisher hatten Musiktheoretiker und Komponisten mit verschiedenen Methoden, oder durch Ausprobieren, solche Anzahlen bestimmt. Durch Verwendung der Theorie von Pólya soll ein System in diese Untersuchungen gebracht werden. Für den Anwender ist es nicht nötig, die Beweise in allen Einzelheiten zu verstehen. Er sollte jedoch mit mathematischen Schreib- und Sprechweisen vertraut sein. Da diese Arbeit auch von Mathematikern gelesen wird, muß sie auch allen mathematischen Forderungen nach Exaktheit und Genauigkeit der Beweise gerecht werden.

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1 Preliminaries

There is a lot of literature about Pólya's counting theory. For instance see [1], [2], [3], [9] or [10].

Definition 1.1 (Type of a Permutation) Let M be a set with $|M| = m$. A permutation $\pi \in S_M$ is of the type $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, iff π can be written as the composition of λ_i disjointed cycles of length i, for $i = 1, \ldots, m$.

Definition 1.2 (Cycle Index) Let P be a set of $|P| = n$ elements and let Γ be a subgroup of S_P, denoted furtheron by $\Gamma \leq S_P$. The cycle index of Γ is defined as a polynomial in n indeterminates $x_1, \ldots, x_n$, defined as:

$$\mathrm{CI}(\Gamma; x_1, \ldots, x_n) := |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \prod_{i=1}^{n} x_i^{\lambda_i(\gamma)}.$$

Lemma 1.1 (Cycle Index of the Cyclic Group) *Let $\zeta_n^{(E)}$ be the cyclic group of order n generated by a cyclic permutation of n objects, then the cycle index of $\zeta_n^{(E)}$ is*

$$\mathrm{CI}(\zeta_n^{(E)}; x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{t|n} \varphi(t) {x_t}^{\frac{n}{t}},$$

where φ is Euler's φ-function.

Lemma 1.2 (Cycle Index of the Dihedral Group) *Let $\vartheta_n^{(E)}$ be the dihedral group of order $2n$ and degree n containing the permutations which coincide with the $2n$ deck transformations of a regular polygon with n vertices.*

1. *If $n \equiv 1 \bmod 2$, then*

$$\mathrm{CI}(\vartheta_n^{(E)}; x_1, x_2, \ldots, x_n) = \frac{1}{2} x_1 {x_2}^{\frac{n-1}{2}} + \frac{1}{2n} \sum_{t|n} \varphi(t) {x_t}^{\frac{n}{t}}.$$

2. *If $n \equiv 0 \bmod 2$, then*

$$\mathrm{CI}(\vartheta_n^{(E)}; x_1, x_2, \ldots, x_n) = \frac{1}{4} ({x_2}^{\frac{n}{2}} + {x_1}^2 {x_2}^{\frac{n-2}{2}}) + \frac{1}{2n} \sum_{t|n} \varphi(t) {x_t}^{\frac{n}{t}}.$$

The main lemma in Pólya's counting theory is

Theorem 1.1 (Lemma of Burnside) *Let P be a finite set and $\Gamma \leq S_P$. Furthermore let $\mathcal{B}$ be the set of the orbits of P under Γ, then*

$$|\mathcal{B}| = |\Gamma|^{-1} \sum_{\gamma \in \Gamma} \chi(\gamma),$$

where $\chi(\gamma)$ is defined as $\chi(\gamma) := |\{p \in P \mid \gamma(p) = p\}|$.

Theorem 1.2 (Pólya's Theorem) *Let P and F be finite sets with $|P| = n$, and let $\Gamma \leq S_P$. Furthermore let $\mathcal{R}$ be a commutative ring over the rationals $\mathbf{Q}$ and let w be a mapping $w: F \to \mathcal{R}$. Two mappings $f_1, f_2 \in F^P$ are called equivalent, iff there exists some $\gamma \in \Gamma$ such that $f_1 \circ \gamma = f_2$. The equivalence classes are called mapping patterns and are written as $[f]$. For every $f \in F^P$ we define the weight $W(f)$ as product weight*

$$W(f) := \prod_{p \in P} w\big(f(p)\big).$$

Any two equivalent f's have the same weight. Thus we may define $W([f]) := W(f)$. Then the sum of the weights of the patterns is

$$\sum_{[f]} W\big([f]\big) = \mathrm{CI}\Big(\Gamma; \sum_{y \in F} w(y), \sum_{y \in F} w(y)^2, \ldots, \sum_{y \in F} w(y)^n\Big).$$

Theorem 1.3 (Power Group Enumeration Theorem) *Let P and F be finite sets, with $|P| = n$ and $|F| = k$, let $\Pi \leq S_P$ and $\Phi \leq S_F$. We will call two mappings $f_1, f_2 \in F^P$ equivalent:*

$$f_1 \sim f_2 :\Longleftrightarrow \exists \pi \in \Pi \quad \exists \varphi \in \Phi \text{ with } f_1 \circ \pi = \varphi \circ f_2.$$

The equivalence classes $[f]$ are called mapping patterns. Let w be a mapping $w: F \to \mathcal{R}$ with $\mathbf{Q} \subseteq \mathcal{R}$ such that

$$W(f) := \prod_{p \in P} w\bigl(f(p)\bigr)$$

is constant on each pattern. Then:

$$\sum_{[f]} W\bigl([f]\bigr) = |\Phi|^{-1} \sum_{\delta \in \Phi} \mathrm{CI}\bigl(\Pi; \kappa_1(\delta), \kappa_2(\delta), \ldots, \kappa_n(\delta)\bigr),$$

where

$$\kappa_i(\delta) := \sum_{\substack{y \in F \\ \delta^i(y) = y}} w(y) \cdot w\bigl(\delta(y)\bigr) \cdot \ldots \cdot w\bigl(\delta^{i-1}(y)\bigr).$$

This is the Power Group Enumeration Theorem in polynomial Form *of [7].*

Theorem 1.4 (de Bruijn) *Let P and F be finite sets with $|P| = n$ and $|F| = k$, let $\Pi \leq S_P$ and $\Phi \leq S_F$. We will call two mappings $f_1, f_2 \in F^P$ equivalent:*

$$f_1 \sim f_2 :\Longleftrightarrow \exists \pi \in \Pi \quad \exists \varphi \in \Phi \text{ with } f_1 \circ \pi = \varphi \circ f_2.$$

The equivalence classes $[f]$ are called mapping patterns. The weight of a function $f \in F^P$ is defined as:

$$W(f) := \begin{cases} 1 & \textit{if } f \textit{ is injective} \\ 0 & \textit{else.} \end{cases}$$

The number of patterns of injective functions is

$$\mathrm{CI}\Bigl(\Pi; \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots \frac{\partial}{\partial x_n}\Bigr)\, \mathrm{CI}(\Phi; 1 + x_1, 1 + 2x_2, \ldots 1 + kx_k)\,\big|_{\,x_1 = x_2 = \ldots = x_k = 0}.$$

Theorem 1.5 (de Bruijn [1]) *Let P and F be finite sets with $|P| = n$ and $|F| = k$, let $\Pi \leq S_P$ and $\phi \in S_F$. We will call two mappings $f_1, f_2 \in F^P$ equivalent:*

$$f_1 \sim f_2 :\Longleftrightarrow \exists \pi \in \Pi \text{ with } f_1 \circ \pi = f_2.$$

The equivalence classes $[f]$ are called mapping patterns. Let

$$Y := \{[f] \mid \phi[f] = [f]\}.$$

Furthermore let $\mathcal{R}$ be a commutative ring over the rationals $\mathbf{Q}$ and let w be a mapping $w: F \to \mathcal{R}$. For every $f \in F^P$ we define the weight $W(f)$ as

$$W(f) := \prod_{p \in P} w\bigl(f(p)\bigr).$$

Then

$$\sum_{[f] \in Y} W\bigl([f]\bigr) = \mathrm{CI}(\Pi; \kappa_1, \kappa_2, \ldots, \kappa_n),$$

where

$$\kappa_i := \sum_{\substack{y \in F \\ \phi^i(y) = y}} w(y) \cdot w\bigl(\phi(y)\bigr) \cdot \ldots \cdot w\bigl(\phi^{i-1}(y)\bigr).$$

2 Applications of Pólya's Theory in Musical Theory

Some parts of this chapter were already discussed by D.L.Reiner in [11]. Now we are going to calculate the number of patterns of chords, intervals, tone-rows, all-interval-rows, rhythms, motifs and tropes. Proving any detail would carry me too far. For further information see [6].

2.1 Patterns of Intervals and Chords

2.1.1 Number of Patterns of Chords

Definition 2.1 (n-Scale) 1. If we divide one octave into n parts, we will speak of an n-scale. The objects of an n-scale are designated as

$$0, 1, \ldots, n-1.$$

2. In twelve tone music we usually identify two tones which are 12 semi-tones apart. For that reason we define an n-scale as the cyclic group $(Z_n, +)$ of order n.

Definition 2.2 (Transposing, Inversion) 1. Let us define T the operation of transposing as a permutation

$$T\colon Z_n \to Z_n$$

$$a \mapsto T(a) := 1 + a.$$

The group $\langle T \rangle$ is the cyclic group $\zeta_n^{(E)}$.

2. Let us define I the operation of inversion as

$$I\colon Z_n \to Z_n$$

$$a \mapsto I(a) := -a.$$

The group $\langle T, I \rangle$ is the dihedral group $\vartheta_n^{(E)}$.

Definition 2.3 (k-Chord) 1. Let $k \leq n$. A k-chord in an n-scale is a subset of k elements of Z_n. An interval is a 2-chord.

2. Let $G = \zeta_n^{(E)}$ or $G = \vartheta_n^{(E)}$. Two k-chords A_1, A_2 are called equivalent iff there is some $\gamma \in G$ such that $A_2 = \gamma(A_1)$.

Remark 2.1 1. We want to work with Pólya's Theorem, therefore I identify each k-chord A with its characteristic function χ_A. Two k-chords A_1, A_2 are equivalent iff the two functions χ_{A_1} and χ_{A_2} are equivalent in the sense of Theorem 1.2.

2. Let us define two finite sets: $P\colon= Z_n$ and $F\colon= \{0, 1\}$. Each function $f \in F^P$ will be identified with

$$A_f\colon= \{k \in Z_n \mid f(k) = 1\}.$$

3. Let $w\colon F \to \mathcal{R} := \mathbf{Q}[x]$ be a mapping with $w(1)\colon= x$ and $w(0)\colon= 1$, where x is an indeterminate. Define the weight $W(f)$ of a function $f \in F^P$ as

$$W(f)\colon= \prod_{k \in Z_n} w\big(f(k)\big).$$

We see that the weight of a k-chord is x^k. The weight of a pattern $W([f]) := W(f)$ is well defined.

Theorem 2.1 (Patterns of k-Chords) *1. Let G be a permutation group on Z_n. The number of patterns of k-chords in the n-scale Z_n is the coefficient of x^k in*

$$\mathrm{CI}(G; 1+x, 1+x^2, \ldots, 1+x^n).$$

2. If $G = \zeta_n^{(E)}$, the number of patterns of k-chords is

$$\frac{1}{n} \sum_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}},$$

where φ is Euler's φ-function.

3. If $G = \vartheta_n^{(E)}$, the number of patterns of k-chords is

$$\begin{cases} \frac{1}{2n}\Big(\sum\limits_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}} + n\binom{\frac{(n-1)}{2}}{[\frac{k}{2}]}\Big) & \text{if } n \equiv 1 \bmod 2 \\ \frac{1}{2n}\Big(\sum\limits_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}} + n\binom{\frac{n}{2}}{\frac{k}{2}}\Big) & \text{if } n \equiv 0 \bmod 2 \text{ and } k \equiv 0 \bmod 2 \\ \frac{1}{2n}\Big(\sum\limits_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}} + n\binom{\frac{n}{2}-1}{[\frac{k}{2}]}\Big) & \text{if } n \equiv 0 \bmod 2 \text{ and } k \equiv 1 \bmod 2. \end{cases}$$

4. In the case $n = 12$ and $G = \zeta_n^{(E)}$, we get the numbers in table 1 on page 23.

5. In the case $n = 12$ and $G = \vartheta_n^{(E)}$, we get the numbers in table 2 on page 23.

Proof:

1. Application of Theorem 1.2.
2. Let us calculate the coefficient of x^k in

$$\mathrm{CI}(\zeta_n^{(E)}; 1+x, 1+x^2, \ldots, 1+x^n) =$$

$$= \frac{1}{n} \sum_{t \mid n} \varphi(t)(1+x^t)^{\frac{n}{t}} = \frac{1}{n} \sum_{t \mid n} \varphi(t) \sum_{i=0}^{\frac{n}{t}} \binom{\frac{n}{t}}{i} x^{t \cdot i}. \qquad (\circ)$$

Let $k := t \cdot i$, then $i = \frac{k}{t}$. With $(\circ)$ we have:

$$\frac{1}{n} \sum_{t \mid n} \varphi(t) \sum_{\substack{k=0 \\ t \mid k}}^{n} \binom{\frac{n}{t}}{\frac{k}{t}} x^k = \frac{1}{n} \sum_{k=0}^{n} \sum_{\substack{t \mid n \\ t \mid k}} \varphi(t) \binom{\frac{n}{t}}{\frac{k}{t}} x^k =$$

$$= \sum_{k=0}^{n} \frac{1}{n} \sum_{t \mid \gcd(n,k)} \varphi(t) \binom{\frac{n}{t}}{\frac{k}{t}} x^k.$$

3. Same proof as 2.

q.e.d. Theorem 2.1

2.1.2 The Complement of a k-Chord

Definition 2.4 (Complement of a k-Chord) Let $A \subseteq Z_n$ with $|A| = k$ be a k-chord. The complement of A is the $(n-k)$-chord $Z_n \setminus A$.

Remark 2.2 1. Let $G = \zeta_n^{(E)}$ or $G = \vartheta_n^{(E)}$ be a permutation group on Z_n and let $1 \leq k < n$. There exists a bijection between the sets of patterns of k-chords and $(n-k)$-chords.

PROOF:

The following general result holds:
Let M_1 and M_2 be two finite sets and f a bijective mapping $f: M_1 \to M_2$. Furthermore let $\sim_i$ be an equivalence relation on M_i and π_i the canonical projection

$$\pi_i: M_i \to M_i|_{\sim_i}$$

$$x \mapsto \pi_i(x) := [x]$$

for $i = 1, 2$. In addition to this the function f satisfies

$$x \sim_1 y \Longleftrightarrow f(x) \sim_2 f(y).$$

Then the function $\bar{f}: M_1|_{\sim_1} \to M_2|_{\sim_2}$ defined by $\bar{f}([x]) := [f(x)]$ is well defined and bijective.

In our context we have the case that M_1 is the set of all k-chords, M_2 is the set of all $(n-k)$-chords, $\sim_i$ is induced by G and $f(A) := Z_n \setminus A$, then $\bar{f}$ is a bijection between the sets of patterns of k-chords and $(n-k)$-chords.

q.e.d. Remark 2.2

2. If $n \equiv 0 \bmod 2$, the complement of an $\frac{n}{2}$-chord is an $\frac{n}{2}$-chord. Now I want to figure out the number of patterns of $\frac{n}{2}$-chords $[A]$ with the property $A \sim Z_n \setminus A$. Applying Theorem 1.5 we get:

Theorem 2.2 *1. Let $n \equiv 0 \bmod 2$. The number of patterns of $\frac{n}{2}$-chords which are equivalent to their complement, is*

$$\mathrm{CI}(G; 0, 2, 0, 2, \ldots).$$

2. If $n = 12$ and $G = \zeta_n^{(E)}$, there are 20 patterns of 6-chords which are equivalent to their complement.

3. If $n = 12$ and $G = \vartheta_n^{(E)}$, there are 8 patterns of 6-chords which are equivalent to their complement.

PROOF:

Let us define two finite sets $P := Z_n$ and $F := \{0, 1\}$ and define a weight function by $W(f) := 1$ for all $f \in F^P$. Each function $f \in F^P$ will be identified with $M_f := \{k \in Z_n \mid f(k) = 1\} = f^{-1}(\{1\})$. The group G defines an equivalence relation on P. Furthermore let $\phi := (0, 1)$ be a transposition in S_F. To determine the number of patterns of $\frac{n}{2}$-chords which are equivalent to their complement, we have to calculate the number of patterns of functions $f \in F^P$ which are invariant under ϕ. Using a special case of Theorem 1.5 we get that this number is given by

$$\mathrm{CI}(G; \kappa_1, \kappa_2, \ldots, \kappa_n)$$

where

$$\kappa_i := \sum_{j|i} j \cdot \mu_j$$

and $(\mu_1, \mu_2, \ldots)$ is the type of the permutation ϕ. Since ϕ is of the type $(0, 1)$, this is

$$\mathrm{CI}(G; 0, 2, 0, 2, \ldots).$$

q.e.d. Theorem 2.2

2.1.3 The Interval Structure of a k-Chord

In this section we use $\vartheta_n^{(E)}$ as the permutation group acting on Z_n. The set of all possible intervals between two differnet tones in n-tone music will be called $\mathrm{Int}(n)$, thus

$$\mathrm{Int}(n) := \{x - y \mid x, y \in Z_n,\ x \neq y\} = \{1, 2, \ldots, n-1\}.$$

Definition 2.5 (Interval Structure) On Z_n we define a linear order $0 < 1 < 2 < \ldots < n-1$. Let $A := \{i_1, i_2, \ldots, i_k\}$ be a k-chord. Without loss of generality let $i_1 < i_2 < \ldots < i_k$. The interval structure of A is defined as the pattern $[f_A]$, wherein the function f_A is defined as

$$f_A \colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$$

$$\begin{aligned}
f_A(1) &:= i_2 - i_1, \\
f_A(2) &:= i_3 - i_2, \\
&\ldots \\
f_A(k-1) &:= i_k - i_{k-1}, \\
f_A(k) &:= i_1 - i_k,
\end{aligned}$$

and two functions $f_1, f_2 \colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$ are called equivalent, iff there exists some $\varphi \in \vartheta_k^{(E)}$ such that $f_2 = f_1 \circ \varphi$. The group $\vartheta_k^{(E)}$ is generated by $\tilde{T}$ and $\tilde{I}$ with $\tilde{T}(i) := i + 1 \bmod k$ and $\tilde{I}(i) := k + 1 - i$ for $i = 1, \ldots, k$. The differences $i_{j+1} - i_j$ must be interpreted as differences in Z_n. They are the intervals between the tones i_j and i_{j+1}.

Theorem 2.3 *Let $A_1 := \{i_1, i_2, \ldots, i_k\}$ and $A_2 := \{j_1, j_2, \ldots, j_k\}$ be two k-chords with $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$. Furthermore let $f := f_{A_1}$ and $g := f_{A_2} \colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$ be constructed as in Definition 2.5. Then*

$$[f] = [g] \Longleftrightarrow [\{i_1, i_2, \ldots, i_k\}] = [\{j_1, j_2, \ldots, j_k\}].$$

PROOF:

$\Longrightarrow$: From $[f] = [g]$ we derive that there exists a $\varphi \in \vartheta_k^{(E)}$ such that $g = f \circ \varphi$. Since $\vartheta_k^{(E)}$ is generated of $\tilde{T}$ and $\tilde{I}$, we have to investigate two cases:

1st **case:** Let $g = f \circ \tilde{T}$, then $f(2) = g(1),\ f(3) = g(2), \ldots,\ f(k) = g(k-1)$ and $f(1) = g(k)$. Hence:

$$\begin{aligned}
i_3 - i_2 &= j_2 - j_1 \\
i_4 - i_3 &= j_3 - j_2 \\
&\ldots \\
i_k - i_{k-1} &= j_{k-1} - j_{k-2} \\
i_1 - i_k &= j_k - j_{k-1} \\
i_2 - i_1 &= j_1 - j_k.
\end{aligned}$$

This can be written as

$$i_3 = j_2 + (i_2 - j_1) \qquad (*)$$

$$i_4 = j_3 + (i_3 - j_2)$$

$$\ldots$$

$$i_k = j_{k-1} + (i_{k-1} - j_{k-2}) \qquad (**)$$

$$i_1 = j_k + (i_k - j_{k-1}) \qquad (***)$$
$$i_2 = j_1 + (i_1 - j_k).$$

Now I want to prove that the terms in brackets are all the same, which means:

$$i_2 - j_1 = i_3 - j_2 = \ldots = i_{k-1} - j_{k-2} = i_k - j_{k-1} = i_1 - j_k.$$

From $(*)$ we get $i_3 - j_2 = i_2 - j_1$. Let us assume that we already know that $i_2 - j_1 = i_{k-1} - j_{k-2}$, then $(**)$ implies that

$$i_k - j_{k-1} = i_{k-1} - j_{k-2} = i_2 - j_1.$$

Rewriting $(***)$ leads to

$$i_1 - j_k = i_k - j_{k-1} = i_2 - j_1.$$

Using this we get $i_{l+1(\mathrm{mod}\, k)} = T^{(i_2 - j_1)} j_l$ for $l = 1, 2, \ldots, k$ and finally

$$[\{i_1, i_2, \ldots, i_k\}] = [\{j_1, j_2, \ldots, j_k\}].$$

2$^{\text{nd}}$ **case:** Let $g = f \circ \tilde{T}^{k-1} \circ \tilde{I}$, then $f(k-1) = g(1)$, $f(k-2) = g(2), \ldots,$ $f(1) = g(k-1)$ and $f(k) = g(k)$. Hence:

$$i_k - i_{k-1} = j_2 - j_1$$
$$i_{k-1} - i_{k-2} = j_3 - j_2$$
$$\ldots$$
$$i_3 - i_2 = j_{k-1} - j_{k-2}$$
$$i_2 - i_1 = j_k - j_{k-1}$$
$$i_1 - i_k = j_1 - j_k.$$

This can be written as:

$$i_k = -j_1 + (i_{k-1} + j_2)$$
$$i_{k-1} = -j_2 + (i_{k-2} + j_3)$$
$$\ldots$$
$$i_3 = -j_{k-2} + (i_2 + j_{k-1})$$
$$i_2 = -j_{k-1} + (i_1 + j_k)$$
$$i_1 = -j_k + (i_k + j_1).$$

In the same way as in the first case we get

$$i_{k-1} + j_2 = i_{k-2} + j_3 = \ldots = i_2 + j_{k-1} = i_1 + j_k = i_k + j_1$$

and this implies

$$i_l = (T^{i_k + j_1} \circ I)(j_{k+1-l})$$

for $l = 1, 2, \ldots, k$, from which we get $[\{i_1, i_2, \ldots, i_k\}] = [\{j_1, j_2, \ldots, j_k\}]$.

Since $\tilde{T}$ and $\tilde{T}^{k-1} \circ \tilde{I}$ generate $\vartheta_k^{(E)}$, the first part of this proof is finished.

$\Longleftarrow$: Assuming that $[\{i_1, i_2, \ldots, i_k\}] = [\{j_1, j_2, \ldots, j_k\}]$ we have to investigate two cases:

1$^{\text{st}}$ **case:** Let $\{j_1, j_2, \ldots, j_k\} = T\{i_1, i_2, \ldots, i_k\}$. Again we have two cases:

1. Let $i_1 < i_2 < \ldots < i_k < n-1$, then $T(i_1) < T(i_2) < \ldots < T(i_k) \le n-1$. This means $j_1 = T(i_1),\ j_2 = T(i_2), \ldots,\ j_k = T(i_k)$. Let the interval structure of $\{i_1, i_2, \ldots, i_k\}$ be $[f]$. For the interval structure $[g]$ of $\{j_1, j_2, \ldots, j_k\}$ we get
$$g(1) = j_2 - j_1 = T(i_2) - T(i_1) = (i_2+1) - (i_1+1) = i_2 - i_1 = f(1)$$
$$g(2) = j_3 - j_2 = T(i_3) - T(i_2) = (i_3+1) - (i_2+1) = i_3 - i_2 = f(2)$$
$$\ldots$$
$$\begin{aligned} g(k-1) = j_k - j_{k-1} = T(i_k) - T(i_{k-1}) &= (i_k+1) - (i_{k-1}+1) = \\ &= i_k - i_{k-1} = f(k-1) \end{aligned}$$
$$g(k) = j_1 - j_k = T(i_1) - T(i_k) = (i_1+1) - (i_k+1) = i_1 - i_k = f(k).$$
Immediately we see that $f = g$ and $[f] = [g]$.
2. Let $i_1 < i_2 < \ldots < i_k = n-1$, then $T(i_k) = 0$, and $T(i_k) < T(i_1) < T(i_2) < \ldots < T(i_{k-1})$, consequently $j_1 = T(i_k),\ j_2 = T(i_1), \ldots,\ j_k = T(i_{k-1})$. Let the interval structure of $\{i_1, i_2, \ldots, i_k\}$ be $[f]$. For the interval structure $[g]$ of $\{j_1, j_2, \ldots, j_k\}$ we get
$$g(1) = j_2 - j_1 = T(i_1) - T(i_k) = (i_1+1) - (i_k+1) = i_1 - i_k = f(k)$$
$$g(2) = j_3 - j_2 = T(i_2) - T(i_1) = (i_2+1) - (i_1+1) = i_2 - i_1 = f(1)$$
$$g(3) = j_4 - j_3 = T(i_3) - T(i_2) = (i_3+1) - (i_2+1) = i_3 - i_2 = f(2)$$
$$\ldots$$
$$\begin{aligned} g(k-1) = j_k - j_{k-1} = T(i_{k-1}) - T(i_{k-2}) &= (i_{k-1}+1) - (i_{k-2}+1) = \\ &= i_{k-1} - i_{k-2} = f(k-2) \end{aligned}$$
$$g(k) = j_1 - j_k = T(i_k) - T(i_{k-1}) = (i_k+1) - (i_{k-1}+1) = i_k - i_{k-1} = f(k-1).$$
Thus $g = f \circ \tilde{T}$ and $[f] = [g]$.

2^{nd} **case:** Let $\{j_1, j_2, \ldots, j_k\} = I\{i_1, i_2, \ldots, i_k\}$. There are two cases:

1. Let $0 < i_1 < i_2 < \ldots < i_k$, then $I(i_k) < I(i_{k-1}) < \ldots < I(i_1)$, thus $j_1 = I(i_k),\ j_2 = I(i_{k-1}), \ldots,\ j_k = I(i_1)$. Let the interval structure of $\{i_1, i_2, \ldots, i_k\}$ be $[f]$. For the interval structure $[g]$ of $\{j_1, j_2, \ldots, j_k\}$ we get
$$g(1) = j_2 - j_1 = I(i_{k-1}) - I(i_k) = i_k - i_{k-1} = f(k-1)$$
$$g(2) = j_3 - j_2 = I(i_{k-2}) - I(i_{k-1}) = i_{k-1} - i_{k-2} = f(k-2)$$
$$\ldots$$
$$g(k-1) = j_k - j_{k-1} = I(i_1) - I(i_2) = i_2 - i_1 = f(1)$$
$$g(k) = j_1 - j_k = I(i_k) - I(i_1) = i_1 - i_k = f(k).$$
Hence $g = f \circ \tilde{T}^{k-1} \circ \tilde{I}$ and $[f] = [g]$.
2. Let $0 = i_1 < i_2 < \ldots < i_k$, then $0 = I(i_1) < I(i_k) < I(i_{k-1}) < \ldots < I(i_2)$, thus $j_1 = I(i_1),\ j_2 = I(i_k),\ j_3 = I(i_{k-1}), \ldots, j_k = I(i_2)$. Let the interval structure of $\{i_1, i_2, \ldots, i_k\}$ be $[f]$. For the interval structure $[g]$ of $\{j_1, j_2, \ldots, j_k\}$ we get
$$g(1) = j_2 - j_1 = I(i_k) - I(i_1) = i_1 - i_k = f(k)$$
$$g(2) = j_3 - j_2 = I(i_{k-1}) - I(i_k) = i_k - i_{k-1} = f(k-1)$$
$$g(3) = j_4 - j_3 = I(i_{k-2}) - I(i_{k-1}) = i_{k-1} - i_{k-2} = f(k-2)$$
$$\ldots$$
$$g(k-1) = j_k - j_{k-1} = I(i_2) - I(i_3) = i_3 - i_2 = f(2)$$
$$g(k) = j_1 - j_k = i(i_1) - I(i_2) = i_2 - i_1 = f(1).$$
Hence $g = f \circ \tilde{I}$ and consequently $[f] = [g]$.

Since T and I generate $\vartheta_n^{(E)}$, everything is proved.

q.e.d. Theorem 2.3

Remark 2.3 If the permutation group acting on Z_n is the cyclic group $\zeta_n^{(E)}$, then the interval structure of $A := \{i_1, i_2, \ldots, i_k\}$ must be defined as the pattern $[f_A]$ in regard to $\zeta_k^{(E)} := \langle \tilde{T} \rangle$ with $\tilde{T}(i) := i + 1 \bmod k$. The function f_A is defined as in Definition 2.5.

Remark 2.4 Let f be a function $f\colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$. The pattern $[f]$ is the interval structure of a k-chord, iff

$$\sum_{i=1}^{k} f(i) = n.$$

One must interpret this sum as a sum of intervals, thus as a sum of positive integers.

PROOF:

$\Longrightarrow$: Let f_A be the interval structure of $A := \{i_1, i_2, \ldots, i_k\}$, with $i_1 < i_2 < \ldots < i_k$, then

$$\begin{aligned} f_A(1) &= i_2 - i_1 \\ f_A(2) &= i_3 - i_2 \\ &\ldots \\ f_A(k-1) &= i_k - i_{k-1} \\ f_A(k) &= i_1 - i_k. \end{aligned}$$

Because of the fact that these differences are differences in Z_n and $i_1 < i_k$ we rewrite $f_A(k) = (i_1 + n) - i_k$. Now we get:

$$\sum_{j=1}^{k} f_A(j) = \sum_{j=1}^{k-1} (i_{j+1} - i_j) + (i_1 + n) - i_k = (-i_1 + i_k) + (i_1 + n) - i_k = n.$$

$\Longleftarrow$: Let f be a function $f\colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$ such that

$$\sum_{i=1}^{k} f(i) = n,$$

then we define

$$\begin{aligned} i_1 &:= 0 \\ i_j &:= \sum_{i=1}^{j-1} f(i) \text{ for } 2 \le j \le k. \end{aligned}$$

It is easily seen, that $[f]$ is the interval structure of $\{i_1, i_2, \ldots, i_k\}$.

q.e.d. Remark 2.4

Remark 2.5 Let $x, y_1, y_2, \ldots, y_n$ be indeterminates over $\mathbf{Q}$ and let $\mathcal{R}$ be the ring

$$\mathcal{R} := \mathbf{Q}[x, y_1, y_2, \ldots, y_n].$$

Now I want to define a weight function

$$w\colon \mathrm{Int}(n) \to \mathcal{R}$$

$$i \mapsto w(i) := x^i y_i.$$

The weight of a function $f\colon \{1, 2, \ldots, k\} \to \mathrm{Int}(n)$ is the product weight

$$W(f) := \prod_{i=1}^{k} w(f(i)) = \prod_{i=1}^{k} x^{f(i)} y_{f(i)} = x^{\sum_{i=1}^{k} f(i)} \prod_{i=1}^{k} y_{f(i)}.$$

Now we can define $W([f]) := W(f)$. According to Remark 2.4 the pattern $[f]$ is the interval structure of a k-chord, iff

$$\sum_{i=1}^{k} f(i) = n.$$

This is true, iff

$$W(f) = x^n \prod_{i=1}^{k} y_{f(i)}.$$

The indices of the y's in $W(f)$ show, which intervals occur in the k-chord.

Theorem 2.4 *The inventory of interval structures of k-chords in n-tone music is the coefficient of x^n in*

$$\mathrm{CI}\Big(\vartheta_k^{(E)}; \sum_{i=1}^{n-1} x^i y_i, \sum_{i=1}^{n-1} x^{2i} {y_i}^2, \sum_{i=1}^{n-1} x^{3i} {y_i}^3, \ldots, \Big).$$

PROOF:
Application of Theorem 1.2.

q.e.d. Theorem 2.4

Example 2.1 The inventory of the interval structures of 3-chords in 12-tone music is the coefficient of x^{12} in

$$\mathrm{CI}\Big(\vartheta_3^{(E)}; \sum_{i=1}^{11} x^i y_i, \sum_{i=1}^{11} x^{2i} {y_i}^2, \sum_{i=1}^{11} x^{3i} {y_i}^3\Big).$$

This is

$${y_1}^2 y_{10} + y_1(y_2 y_9 + y_3 y_8 + y_4 y_7 + y_5 y_6) + {y_2}^2 y_8 + y_2(y_3 y_7 + y_4 y_6 + {y_5}^2) + {y_3}^2 y_6 + y_3 y_4 y_5 + {y_4}^3$$

If you are interested in the number of patterns of 3-chords with intervals $\geq k$, then put $y_1 := y_2 := \ldots := y_{k-1} := 0$ and $y_k := y_{k+1} := \ldots := y_n := 1$. In the case $k = 2$ there are 7 patterns of 3-chords with intervals greater or equal 2.

If the permutation group $\zeta_{12}^{(E)}$ is acting on Z_{12}, then the interval structures of 3-chords in 12-tone music is the coefficient of x^{12} in

$$\mathrm{CI}\Big(\zeta_3^{(E)}; \sum_{i=1}^{12} x^i y_i, \sum_{i=1}^{12} x^{2i} {y_i}^2, \sum_{i=1}^{12} x^{3i} {y_i}^3\Big).$$

This is ${y_1}^2 y_{10} + 2y_1(y_2 y_9 + y_3 y_8 + y_4 y_7 + y_5 y_6) + {y_2}^2 y_8 + y_2(2y_3 y_7 + 2y_4 y_6 + {y_5}^2) + {y_3}^2 y_6 + 2y_3 y_4 y_5 + {y_4}^3$.

2.2 Patterns of Tone-Rows

Definition 2.6 (Tone-Row, k-Row) 1. Arnold Schönberg introduced the so called tone-rows. In this paper I am going to give a mathematical form of his definition. Let $n \geq 3$. A tone-row in an n-scale is a bijectiv mapping

$$\begin{aligned} f\colon \{0,1,\ldots,n-1\} &\to Z_n \\ i &\mapsto f(i). \end{aligned}$$

$f(i)$ is the tone which occurs in i^{th} position in the tone-row.

2. Let $n \geq 3$ and $2 \leq k \leq n$. A k-row in n-tone music is an injective mapping $f\colon \{0,1,\ldots,k-1\} \to Z_n$.

Remark 2.6 1. A k-row with $k = n$ is a tone-row.

2. Two k-rows f_1, f_2 are equivalent if f_1 can be written as transposing, inversion, retrogradation or an arbitrary sequence of these operations of f_2.

   Transposing of a k-row f is $T \circ f$, Inversion of f is $I \circ f$. According to Definition 2.2, we know that T and I are permutations on Z_n, and that $\langle T, I\rangle = \vartheta_n^{(E)}$. Actually inversion of a k-row f should be defined as

$$T^{f(0)} \circ I \circ T^{-f(0)} \circ f.$$

   Retrogradation R, is a permutation $R \in S_{\{0,1,\ldots,k-1\}}$ defined as:

$$R := \begin{cases} (0,k-1) \circ (1,k-2) \circ \ldots \circ (\frac{k}{2}-1, \frac{k}{2}) & \text{if } k \equiv 0 \bmod 2 \\ (0,k-1) \circ (1,k-2) \circ \ldots \circ (\frac{k-3}{2}, \frac{k+1}{2}) \circ (\frac{k-1}{2}) & \text{if } k \equiv 1 \bmod 2. \end{cases}$$

   Let $\Pi := \langle R\rangle \leq S_{\{0,1,\ldots,k-1\}}$, then $|\Pi| = 2$. Retrogradation of a k-row f is defined as $f \circ R$.

3. Since $\Pi\colon = \langle R\rangle$, the cycle index of Π is

$$\mathrm{CI}(\Pi; y_1, y_2, \ldots, y_k) = \begin{cases} \frac{1}{2}({y_1}^k + {y_2}^{\frac{k}{2}}) & \text{if } k \equiv 0 \bmod 2 \\ \frac{1}{2}({y_1}^k + y_1 {y_2}^{\frac{k-1}{2}}) & \text{if } k \equiv 1 \bmod 2. \end{cases}$$

   Thus two k-rows f_1, f_2 are equivalent

$$\iff \exists \varphi \in \vartheta_n^{(E)} \exists \sigma \in \Pi \text{ such that } f_1 = \varphi \circ f_2 \circ \sigma.$$

Theorem 2.5 (Number of Patterns of k-Rows) *The number of patterns of k-rows in Z_n is*

$$\mathrm{CI}\Big(\Pi; \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k}\Big)\, \mathrm{CI}(\vartheta_n^{(E)}; 1+x_1, 1+2x_2, \ldots, 1+nx_n)\,\Big|_{x_1=x_2=\ldots=x_n=0}.$$

This is

1.

$$\frac{1}{2}\left(\frac{1}{4}\left((2)_k + 2^{\frac{k}{2}}(\frac{k}{2})!\left(\binom{\frac{n}{2}}{\frac{k}{2}} + \binom{\frac{n-2}{2}}{\frac{k}{2}}\right)\right) + \frac{1}{2n}\left(\binom{n}{k}k! + 2^{\frac{k}{2}}(\frac{k}{2})!\binom{\frac{n}{2}}{\frac{k}{2}}\right)\right),$$

if $n \equiv 0 \bmod 2$ and $k \equiv 0 \bmod 2$. For integers k, v, $v \geq 0$ the expression $(k)_v$ is definied as:

$$(k)_v := k \cdot (k-1) \cdot \ldots \cdot \big(k-(v-1)\big).$$

2.

$$\frac{1}{2}\left(\frac{1}{4} \cdot 2 \cdot 2^{\frac{k-1}{2}} \binom{\frac{n-2}{2}}{\frac{k-1}{2}} (\frac{k-1}{2})! + \frac{1}{2n}\binom{n}{k}k!\right),$$

if $n \equiv 0 \bmod 2$ and $k \equiv 1 \bmod 2$.

3.

$$\frac{1}{2}\left(\frac{1}{2n}\binom{n}{k}k! + \frac{1}{2}2^{\frac{k}{2}}\binom{\frac{n-1}{2}}{\frac{k}{2}}(\frac{k}{2})!\right),$$

if $n \equiv 1 \bmod 2$ *and* $k \equiv 0 \bmod 2$.

4.

$$\frac{1}{2}\left(\frac{1}{2n}\binom{n}{k}k! + \frac{1}{2}2^{\frac{k-1}{2}}\binom{\frac{n-1}{2}}{\frac{k-1}{2}}(\frac{k-1}{2})!\right),$$

if $n \equiv 1 \bmod 2$ *and* $k \equiv 1 \bmod 2$.

In the case $n = 12$ *the number of patterns of* k*-rows is in table 3 on page 23.*

Proof:
Application of Theorem 1.4.

q.e.d. Theorem 2.5

Theorem 2.6 (Number of patterns of Tone-Rows) *Let* $n \geq 3$. *The number of patterns of tone-rows in* n*-tone music is*

$$\begin{cases} \frac{1}{4}\Big((n-1)! + (n-1)!!\Big) & \text{if } n \equiv 1 \bmod 2 \\ \frac{1}{4}\Big((n-1)! + (n-2)!!(\frac{n}{2}+1)\Big) & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

If n *is in* $\mathbf{N}$ *then*

$$n!! = \begin{cases} n \cdot (n-2) \cdot \ldots \cdot 2 & \text{if } n \equiv 0 \bmod 2 \\ n \cdot (n-2) \cdot \ldots \cdot 1 & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

Especially there are 9 985 920 patterns of tone-rows in 12-tone music.

Proof:
This is a special case of Theorem 2.5.

1. If $n \equiv 0 \bmod 2$, then the number of patterns of n-rows is

$$\frac{1}{2}\Big(\frac{1}{4}\big((2)_n + 2^{\frac{n}{2}}(\frac{n}{2})!\big) + \frac{1}{2n}\big(n! + 2^{\frac{n}{2}}(\frac{n}{2})!\big)\Big). \qquad (*)$$

   Since $n \geq 3$, we have $(2)_n = 0$. Furthermore

$$2^{\frac{n}{2}}(\frac{n}{2})! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (n-2) \cdot n = n!!.$$

   $(*)$ can be written as

$$\frac{1}{4}(\frac{1}{2}n!!) + \frac{1}{4}\big(\frac{1}{n}(n! + n!!)\big) = \frac{1}{4}\big((n-1)! + (n-2)!!(\frac{n}{2}+1)\big).$$

2. If $n \equiv 1 \bmod 2$, then the number of patterns of n-rows is

$$\frac{1}{2}\big(\frac{1}{2n}n! + \frac{1}{2}2^{\frac{n-1}{2}}(\frac{n-1}{2})!\big) = \frac{1}{4}\big((n-1)! + (n-1)!!\big).$$

q.e.d. Theorem 2.6

2.3 Patterns of All-Interval-Rows

Let A and B be two finite sets. The set of all injective functions $f\colon A \to B$ will be denoted by $\mathrm{Inj}(A, B)$. For that reason the set of all tone-rows is $\mathrm{Inj}\bigl(\{0, 1, \ldots, n-1\}, Z_n\bigr)$. In this chapter let $n \geq 3$.

Definition 2.7 (All-Interval-Rows) Let us define a mapping

$$\alpha\colon \mathrm{Inj}\bigl(\{0, 1, \ldots, n-1\}, Z_n\bigr) \to \{g \mid g\colon \{1, 2, \ldots, n-1\} \to \mathrm{Int}(n)\}$$

$$f \mapsto \alpha(f)$$

and $\alpha(f)(i) := f(i) - f(i-1)$ for $i = 1, 2, \ldots, n-1$. This is subtraction in Z_n. The function $\alpha(f)$ is called all-interval-row, iff $\alpha(f)$ is injective, that means $\alpha(f) \in \mathrm{Inj}\bigl(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n)\bigr)$. In other words a tone-row induces an all-interval-row, iff all possible intervals occur as differences between two successive tones of the tone-row. The set of all all-interval-rows will be denoted as $\mathrm{Allint}(n)$.

Let's define some mappings:

1. $$\beta\colon \mathrm{Inj}\bigl(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n)\bigr) \to \{g \mid g\colon \{0, 1, \ldots, n-1\} \to Z_n\}$$

   $$f \mapsto \beta(f)$$

   $\beta(f)(0) := 0$ and $\beta(f)(i) := \beta(f)(i-1) + f(i) \bmod n$ for $i = 1, 2, \ldots, n-1$. You can easily derive that

   $$\beta(f)(i) \equiv \sum_{j=1}^{i} f(j) \bmod n$$

   for $i = 0, 1, \ldots, n-1$.

2. Let $l \in Z_n$.

   $$\tilde{\beta}\colon \mathrm{Inj}\bigl(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n)\bigr) \to \{g \mid g\colon \{0, 1, \ldots, n-1\} \to Z_n\}$$

   $$f \mapsto \tilde{\beta}(f)$$

   $$\tilde{\beta}(f)(i) \equiv \sum_{j=1}^{i} f(j) + l \bmod n.$$

3. There is another possibility to generalize β by expanding its domain.

   $$\hat{\beta}\colon \bigl\{f \mid f\colon \{1, 2, \ldots, n-1\} \to \mathrm{Int}(n)\bigr\} \to \bigl\{g \mid g\colon \{0, 1, \ldots, n-1\} \to Z_n\bigr\}$$

   $$f \mapsto \hat{\beta}(f),$$

   $$\hat{\beta}(f)(i) \equiv \sum_{j=1}^{i} f(j) \bmod n$$

   for $i = 0, 1, \ldots, n-1$.

Theorem 2.7 *Let f be a mapping $f\colon \{1, 2, \ldots, n-1\} \to \mathrm{Int}(n)$. The following statements are equivalent:*

1. *f is an all-interval-row.*
2. *$f \in \mathrm{Inj}\bigl(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n)\bigr)$ and $\beta(f) \in \mathrm{Inj}\bigl(\{0, 1, \ldots, n-1\}, Z_n\bigr)$.*
3. *$f \in \mathrm{Inj}\bigl(\{1, 2, \ldots, n-1\}, \mathrm{Int}(n)\bigr)$ and $\tilde{\beta}(f) \in \mathrm{Inj}\bigl(\{0, 1, \ldots, n-1\}, Z_n\bigr)$.*

4. *f is injective and $\hat{\beta}(f) \in \operatorname{Inj}(\{0,1,\ldots,n-1\}, Z_n)$.*

PROOF:
I only want to prove that 1 is equivalent to 2.

<u>$1 \Longrightarrow 2$</u>: Let $g \in \operatorname{Inj}(\{0,1,\ldots,n-1\}, Z_n)$ and $f = \alpha(g)$, then $\alpha(g) \in \operatorname{Inj}(\{1,2,\ldots,n-1\}, \operatorname{Int}(n))$. For $0 \leq i < n$ we calculate

$$\beta(\alpha(g))(i) \equiv \sum_{j=1}^{i} \alpha(g)(j) = \sum_{j=1}^{i} (g(j) - g(j-1)) = g(i) - g(0) \bmod n.$$

Thus $\beta(f) = \beta(\alpha(g)) = (T^{-g(0)} \circ g)$. Consequently it is injective and $\beta(f) \in \operatorname{Inj}(\{0,1,\ldots, n-1\}, Z_n)$.

<u>$2 \Longrightarrow 1$</u>: Since $f \in \operatorname{Inj}(\{1,2,\ldots,n-1\}, \operatorname{Int}(n))$ and $\beta(f) \in \operatorname{Inj}(\{0,1,\ldots,n-1\}, Z_n)$ let us calculate

$$\alpha(\beta(f))(i) = \beta(f)(i) - \beta(f)(i-1) \equiv \sum_{j=1}^{i} f(j) - \sum_{j=1}^{i-1} f(j) = f(i) \bmod n.$$

We conclude that $\alpha(\beta(f)) = f \in \operatorname{Inj}(\{1,2,\ldots,n-1\}, \operatorname{Int}(n))$, hence f is an all-interval-row.

q.e.d. Theorem 2.7

You can easily prove the following results:

1. If $n \equiv 1 \bmod 2$, there are no all-interval-rows.
2. If $n \equiv 0 \bmod 2$ the function f defined as

$$f(i) := \begin{cases} i & \text{if } i \equiv 1 \bmod 2 \\ -i & \text{if } i \equiv 0 \bmod 2 \end{cases}$$

is an all-interval-row.

For the rest of this chapter let $n \geq 4$ and $n \equiv 0 \bmod 2$.

3. $f \in \operatorname{Allint}(n)$ implies $\beta(f)(n-1) = \frac{n}{2}$.
4. $f \in \operatorname{Allint}(n)$ implies $f(1) \neq \frac{n}{2}$ and $f(n-1) \neq \frac{n}{2}$.

Remark 2.7 1. On $\operatorname{Int}(n)$ we have the following permutations:

$$I\colon \operatorname{Int}(n) \to \operatorname{Int}(n)$$
$$j \mapsto I(j) := n - j.$$

I stands for inversion. I is of the type $(1, \frac{n}{2} - 1, 0, \ldots)$.

In the case $n = 12$ there is a further permutation called

$$Q\colon \operatorname{Int}(n) \to \operatorname{Int}(n)$$
$$j \mapsto Q(j) :\equiv 5 \cdot j \bmod 12.$$

Q stands for quartcircle symmetry. Since $\gcd(5, 12) = 1$, Q is a permutation on Z_n, and since $5 \cdot 0 = 0$, Q is a permutation on $\operatorname{Int}(n)$. Q is of the type $(3, 4, 0, \ldots, 0)$. You can easily prove that $(I \circ Q)(j) = (Q \circ I)(j) = 7 \cdot j \bmod 12$ and that it is of the type $(5, 3, 0, \ldots, 0)$. $I \circ Q$ is called quintcircle symmetry.

2. On the set $\{1, 2, \dots, n-1\}$ retrogradation R is a permutation, defined as
$$R := (1, n-1) \circ (2, n-2) \circ \dots \circ (\frac{n}{2} - 1, \frac{n}{2} + 1) \circ (\frac{n}{2}).$$
3. If $f \in \mathrm{Allint}(n)$, then $I \circ f$, $f \circ R$ are in $\mathrm{Allint}(n)$. Furthermore if $n = 12$ then $Q \circ f \in \mathrm{Allint}(12)$.
4. For that reason we can define the following permutations on $\mathrm{Allint}(n)$.
$$\varphi_I, \varphi_R, \varphi_Q \colon \mathrm{Allint}(n) \to \mathrm{Allint}(n)$$
$$f \mapsto \varphi_I(f) := I \circ f$$
$$f \mapsto \varphi_R(f) := f \circ R$$
$$f \mapsto \varphi_Q(f) := Q \circ f.$$
For φ_Q we need the assumption that $n = 12$.
5. It is easy to prove that these permutations commute in pairs and that ${\varphi_I}^2 = {\varphi_R}^2 = {\varphi_Q}^2 = \mathrm{id}$.
6. In [4] there is a further permutation E called exchange at $\frac{n}{2}$. It is defined as
$$E \colon \mathrm{Allint}(n) \to \mathrm{Allint}(n)$$
$$f \mapsto E(f)$$
and
$$E(f)(i) := \begin{cases} f\big(f^{-1}(\frac{n}{2}) + i\big) & \text{if } i < n - f^{-1}(\frac{n}{2}) \\ \frac{n}{2} & \text{if } i = n - f^{-1}(\frac{n}{2}) \\ f\big(i - n + f^{-1}(\frac{n}{2})\big) & \text{if } i > n - f^{-1}(\frac{n}{2}). \end{cases}$$
I have already mentioned, that $f(1) \neq \frac{n}{2}$ and $f(n-1) \neq \frac{n}{2}$. Since $f \in \mathrm{Allint}(n)$ is bijective, there exists exactly one j, such that $1 < j < n-1$ and $f(j) = \frac{n}{2}$. The values of the function $E(f)(i)$ for $i = 1, 2, \dots, n-1$ are $f(j+1), f(j+2), \dots, f(n-1), f(j) = \frac{n}{2}, f(1), f(2), \dots, f(j-1)$. The permutation E is defined for $n \geq 4$, but in the case $n = 4$ we have $E = \varphi_R$.
Now I want to prove that E is well defined. According to Theorem 2.7 we have to prove that $\tilde\beta\big(E(f)\big)$ is injective, in the course of which $\tilde\beta\big(E(f)\big)(0) := \beta(f)\big(f^{-1}(\frac{n}{2})\big)$. For $i < n - f^{-1}(\frac{n}{2})$ we derive
$$\tilde\beta\big(E(f)\big)(i) \equiv \sum_{j=1}^{i} E(f)(j) + \beta(f)\big(f^{-1}(\frac{n}{2})\big) =$$
$$= \sum_{j=1}^{i} f\big(f^{-1}(\frac{n}{2}) + j\big) + \beta(f)\big(f^{-1}(\frac{n}{2})\big) \equiv \beta(f)\big(f^{-1}(\frac{n}{2}) + i\big).$$
Especially
$$\tilde\beta\big(E(f)\big)\big(n - f^{-1}(\frac{n}{2}) - 1\big) = \beta(f)(n-1) = \frac{n}{2},$$
and for that reason
$$\tilde\beta\big(E(f)\big)\big(n - f^{-1}(\frac{n}{2})\big) \equiv \tilde\beta\big(E(f)\big)\big(n - f^{-1}(\frac{n}{2}) - 1\big) + E(f)\big(n - f^{-1}(\frac{n}{2})\big) =$$
$$= \frac{n}{2} + \frac{n}{2} \equiv 0 = \beta(f)(0).$$
For $i > n - f^{-1}(\frac{n}{2})$ we calculate
$$\tilde\beta\big(E(f)\big)(i) \equiv \tilde\beta\big(E(f)\big)\big(n - f^{-1}(\frac{n}{2})\big) + \sum_{j=1}^{i-(n-f^{-1}(\frac{n}{2}))} E(f)\big(n - f^{-1}(\frac{n}{2}) + j\big) =$$

$$= 0 + \sum_{j=1}^{i-(n-f^{-1}(\frac{n}{2}))} f(j) = \beta(f)\big(i - n + f^{-1}(\frac{n}{2})\big).$$

Thus everything is proved.

7. The following formulas hold: $E \circ \varphi_I = \varphi_I \circ E$, $E \circ \varphi_Q = \varphi_Q \circ E$, $E \circ \varphi_R = \varphi_R \circ E$ and $E^2 = \mathrm{id}$.

8. Let us define three permutation groups on $\mathrm{Allint}(n)$.
   $G_1 := \langle \varphi_I, \varphi_R \rangle, G_2 := \langle \varphi_I, \varphi_R, E \rangle$ und $G_3 := \langle \varphi_I, \varphi_R, E, \varphi_Q \rangle$. For G_2 we must assume $n \geq 6$, and for G_3 we must assume $n = 12$. We calculate that $|G_1| = 4,\ |G_2| = 8,\ |G_3| = 16$.

Remark 2.8 (Counting of All-Interval-Rows) Let

$$x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{n-1}, z_1, z_2, \ldots, z_{n-1}$$

be indeterminates over $\mathbf{Q}$. Furthermore let f be a mapping $f\colon \{1, 2, \ldots, n-1\} \to \mathrm{Int}(n)$. We define:

$$\mathcal{R} := \mathbf{Q}[x_1, x_2, \ldots, x_{n-1}, z_1, z_2, \ldots, z_{n-1}]$$

and

$$W(f) := \prod_{i:=1}^{n-1} w_i\big(f(i)\big).$$

The functions w_i are defined as

$$w_i\colon \mathrm{Int}(n) \to \mathcal{R}$$

$$j \mapsto w_i(j) := z_j \prod_{\nu:=i}^{n-1} {x_\nu}^j.$$

After calculating $W(f)$ you have to replace terms of the form ${x_\nu}^j$ by $y_{j \bmod n}$. Then you get $\tilde{W}(f) \in \mathbf{Q}[y_1, y_2, \ldots, y_{n-1}, z_1, z_2, \ldots, z_{n-1}]$. According to Theorem 2.7 f is an all-interval-row, if and only if, $\tilde{W}(f) = \prod_{i=1}^{n-1} y_i z_i$.

PROOF:
f is an all-interval-row, if and only if, f and $\hat{\beta}(f)$ are injective. The function f is injective, iff $\tilde{W}(f)$ is divisible by $\prod_{i=1}^{n-1} z_i$. According to the construction of $\tilde{W}(f)$ the power of x_i is $\sum_{j=1}^{i} f(j) \equiv \hat{\beta}(f)(i) \bmod n$. Thus

$$\tilde{W}(f) = \prod_{i=1}^{n-1} z_{f(i)} y_{\hat{\beta}(f)(i)}$$

and the function $\hat{\beta}(f)$ is injective, iff $\tilde{W}(f)$ is divisible by $\prod_{i=1}^{n-1} y_i$. Consequently the number of all-interval-rows in n-tone music is the coefficient of $\prod_{i=1}^{n-1} y_i z_i$ in

$$\prod_{i=1}^{n-1} \Big(\sum_{j=1}^{n-1} z_j \prod_{k=i}^{n-1} {x_k}^j \Big) \Big|_{{x_\nu}^j = y_{j \bmod n}}.$$

q.e.d. Remark 2.8

Remark 2.9 For $\varphi \in G_1$ or G_2 or G_3 we want to calculate

$$\chi(\varphi) := |\{ f \in \mathrm{Allint}(n) \mid \varphi(f) = f \}|.$$

After some calculations we can derive that there are only 4 permutions φ such that $\chi(\varphi) \neq 0$. In Remark 2.8 we calculated $\chi(\mathrm{id})$. The value of $\chi(\varphi_I \circ \varphi_R)$ is the coefficient of $\prod_{i=1}^{n-1} y_i z_i$ in

$$\prod_{i=1}^{\frac{n}{2}-1} \Big(\sum_{\substack{j=1 \\ j \neq \frac{n}{2}}}^{n-1} z_j z_{n-j} \prod_{k=i}^{n-1} x_k{}^j \prod_{k=n-i}^{n-1} x_k{}^{n-j} \Big) z_{\frac{n}{2}} \prod_{k=\frac{n}{2}}^{n-1} x_k{}^{\frac{n}{2}} \Big|_{x_\nu{}^j = y_{j \bmod n}}.$$

Now let $n \geq 6$. The value of $\chi(\varphi_I \circ V)$ is the coefficient of $\prod_{i=1}^{n-1} y_i z_i$ in

$$\prod_{i=1}^{\frac{n}{2}-1} \Big(\sum_{\substack{j=1 \\ j \neq \frac{n}{2}}}^{n-1} z_j z_{n-j} \prod_{k=i}^{n-1} x_k{}^j \prod_{k=(\frac{n}{2}+i)}^{n-1} x_k{}^{n-j} \Big) z_{\frac{n}{2}} \prod_{k=\frac{n}{2}}^{n-1} x_k{}^{\frac{n}{2}} \Big|_{x_\nu{}^j = y_{j \bmod n}}.$$

Now let $n = 12$. In order to calculate $\chi(\varphi_Q \circ V \circ \varphi_R)$ you must compute

$$\sum_{i=1}^{5} \Bigg(z_6 \prod_{j=2i}^{11} x_j{}^6 z_3 z_9 \Big(\prod_{j=i}^{11} x_j{}^3 \prod_{j=i+6}^{11} x_j{}^9 + \prod_{j=i}^{11} x_j{}^9 \prod_{j=i+6}^{11} x_j{}^3 \Big) \cdot$$

$$\cdot \prod_{j=1}^{i-1} \Big(\sum_{\substack{k=1 \\ k \notin \{3,6,9\}}}^{n-1} z_k z_{5k \bmod 12} \prod_{l=j}^{11} x_l{}^k \prod_{l=2i-j}^{11} x_l{}^{5k \bmod 12} \Big) \cdot$$

$$\cdot \prod_{j=2i+1}^{i+5} \Big(\sum_{\substack{k=1 \\ k \notin \{3,6,9\}}}^{n-1} z_k z_{5k \bmod 12} \prod_{l=j}^{11} x_l{}^k \prod_{l=12+2i-j}^{11} x_l{}^{5k \bmod 12} \Big) \Bigg).$$

Then substitute $y_{j \bmod 12}$ for $x_\nu{}^j$ and find the coefficient of $\prod_{i=1}^{11} y_i z_i$.

Theorem 2.8 (Number of Patterns of All-Interval-Rows) *The number of patterns of all-interval-rows in regard to G_i for $i = 1, 2, 3$ is*

1. $\frac{1}{4}\big(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R)\big)$ *for $i = 1$.*
2. $\frac{1}{8}\big(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R) + \chi(\varphi_I \circ V)\big)$ *for $i = 2$.*
3. *For $i = 3$ we calculate*

$$\frac{1}{16}\big(\chi(\mathrm{id}) + \chi(\varphi_I \circ \varphi_R) + \chi(\varphi_I \circ V) + \chi(\varphi_Q \circ \varphi_R \circ V)\big) =$$

$$= \frac{1}{16}(3\,856 + 176 + 120 + 120) = 267.$$

PROOF:
Application of the Lemma of Bunside, Theorem 1.1.

q.e.d. Theorem 2.8

2.4 Patterns of Rhythms

Definition 2.8 (n-Bar, Entry-time, k-Rhythm) An important contribution in a composition is a bar. Usually a lot of bars of the same form follow one another. If you know the smallest rhythmical subdivision of a bar, you can figure out how many entry-times (think of rhythmical accents played on a drum) a bar holds. If there are n entry-times in a bar, I call it an n-bar. In mathematical terms an n-bar is expressed as the cyclic group Z_n. We can define cyclic temporal shifting S as a permutation

$$S \colon Z_n \to Z_n$$

$$t \mapsto S(t) := t + 1.$$

Retrogradation R (temporal inversion) is defined as

$$R: Z_n \to Z_n$$

$$t \mapsto R(t) := -t.$$

The group $\langle S \rangle$ is $\zeta_n^{(E)}$ and $\langle S, R \rangle = \vartheta_n^{(E)}$. A k-rhythm in an n-bar is a subset of k elements of Z_n. The permutation groups $\zeta_n^{(E)}$ or $\vartheta_n^{(E)}$ induce an equivalence relation on the set of all k-rhythms. Now we want to calculate the number of patterns of k-rhythms. We get the same numbers as in Theorem 2.1.

Theorem 2.9 (Patterns of k-Rhythms) *1. Let G be a permutation group on Z_n. The number of patterns of k-rhythms in the n-bar Z_n is the coefficient of x^k in*

$$\mathrm{CI}(G; 1 + x, 1 + x^2, \ldots, 1 + x^n).$$

2. If $G = \zeta_n^{(E)}$, the number of patterns of k-rhythms is

$$\frac{1}{n} \sum_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}},$$

where φ is Euler's φ-function.

3. If $G = \vartheta_n^{(E)}$, the number of patterns of k-rhythms is

$$\begin{cases} \frac{1}{2n}\Big(\sum\limits_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}} + n \binom{\frac{(n-1)}{2}}{[\frac{k}{2}]} \Big) & \text{if } n \equiv 1 \bmod 2 \\ \frac{1}{2n}\Big(\sum\limits_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}} + n \binom{\frac{n}{2}}{\frac{k}{2}} \Big) & \text{if } n \equiv 0 \bmod 2 \text{ and } k \equiv 0 \bmod 2 \\ \frac{1}{2n}\Big(\sum\limits_{j \mid \gcd(n,k)} \varphi(j) \binom{\frac{n}{j}}{\frac{k}{j}} + n \binom{\frac{n}{2}-1}{[\frac{k}{2}]} \Big) & \text{if } n \equiv 0 \bmod 2 \text{ and } k \equiv 1 \bmod 2. \end{cases}$$

2.5 Patterns of Motifs

Definition 2.9 (k-Motif) 1. Now I want to combine both rhythmical and tonal aspects of music.

2. Assume we have an n-scale and an m-bar, then the set M

$$M := \{(x, y) \mid x \in Z_m, y \in Z_n\} = Z_m \times Z_n$$

is the set of all possible combinations of entry-times in the m-bar Z_m and pitches in the n-scale Z_n. Furthermore let G be a permutation group on M. In Remark 2.11 we are going to study two special groups G. The group G defines an equivalence relation on M:

$$(x_1, y_1) \sim (x_2, y_2) :\Longleftrightarrow \exists g \in G \text{ with } (x_2, y_2) = g(x_1, y_1).$$

In addition to this we have $|M| = m \cdot n$.

3. Let $1 \le k \le m \cdot n$. A **k-motif** is a subset of k elements of M.

Remark 2.10 Let f be a mapping $f: M \to \{0, 1\}$. Now we identify f with the set

$$M_f := \{(x, y) \in M \mid f(x, y) = 1\}.$$

This means: f is the characteristic function of M_f. The function f is the characteristic function of a k-motif $\iff |M_f| = k$. Two functions $f_1, f_2\colon M \to \{0,1\}$ are defined equivalent

$$f_1 \sim f_2 :\iff \exists g \in G \text{ with } f_2 = f_1 \circ g.$$

Let f_i be the characteristic function of the k-motif M_i for $i = 1, 2$, then we have:

$$f_1 \sim f_2 \iff \exists g \in G \text{ with } g(M_2) = M_1.$$

Now $w(0) := 1$ and $w(1) := x$ define a weight function $w\colon \{0,1\} \to \mathbf{Q}[x]$, where x is an indeterminate over $\mathbf{Q}$. In addition to this let

$$W(f) := \prod_{p \in M} w\big(f(p)\big).$$

Now we can say: f is the characteristic function of a k-motif $\iff W(f) = x^k$.

Theorem 2.10 (Number of Patterns of k-Motifs) *The number of patterns of k-motifs in an n-scale and in an m-bar is the coefficient of x^k in*

$$\mathrm{CI}(G; 1+x, 1+x^2, \ldots, 1+x^{m\cdot n}).$$

PROOF:
This completely follows Pólya's Theorem 1.2.

q.e.d. Theorem 2.10

Remark 2.11 (Special Permutation Groups) Now I want to demonstrate two examples for group G.

1. In Definition 2.2 we had a permutation group $G_2 = \zeta_n^{(E)}$ or $G_2 = \vartheta_n^{(E)}$ acting on the n-scale Z_n. Moreover in Definition 2.8 there was a permutation group $G_1 = \zeta_m^{(E)}$ or $G_1 = \vartheta_m^{(E)}$ defined on the m-bar Z_m. For that reason, we define the group G as $G := G_1 \otimes G_2$. Two elements $(x_1, y_1), (x_2, y_2) \in M$ are called equivalent with respect to G, iff there exist $\varphi \in G_1$ and $\psi \in G_2$, with
$$(x_2, y_2) = (\varphi, \psi)(x_1, y_1) = (\varphi(x_1), \psi(y_1)).$$
Because of the fact that we know how to calculate the cycle index of $G_1 \otimes G_2$, we can compute the number of patterns of k-motifs.

2. In the case $m = n$, we can define another permutation group G, as it is done in [8]. The group G is defined as
$$G := \langle T, S, \varphi_A \bigm| A \in \mathrm{Gl}\,(2, Z_n) \rangle,$$
with
$$T\colon M \to M$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x \\ y+1 \end{pmatrix}$$
$$S\colon M \to M$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto S\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x+1 \\ y \end{pmatrix}$$
$$\varphi_A\colon M \to M$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \varphi_A\begin{pmatrix} x \\ y \end{pmatrix} := A\begin{pmatrix} x \\ y \end{pmatrix}.$$
The multiplication $A \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ stands for matrix multiplication. The set $\mathrm{Gl}\,(2, Z_n)$ is the group of all regular 2×2-matrices over Z_n.
You can easily derive the following results:

(a) $T^n = S^n = \mathrm{id}_M$ and $T^j \neq \mathrm{id}_M$ and $S^j \neq \mathrm{id}_M$ for $1 \leq j < n$.

(b) $T \circ S = S \circ T$. In addition to this $T \notin \langle S \rangle$ and $S \notin \langle T \rangle$.

(c) Let $0 \leq i, j < n$, then: $T^i \circ S^j \notin \langle \varphi_A \bigm| A \in \mathrm{Gl}\,(2, Z_n) \rangle$, iff $i \neq 0$ or $j \neq 0$.

(d) Let $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then: $\varphi_A \circ T^k \circ S^l = T^{(cl+dk)} \circ S^{(al+bk)} \circ \varphi_A$.

(e) G is the group of all affine mappings ${Z_n}^2 \to {Z_n}^2$.

Although we know quite a lot about the group G, I could not find a formula for the cycle index of G for arbitrary n.

Example 2.2 Let us consider the case, that $n = m = 12$.

1. If G is defined as $G := \vartheta_n^{(E)} \otimes \vartheta_n^{(E)}$, then we derive

$$\mathrm{CI}(G; x_1, x_2, \ldots, x_{144}) =$$

$$= \tfrac{1}{576}(x_1^{144} + 12x_1^{24}x_2^{60} + 36x_1^4x_2^{70} + 147x_2^{72} + 8x_3^{48} + 24x_3^8x_6^{20} + 60x_4^{36} + 96x_6^{24} + 192x_{12}^{12}).$$

By applying Theorem 2.10, the number of patterns of k-motifs is the coefficient of x^k in $1 + x + 48x^2 + 937x^3 + 31\,261x^4 + 840\,006x^5 19\,392\,669x^6 + 381\,561\,281x^7 + 6\,532\,510\,709x^8 + 98\,700\,483\,548x^9 + 1\,332\,424\,197\,746x^{10} + \ldots$.

2. If $G := \langle T, S, \varphi_A \bigm| A \in \mathrm{Gl}\,(2, Z_n) \rangle$, I computed the cycle index of G with a Turbo Pascal program as

$$\mathrm{CI}(G; x_1, x_2, \ldots, x_{144}) =$$

$$\begin{aligned}
&= \tfrac{1}{663\,552}(x_1^{144} + 18x_1^{72}x_2^{36} + 36x_1^{48}x_2^{48} + 24x_1^{48}x_3^{32} + 72x_1^{36}x_2^{54} + 48x_1^{36}x_2^{18}x_4^{18} + 648x_1^{24}x_2^{60} + \\
&432x_1^{24}x_2^{12}x_3^{16}x_6^8 + 192x_1^{18}x_2^9x_4^{27} + 9x_1^{16}x_2^{64} + 72x_1^{16}x_2^{16}x_6^{16} + 54x_1^{16}x_4^{32} + 108x_1^{16}x_8^{16} + 2\,592x_1^{12}x_2^{66} + \\
&1\,728x_1^{12}x_2^{30}x_4^{18} + 1\,728x_1^{12}x_2^{18}x_3^8x_6^{12} + 1\,152x_1^{12}x_2^6x_3^8x_4^6x_6^4x_{12}^4 + 128x_1^9x_3^{45} + 384x_1^9x_3^9x_6^{18} + \\
&162x_1^8x_2^{68} + 1\,296x_1^8x_2^{20}x_6^{16} + 972x_1^8x_2^4x_4^{32} + 1\,944x_1^8x_2^4x_8^{16} + 6\,912x_1^6x_2^{15}x_4^{27} + 4\,608x_1^6x_2^3x_3^4x_4^9x_6^2x_{12}^6 + \\
&648x_1^4x_2^{70} + 432x_1^4x_2^{34}x_4^{18} + 5\,184x_1^4x_2^{22}x_6^{16} + 3\,456x_1^4x_2^{10}x_4^6x_6^8x_{12}^4 + 3\,888x_1^4x_2^6x_4^{32} + 7\,776x_1^4x_2^6x_8^{16} + \\
&2\,592x_1^4x_2^2x_4^{34} + 5\,184x_1^4x_2^2x_4^2x_8^{16} + 4\,608x_1^3x_2^3x_3^{15}x_6^{15} + 13\,824x_1^3x_2^3x_3^3x_6^{21} + 3\,072x_1^3x_3^{47} + \\
&9\,216x_1^3x_3^{11}x_6^{18} + 1\,728x_1^2x_2^{17}x_4^{27} + 13\,824x_1^2x_2^5x_4^9x_6^4x_{12}^6 + 10\,368x_1^2x_2x_4^{35} + 20\,736x_1^2x_2x_4^3x_8^{16} + \\
&1\,152x_1x_2^4x_3^5x_6^{20} + 3\,456x_1x_2^4x_3x_6^{22} + 9\,216x_1x_2x_3^5x_6^{21} + 27\,648x_1x_2x_3x_6^{23} + 6\,912x_1x_3^5x_4^2x_{12}^{10} + \\
&13\,824x_1x_3^5x_8x_{24}^5 + 20\,736x_1x_3x_4^2x_6^2x_{12}^{10} + 41\,472x_1x_3x_6^2x_8x_{24}^5 + 3\,174x_2^{72} + 2\,208x_2^{36}x_4^{18} + \\
&6\,624x_2^{24}x_6^{16} + 4\,608x_2^{12}x_4^6x_6^8x_{12}^4 + 3\,726x_2^8x_4^{32} + 7\,452x_2^8x_8^{16} + 2\,592x_2^4x_4^{34} + 5\,184x_2^4x_4^2x_8^{16} + \\
&7\,224x_3^{48} + 1\,008x_3^{24}x_6^{12} + 9\,288x_3^{16}x_6^{16} + 25\,536x_3^{12}x_6^{18} + 2\,688x_3^{12}x_6^6x_{12}^6 + 1\,296x_3^8x_6^{20} + \\
&10\,752x_3^6x_6^3x_{12}^9 + 32\,832x_3^4x_6^{22} + 3\,456x_3^4x_6^{10}x_{12}^6 + 13\,824x_3^2x_6^5x_{12}^9 + 38\,400x_4^{36} + 36\,864x_4^{12}x_{12}^8 + \\
&41\,472x_4^4x_8^{16} + 8\,832x_6^{24} + 6\,144x_6^{12}x_{12}^6 + 39\,936x_8^{18} + 18\,432x_8^6x_{24}^4 + 49\,152x_{12}^{12} + 24\,576x_{24}^6).
\end{aligned}$$

By applying Theorem 2.10, the number of patterns of k-motifs is the coefficient of x^k in $1+x+5x^2+26x^3+216x^4+2\,024x^5+27\,806x^6+417\,209x^7+6\,345\,735x^8+90\,590\,713x^9+1\,190\,322\,956x^{10}+\ldots$.

For $k = 1, 2, 3, 4$ these numbers are the same as in [8]. In the case $k = 5$ however, it is stated that there exist 2 032 different patterns of 5-motifs, while here we get 2 024 of these patterns.

2.6 Patterns of Tropes

Definition 2.10 (Trope) 1. If you divide the set of 12 tones in 12-tone music into 2 disjointed sets, each containing 6 elements, and if you label these sets as a first and a second set, we will speak of a trope. This definition goes back to Josef Matthias Hauer. Two tropes are called equivalent, iff transposing, inversion, changing the labels of the two sets or arbitrary sequences of these operations transform one trope into the other.

2. For a mathematical definition let $n \geq 4$ and $n \equiv 0 \bmod 2$. A trope in n-tone music is a function
$$f\colon Z_n \to F\colon = \{1,2\} \text{ such that } |f^{-1}(\{1\})| = |f^{-1}(\{2\})| = \frac{n}{2}.$$
$f(i) = k$ is translated into: The tone i lies in the set with label k. Furthermore T and I are permutations on Z_n as in Definition 2.2. The group $\langle T, I\rangle$ is $\vartheta_n^{(E)}$. Two tropes f_1, f_2 are called equivalent, if and only if,
$$\exists \pi \in \vartheta_n^{(E)}\ \exists \varphi \in S_2 \text{ such that } f_2 = \varphi^{-1} \circ f_1 \circ \pi.$$

3. Let x and y be indeterminates over $\mathbf{Q}$. Define a function $w\colon F \to \mathbf{Q}[x,y]$ by $w(1)\colon = x$ and $w(2)\colon = y$. For $f \in F^{Z_n}$ the weight of f is defined as product weight
$$W(f)\colon = \prod_{x \in Z_n} w\big(f(x)\big).$$
A function $f\colon Z_n \to F\colon = \{1,2\}$ is a trope, iff $W(f) = x^{\frac{n}{2}} y^{\frac{n}{2}}$.

Theorem 2.11 (Patterns of Tropes) *Let φ be Euler's φ-function. The number of patterns of tropes in regard to $\vartheta_n^{(E)}$ is*
$$\begin{cases} \frac{1}{4}\left(\frac{1}{n}\Big(\sum\limits_{t \mid \frac{n}{2}} \varphi(t)\binom{\frac{n}{t}}{\frac{n}{2t}} + \sum\limits_{\substack{t \mid n \\ t \equiv 0 \bmod 2}} \varphi(t) 2^{\frac{n}{t}}\Big) + \binom{\frac{n}{2}}{\frac{n}{4}} + 2^{\frac{n}{2}-1}\right) & \text{if } n \equiv 0 \bmod 4 \\ \frac{1}{4}\left(\frac{1}{n}\Big(\sum\limits_{t \mid \frac{n}{2}} \varphi(t)\binom{\frac{n}{t}}{\frac{n}{2t}} + \sum\limits_{\substack{t \mid n \\ t \equiv 0 \bmod 2}} \varphi(t) 2^{\frac{n}{t}}\Big) + \binom{\frac{n-2}{2}}{\frac{n-2}{4}} + 2^{\frac{n}{2}-1}\right) & \text{if } n \equiv 2 \bmod 4. \end{cases}$$
In 12-tone music there are 35 patterns of tropes. (See [5].) Hauer himself calculated that there are 44 patterns of tropes, because in his work the permutation group acting on Z_n was the cyclic group $\langle T\rangle$.

PROOF:
We want to use Theorem 1.3 which says:
$$\sum_{[f]} W([f]) = \frac{1}{|S_2|} \sum_{\varphi \in S_2} \mathrm{CI}\big(\vartheta_n^{(E)}; \lambda(1,\varphi), \lambda(2,\varphi), \ldots, \lambda(n,\varphi)\big),$$
where
$$\lambda(m,\varphi)\colon = \sum_{\substack{y \in F \\ \varphi^m(y) = y}} w(y) \cdot w\big(\varphi(y)\big) \cdot \ldots \cdot w\big(\varphi^{m-1}(y)\big).$$
The number of patterns of tropes is the coefficient of $x^{\frac{n}{2}} y^{\frac{n}{2}}$ in
$$\frac{1}{2}\Big(\mathrm{CI}(\vartheta_n^{(E)}; x+y, x^2+y^2, \ldots, x^n+y^n) + \mathrm{CI}(\vartheta_n^{(E)}; 0, 2xy, 0, 2x^2y^2, \ldots, 0, 2x^{\frac{n}{2}}y^{\frac{n}{2}})\Big).$$
This can be transformed into the formula above.

q.e.d. Theorem 2.11

2.7 Special Remarks on 12-tone music

In addition to the operations of transposing T and of inversion I we can study quartcircle- and quintcircle symmetry in 12-tone music.

Remark 2.12 (Quartcircle Symmetry) The quartcircle symmetry Q is defined as

$$Q\colon Z_{12} \to Z_{12}$$
$$x \mapsto Q(x) := 5x.$$

Q is a permutation on Z_{12}, since $\gcd(5,12) = 1$. Furthermore

1. $Q \notin \langle I, T \rangle$.
2. $Q \circ T = T^5 \circ Q$.
3. $Q \circ I = I \circ Q = 7x$, which is called the quintcircle symmetry.
4. $Q^2 = \mathrm{id}_{Z_{12}}$.
5. Let G be $G\colon= \langle I, T, Q \rangle$. Each element $\varphi \in G$ can be written as
$$\varphi = T^k \circ I^j \circ Q^l$$
such that $k \in \{0, 1, \ldots, n-1\}$, $j \in \{0, 1\}$, and $l \in \{0, 1\}$.
6. The cycle index of $G\colon= \langle I, T, Q \rangle$ is
$$\mathrm{CI}(G; x_1, x_2, \ldots, x_{12}) =$$
$$= \frac{1}{48}\Big(\sum_{t|12} \varphi(t) {x_t}^{\frac{12}{t}} + 2x_1^6x_2^3 + 3x_1^4x_2^4 + 6x_1^2x_2^5 + 11x_2^6 + 4x_3^2x_6 + 6x_4^3 + 4x_6^2\Big).$$

This group G is an other permutation group acting on Z_{12} with a musical background. The question arises, how to generalize the quartcircle symmetry of 12-tone music to n-tone music. Should we take any unit in Z_n or only those units e such that $e^2 = 1$?

k	1	2	3	4	5	6	7	8	9	10	11	12
# of patterns	1	6	19	43	66	80	66	43	19	6	1	1

Table 1: Number of patterns of k-Chords in 12-tone music with regard to $\zeta_n^{(E)}$.

k	1	2	3	4	5	6	7	8	9	10	11	12
# of patterns	1	6	12	29	38	50	38	29	12	6	1	1

Table 2: Number of patterns of k-Chords in 12-tone music with regard to $\vartheta_n^{(E)}$.

k	2	3	4	5	6	7
# of patterns	6	30	275	2 000	14 060	83 280

k	8	9	10	11	12
# of patterns	416 880	1 663 680	4 993 440	9 980 160	9 985 920

Table 3: Number of patterns of k-rows in 12-tone music.

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List of Tables

Contents

Working for my doctorial thesis I succeeded in constructing complete lists of orbit representatives of k-motifs for $k = 1, 2, \ldots, 8$ under the action of the permutation group of the second item of example 2.2.

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