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Remarks on Rhythmical Canons

Harald Fripertinger

Abstract

We collect some remarks on rhythmical canons, which are described as pairs of inner and outer rhythms. First we explain a method for creating new canons by inserting a given canon into a given canon. Then we analyze a backtracking algorithm which uses as an input the inner rhythm of a canon and allows to find all outer rhythms so that the resulting canon is a rhythmic tiling canon. Based on this algorithm we provide a complete list of all RCMC-canons of length 72 and 108.

1 Preliminaries

We describe rhythmical canons as discrete structures, therefore, we use mathematical notions. Especially *group actions* will play a central role in our approach. A detailed introduction to combinatorics under finite group actions can be found in [4, 5].

A multiplicative group G with neutral element 1 acts on a set X if there exists a mapping

$$*\colon G \times X \to X, \qquad *(g,x) \mapsto g * x,$$

such that

$$(g_1 g_2) * x = g_1 * (g_2 * x), \qquad g_1, g_2 \in G,\ x \in X,$$

and

$$1 * x = x, \qquad x \in X.$$

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We usually write gx instead of $g * x$. A group action will be indicated as ${}_GX$. If G and X are finite sets, then we call the group action *finite*.

A group action ${}_GX$ determines a group homomorphism ϕ from G to the symmetric group $S_X := \{\pi \mid \pi : X \to X,\ \pi \text{ is bijective}\}$ by

$$\phi : G \to S_X, \qquad g \mapsto \phi(g) := [x \mapsto gx],$$

which is called a *permutation representation* of G on X. Usually we abbreviate $\phi(g)$ by writing $\bar{g}$, which is the permutation of X that maps x to gx. For instance $\bar{1}$ is always the identity on X. Accordingly, the image $\phi(G)$ is indicated by $\bar{G}$. It is a *permutation group* on X, i.e. a subgroup of S_X.

If X is finite then $\bar{G}$ is finite since it is a subgroup of the symmetric group S_X which is of cardinality $|X|!$. Hence, whenever X is finite we can speak of a finite group action.

A group action ${}_GX$ defines the following equivalence relation on X. $x_1 \sim x_2$ if and only if there is some $g \in G$ such that $x_2 = gx_1$. The equivalence classes $G(x)$ with respect to $\sim$ are the *orbits* of G on X. Hence, the orbit of x under the action of G is

$$G(x) = \{gx \mid g \in G\}.$$

The set of orbits of G on X is indicated by

$$G\backslash\backslash X := \{G(x) \mid x \in X\}.$$

It can be shown that the equivalence classes of any equivalence relation can be represented as orbits under a suitable group action (cf. [5]).

Let ${}_GX$ be a group action. For each $x \in X$ the *stabilizer* G_x of x is the set of all group elements which do not change x, in other words

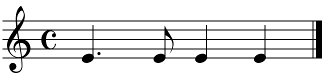
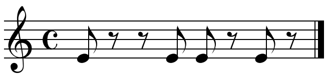
$$G_x := \{g \in G \mid gx = x\}.$$

This is a subgroup of G.

When investigating a rhythm we consider a sequence of beats. We are only interested in the onsets, not in the duration of the different beats. Moreover, we want to forget all information about the pitch.

We also assume that we have found a subdivision of the rhythm, i.e. a regular pulsation, into equidistant beats such that all rhythmical events coincide with some of these beats. If the rhythm is covered by a pulsation of n beats, then it can be described as a $\{0,1\}$-vector of length n, whence as a function $f\colon \{0,\ldots,n-1\} \to \{0,1\}$.

For example, the function $f = (f(0),\ldots,f(7)) = (1,0,0,1,1,0,1,0)$ describes a rhythm of length 8 which can be represented as


or


The cyclic group C_n generated by $\pi_n := (0,1,\ldots,n-1)$ acts on the set of all mappings from $\underline{n} := \{0,1,\ldots,n-1\}$ to $\{0,1\}$ according to

$$C_n \times \{0,1\}^{\underline{n}} \to \{0,1\}^{\underline{n}}, \quad (\sigma, f) \mapsto f \circ \sigma^{-1}.$$

If we write f as a vector $f = (f(0),\ldots,f(n-1))$, then

$$f \circ \pi_n^j = (f(j),\ldots,f(n-1),f(0),\ldots,f(j-1)).$$

Hence, the orbit $C_n(f)$ consists of all cyclic shifts of f. Using the natural order $0 < 1$, the set $\{0,1\}^{\underline{n}}$ is totally ordered by the lexicographical order. For $f, g \in \{0,1\}^{\underline{n}}$ we say

$$f < g :\Longleftrightarrow \exists i \in \underline{n} : f(j) = g(j) \text{ for } j < i \text{ and } f(i) < g(i).$$

Usually we choose the smallest element of an orbit as its canonical representative. For example the orbit of $f = (10011010)$ under C_8 contains the vectors (10011010), (00110101), (01101010), (11010100), (10101001), (01010011), (10100110), (01001101). Therefore, its standard representative is (00110101).

The stabilizer of $f \in \{0,1\}^{\underline{n}}$ is a subgroup of C_n, whence again a cyclic group. We call f *acyclic* if its stabilizer consists of the identity only. If f is acyclic, then the canonical representative of $C_n(f)$ is called a *Lyndon word*.

The function of the last example is acyclic, and (00110101) is a Lyndon word. The vector $f = (01100110)$ has a nontrivial stabilizer, since $\pi_8^4 \neq \mathrm{id}$ and $f = f \circ \pi_8^4$.

We also identify a $\{0,1\}$-vector f with the set $f^{-1}(\{1\})$ of pre-images of 1. Then f is the characteristic function of $f^{-1}(\{1\})$. In this case it is sometimes helpful to assume that $f^{-1}(\{1\}) \subseteq Z_n := \mathbb{Z}/n\mathbb{Z}$, the set of residue classes modulo n.

The *weight* $\mathrm{wt}(f)$ of a $\{0,1\}$-vector f is the number of components of f which are equal to 1, thus

$$\mathrm{wt}(f) = \left|f^{-1}(\{1\})\right|.$$

The concept of a canon is described in [6] and was presented to the author by G. Mazzola as described in [2, 3]. Here we rephrase this definition in the notion of $\{0,1\}$-vectors. A *canon* of length n consisting of $t \geq 1$ voices V_i is a set $\{V_1, \ldots, V_t\}$ of $\{0,1\}$-vectors $V_i \neq 0$ of length n, such that

1. $V_i \in C_n(V_1)$ for $1 \leq i \leq t$,
2. V_1 is acyclic,
3. the set of differences in $K := \bigcup_{i=1}^{t} V_i$ generates Z_n, i.e.

$$\langle K - K\rangle := \langle k - l \mid k, l \in K\rangle = Z_n.$$

Two canons $\{V_1, \ldots, V_t\}$ and $\{W_1, \ldots, W_s\}$ are called *isomorphic* if $s = t$ and if there exists some $\sigma \in C_n$ and a permutation τ in the symmetric group S_t such that $\sigma(V_i) = W_{\tau(i)}$ for $1 \leq i \leq t$. The reader should be aware of the fact that depending on the context we consider the voices of a canon both as $\{0,1\}$-vectors and as subsets of Z_n.

Following the ideas presented in [2] and the notions introduced in [1], the canon $\{V_1, \ldots, V_t\}$ can also be described as a pair $(V_1, f) \in \{0,1\}^{\underline{n}} \times \{0,1\}^{\underline{n}}$, where V_1 is the *inner* and f the *outer rhythm* of the canon. The inner rhythm describes the rhythm of an arbitrary voice. The outer rhythm determines how the different voices are distributed over the n beats of a canon.

For example the canon with the following three voices $V_1 = (10011010)$, $V_2 = (01010011)$, and $V_3 = (11010100)$ yields a score of the form

$$\begin{array}{c} \mathbf{1}0011010 \\ 010\mathbf{1}0011 \\ 11010\mathbf{1}00. \end{array}$$

Here we have marked the starting positions of the different voices with a boldface **1**. Therefore, the outer rhythm of this canon is $f = (10010100)$.

Given a canon (V_1, f) of length n, there exists an isomorphic canon (L, f') where L is a Lyndon word, the canonical representative of $C_n(V_1)$, and f' is the canonical representative of $C_n(f)$. The different voices of the canon given by (L, f') are $j + L$ for $j \in f'$.

However not each pair (L, f) where L is a Lyndon word and f is a $\{0, 1\}$-vector determines a canon of length n. In [2] we have proved the following

Lemma 1. *Let $L \neq 0$ be a Lyndon word, and let f be a $\{0, 1\}$-vector both of length n. The pair (L, f) does not describe a canon if and only if there exists an integer $d > 1$ such that $d \mid n$, $d \mid k - l$ for all $k, l \in L$, and $d \mid k - l$ for all $k, l \in f$.*

Based on this result the number of nonisomorphic canons of length n was determined in [2].

A canon of length n is called a *rhythmic tiling canon* if $Z_n = \dot{\bigcup}_{i=1}^{t} V_i$. In other words, the voices are pairwise disjoint and cover entirely Z_n. The canon (L, f) is a rhythmic tiling canon if and only if $L + f = Z_n$ and $|L|\,|f| = n$, thus Z_n is the direct sum $L \oplus f$ of L and f. Tiling canons were enumerated in [3].

A rhythmic tiling canon described by (L, f) is a *regular complementary canon of maximal category* (RCMC-canon) if both L and f are acyclic. Dan T. Vuza was analyzing these canons in [10, 11, 12, 13]. Maybe the term *twofold acyclic rhythmic tiling canon* would be more suggestive but we stick to the standard terminology.

In general, let G be an abelian group. A subset A of G is called *g-periodic* for $g \in G$ if $A = g + A$, and it is called *periodic* if it is g-periodic for some $g \in G$ which is different from the neutral element. Otherwise A is called *aperiodic.* (Subsets of Z_n are aperiodic if and only if they are acyclic.) The group G is called a *Hajós group*, or has the *2-Hajós property*, if in each factorization of G as $A \oplus B$ at least one factor is periodic. In [8, 9] all finite abelian groups which are Hajós groups are classified. Independently, Vuza described all Hajós groups Z_n in [10, Theorem 2.2].

Vuza showed that RCMC-canons occur only for certain values of n, actually only for those n where Z_n is a *non-Hajós-group.* The smallest n for which Z_n is not a Hajós-group is $n = 72$. Z_n is not a Hajós group if and only if n can be expressed in the form $p_1 p_2 n_1 n_2 n_3$ with p_1, p_2 primes, $n_i \geq 2$ for $1 \leq i \leq 3$, and $\gcd(n_1 p_1, n_2 p_2) = 1$ (cf. [10, Proposition 2.2]).

If Z_n is not a Hajós group, Vuza presents an algorithm for constructing RCMC canons. He describes how to find two acyclic vectors L and f of length n, such that $|L| = n_1 n_2$, $|f| = p_1 p_2 n_3$, and $L + f = Z_n$ (cf. [10, proof of Proposition 2.2]). RCMC-canons which can be constructed by Vuza's algorithm are called *Vuza-constructible*. In [3] we were enumerating Vuza-constructible canons for certain lengths n, and it was also shown that there exist RCMC-canons which are not Vuza-constructible.

2 New Canons From Old Canons

Now we define a composition of two $\{0,1\}$-vectors. For $f \in \{0,1\}^{\underline{n}}$ and $g \in \{0,1\}^{\underline{m}}$ let $f[g]$ be the vector in $\{0,1\}^{\underline{nm}}$ where each 1 in f is replaced by g and each 0 in f is replaced by $0^m = 0 \dots 0$, the sequence of m zeros. Since each $i \in \underline{nm}$ can uniquely be written as $i = qm + r$ with $q \in \underline{n}$ and $r \in \underline{m}$ we obtain

$$f[g](i) = f[g](qm + r) = f(q) \cdot g(r).$$

Lemma 2. *For $f, f' \in \{0,1\}^{\underline{n}}$ and $g \in \{0,1\}^{\underline{m}}$ we have*

1. $\mathrm{wt}(f[g]) = \mathrm{wt}(f)\,\mathrm{wt}(g)$.
2. *If $g \neq 0^m$, then the mapping $\{0,1\}^{\underline{n}} \ni f \mapsto f[g] \in \{0,1\}^{\underline{nm}}$ is injective.*
3. $f[g] \circ \pi_{nm}^{jm} = (f \circ \pi_n^j)[g]$.
4. *$f \leq f'$ implies $f[g] \leq f'[g]$.*
5. *If $g \neq 0^m$ and $f < f'$, then $f[g] < f'[g]$.*
6. *If f and g are the canonical representatives of $C_n(f)$ and $C_m(g)$, then $f[g]$ is the canonical representative of $C_{nm}(f[g])$.*
7. *If f is a Lyndon word of length $n > 1$ and $g \neq 0^m$ is the canonical representative of $C_m(g)$, then $f[g]$ is a Lyndon word.*

Proof. The proof of the first assertion is obvious.
2.) Assume that $f_1[g] = f_2[g]$ for $f_1, f_2 \in \{0,1\}^{\underline{n}}$ and $g \in \{0,1\}^{\underline{m}}$, $g \neq 0^m$. Then for all $(q, r) \in \underline{n} \times \underline{m}$ we have $f_1(q)g(r) = f_2(q)g(r)$. Since there

exists some $r_0 \in \underline{m}$ such that $g(r_0) \neq 0$ we obtain $f_1(q) = f_2(q)$ for all $q \in \underline{n}$, whence $f_1 = f_2$.
3.) Assume that $i \in \underline{nm}$ is expressed as $i = qm + r$ with $q \in \underline{n}$ and $r \in \underline{m}$. Then

$$\begin{aligned}\left(f[g] \circ \pi_{nm}^{jm}\right)(i) &= f[g](i + jm \bmod nm) = f[g]((q+j)m + r \bmod nm)\\ &= f(q + j \bmod n)g(r) = (f \circ \pi_n^j)(q)g(r)\\ &= \left((f \circ \pi_n^j)[g]\right)(qm + r) = \left((f \circ \pi_n^j)[g]\right)(i).\end{aligned}$$

4.) If $g = 0^m$, then $f[g] = f'[g] = 0^{nm}$. Assume that $g \neq 0^m$, then there exists some $r_0 \in \underline{m}$ so that $g(r_0) = 1$ and $g(j) = 0$ for $0 \leq j < r_0$. Since $f < f'$ there exists some $q_0 \in \underline{n}$ so that $0 = f(q_0) < f'(q_0) = 1$ and $f(j) = f'(j)$ for $0 \leq j < q_0$. Consequently $f[g](i) = f'[g](i)$ for $0 \leq i < q_0 m + r_0$ and $0 = f(q_0)g(r_0) = f[g](q_0 m + r_0) < f'[g](q_0 m + r_0) = f'(q_0)g(r_0) = 1$ and, therefore, $f[g] < f'[g]$. This proves also the fifth assertion.
6.) We prove that $f[g] \leq f[g] \circ \pi_{nm}^{qm+r}$ for all $(q, r) \in \underline{n} \times \underline{m}$. The assertion is trivial for $g = 0^m$, thus we restrict our attention to $g \neq 0^m$. Then $g(m-1) = 1$.

Assume that $f = 1^n$, then $f \circ \pi_n^q = f$ and $f[g] = g^n$, the concatenation of n copies of g. Since g is the canonical representative of its orbit, $g \circ \pi_m^r \geq g$ and, consequently, $f[g] \circ \pi_{nm}^{qm+r} = f[g \circ \pi_m^r] = (g \circ \pi_m^r)^n \geq g^n = f[g]$. We still have to consider the case that $f \neq 1^n$. Then $f(0) = 0$ and $f[g](i) = 0$ for $0 \leq i < m$. By assumption $f \leq f \circ \pi_n^q$, whence $f[g] \leq (f \circ \pi_n^q)[g] = f[g] \circ \pi_{nm}^{qm}$.

If $f(q) = (f \circ \pi_n^q)(0) = 1$, then

$$\begin{aligned}\left(f[g] \circ \pi_{nm}^{qm+r}\right)(m-1-r) &= f[g](m-1-r+qm+r)\\ &= f(q)g(m-1) = 1\\ &> 0 = f[g](m-1-r),\end{aligned}$$

whence $f[g] \circ \pi_{nm}^{qm+r} > f[g]$.

Finally, if $f(q) = (f \circ \pi_n^q)(0) = 0$, then $(f[g] \circ \pi_{nm}^{qm})(i) = 0$ for $0 \leq i < m$. Thus,

$$f[g] \leq f[g] \circ \pi_{nm}^{qm} \leq f[g] \circ \pi_{nm}^{qm+1} \leq \ldots \leq f[g] \circ \pi_{nm}^{qm+m-1}$$

since by an application of π_{nm} the leftmost 1 of $f[g]\circ\pi_{nm}^{qm+j}$, $0\leq j<m$, is moving to the left. (Since $f[g]\circ\pi_{nm}^{qm}\neq 0$ there exists some i_0 so that $(f[g]\circ\pi_{nm}^{qm})(i_0)=1$ and $(f[g]\circ\pi_{nm}^{qm})(j)=0$ for $0\leq j<i_0$. By assumption $i_0\geq m$. Then the smallest $i\in\underline{nm}$ so that $(f[g]\circ\pi_{nm}^{qm+r})(i)=1$ is equal to $i_0-r>0$ for $r\in\underline{m}$.)
7.) We just have to prove that $f[g]$ is acyclic. Since f is a Lyndon word we have $f(0)=0$, $f(n-1)=1$, and according to the third and fifth assertion, $f[g]\circ\pi_{nm}^{qm}=(f\circ\pi_n^q)[g]<f[g]$. We still have to prove that $f[g]\neq f[g]\circ\pi_{nm}^{qm+r}$ for $q\in\underline{n}$ and $r\in\underline{m}$, $r\neq 0$. From the definition of $f[g]$ and since $g\neq 0^m$ is a canonical representative it follows that, if $f[g](im+j)=1$ for some $j\in\underline{m}$, then $f[g](im+m-1)=1$.

Since f is a Lyndon word of length $n>1$, there exists some $q_0\in\underline{n}$ so that $f(q_0)=1$ and $f(q_0+1 \bmod n)=0$. Therefore, $f[g](q_0m+m-1)=1$ and moreover $f[g]((q_0+1)m+j \bmod nm)=0$ for $0\leq j<m$. Assuming on the contrary that $f[g]=f[g]\circ\pi_{nm}^{qm+r}$ for some $r\neq 0$, we have for $j_0=q_0m+m-1-(qm+r) \bmod nm$

$$(f[g]\circ\pi_{nm}^{qm+r})(j_0)=f[g](q_0m+m-1)=1.$$

Therefore, also $f[g](j_0)=1$. Since j_0 can be represented as $q'm+r'$ with $r'=m-1-r<m-1$ we obtain from the construction of $f[g]$ that also $f[g](q'm+m-1)=1$. Using the original definition of j_0, this can be written as

$$\begin{aligned}1&=f[g](q'm+m-1)=f[g](j_0+m-1-r')\\&=f[g](q_0m+m-1-(qm+r)+r \bmod nm).\end{aligned}$$

Since $f[g]=f[g]\circ\pi_{nm}^{qm+r}$, this is equal to

$$(f[g]\circ\pi_{nm}^{qm+r})(q_0m+m-1-(qm+r)+r \bmod nm)$$

and consequently

$$f[g]((q_0+1)m+(r-1) \bmod nm)=1.$$

Since $r-1\in\underline{m}$, this is a contradiction to $f[g]((q_0+1)m+j \bmod nm)=0$ for $0\leq j<m$. $\square$

As an immediate consequence we obtain the following

Theorem 3. *For $i = 1, 2$ let (L_i, f_i) be a canon of length n_i.*

1. *Then $(L_1[L_2], f_1[f_2])$ is a canon of length $n_1 n_2$.*
2. *If (L_1, f_1) is a rhythmic tiling canon and (L_2, f_2) is a rhythmic tiling canon, then $(L_1[L_2], f_1[f_2])$ is a rhythmic tiling canon.*
3. *If (L_1, f_1) is an RCMC-canon and (L_2, f_2) is a rhythmic tiling canon, then $(L_1[L_2], f_1[f_2])$ is an RCMC-canon.*

3 A Backtracking Algorithm

Now we describe an algorithm which allows to find for any given inner rhythm L, which is a Lyndon word, all outer rhythms f such that (L, f) is a rhythmic tiling canon. We want to apply this algorithm for finding all RCMC-canons of given length.

In [10, Theorem 2.3] it is shown that if $Z_n = A \oplus B$, then $Z_n = (kA) \oplus B$ for each $k \neq 0$ relatively prime to n. This means that the pair (L, f) indicates an RCMC canon if and only if $(kL + r, f)$ indicates an RCMC-canon for $k, r \in Z_n$, where k and n are relatively prime integers. Thus, instead of L it is possible to use the $\{0, 1\}$-vector $\tilde{L}$ given by $\tilde{L}(i) := L(-i - 1 \bmod n)$, whence $\tilde{L}(i) = L(n - 1 - i)$, $i \in Z_n$. Since L is a Lyndon word, $\tilde{L}(0) = 1$.

The *span* of a rhythm f of length n with $f(0) = 1$ is defined to be $j + 1$ for $j = \max\{i \in \underline{n} \mid f(i) \neq 0\}$. Here is a short example. The rhythm 11000000 has span 2 and its cyclic shift 10000001 has span 8. For the backtracking algorithm it is important to consider rhythms of short span. Due to the construction above it is easy to show that $\tilde{L}$ has the shortest span of all rhythms in the orbit $C_n(\tilde{L})$.

There is still another useful way for representing a rhythm. Assume that f is a $\{0, 1\}$-vector of length n with exactly $k \geq 2$ entries equal to 1. Thus, there are integers $0 \leq i_0 < i_1 < \ldots < i_{k-1} \leq n-1$ so that $f(i_j) = 1$ for $0 \leq j < k$ and $f(i) = 0$ for $i \in \underline{n} \setminus \{i_0, \ldots, i_{k-1}\}$. With f we associate the function $\hat{f}\colon \underline{k} \to \mathbb{Z}_{>0}$ given by

$$\hat{f}(j) := \begin{cases} i_j - i_{j-1} & \text{for } j > 0, \\ n - (i_{k-1} - i_0) & \text{for } j = 0. \end{cases}$$

We call $\hat{f}$ the interval representation of f.

Lemma 4. *Let f be a $\{0,1\}$-vector of length n with $k \geq 2$ entries equal to* 1. *Then*

- $\sum_{i=0}^{k-1} \hat{f}(i) = n$.
- *If $h\colon \underline{k} \to \mathbb{Z}_{>0}$ satisfies $\sum_{i=0}^{k-1} h(i) = n$, then there exists a $\{0,1\}$-vector v so that $\hat{v} = h$.*
- *For each $v \in C_n(f)$ the interval representation $\hat{v}$ belongs to the orbit $C_k(\hat{f})$, where C_k acts on $\mathbb{Z}^k_{>0}$ by*

  $$C_k \times \mathbb{Z}^k_{>0} \to \mathbb{Z}^k_{>0}, \qquad (\sigma, h) \mapsto h \circ \sigma^{-1}.$$

  This is a natural generalization of the group action introduced for $\{0,1\}$-vectors.
- *For each $h \in C_k(\hat{f})$ there exists some $v \in C_n(f)$ so that $h = \hat{v}$.*
- *f is acyclic if and only if $\hat{f}$ is acyclic.*
- *If f is the canonical representative of the orbit $C_n(f)$, then $\hat{f} \geq h$ with respect to the lexicographical order for all $h \in C_k(\hat{f})$. Therefore, $\hat{f}(0) \geq \hat{f}(i)$ for $i \in \underline{k}$.*

Assume that $h\colon \underline{k} \to \mathbb{Z}_{>0}$ satisfies $\sum_{i=0}^{k-1} h(i) = n$. The last assertion of Lemma 4 motivates to choose the greatest vector with respect to the lexicographical order as the canonical representative of the orbit $C_k(h)$.

In the following we use the convention that the empty sum always yields 0. Let (L, f) be an RCMC-canon where f contains exactly k entries equal to 1, and let $\tilde{L}$ and $\hat{f}$ be as above. For $i \in \underline{k}$ the union of the first $i+1$ voices, considered as subsets of Z_n, will be denoted by

$$S_{\tilde{L},\hat{f},i} := \bigcup_{r=0}^{i} \left(\Big(\sum_{j=0}^{r-1} \hat{f}(j) \Big) + \tilde{L} \right).$$

We collect some properties of the canon (L, f):

1. Since each RCMC-canon is a rhythmic tiling canon we have

   $$\left(\Big(\sum_{j=0}^{i} \hat{f}(j) \Big) + \tilde{L} \right) \cap S_{\tilde{L},\hat{f},i} = \emptyset \qquad (\mathrm{T_i})$$

   for all $i \in \underline{k-1}$ and $S_{\tilde{L},\hat{f},k-1} = Z_n$.

2. Let s denote the span of $\tilde{L}$ and assume that $i < k-2$. If $\sum_{j=0}^{i} \hat{f}(j) < s$, then either $\sum_{j=0}^{i+1} \hat{f}(j) < s$ and ($\mathrm{T_{i+1}}$) is satisfied or
$$\sum_{j=0}^{i+1} \hat{f}(j) = \min\left\{t \geq s \mid t \notin S_{\tilde{L},\hat{f},i+1}\right\}$$
and ($\mathrm{T_{i+1}}$) is satisfied.
If $\sum_{j=0}^{i} \hat{f}(j) \geq s$ then
$$\left\{s, s+1, \ldots, \sum_{j=0}^{i} \hat{f}(j)\right\} \subset S_{\tilde{L},\hat{f},i+1}.$$
In this situation it is still possible that there exist some integers in $\underline{s}$ which are not contained in $S_{\tilde{L},\hat{f},i+1}$. Moreover, the value $\hat{f}(i+1)$ satisfies
$$\sum_{j=0}^{i+1} \hat{f}(j) = \min\left\{t \geq s \mid t \notin S_{\tilde{L},\hat{f},i+1}\right\}$$
and ($\mathrm{T_{i+1}}$).
3. From $\hat{f}(0) \geq \hat{f}(i)$, $i \in \underline{k}$, we derive that $k \cdot \hat{f}(0) \geq n$, whence $\hat{f}(0) \geq n/k$.

Now we are in a position to describe the main aspects of the backtracking algorithm. We assume that the Lyndon word L of length $n > 1$ which is a $\{0,1\}$-vector with $\ell > 1$ entries equal to 1 is considered to be the inner rhythm of an RCMC-canon. Then necessarily ℓ is a divisor of n. Let $k := n/\ell$. In the algorithm we use the rhythm $\tilde{L}(i) = L(n-1-i)$, $i \in Z_n$, instead of L. Let s be the span of $\tilde{L}$.

The backtracking tries to find the interval representation of all rhythms f so that (L, f) is an RCMC-canon. Since we are only interested in non-isomorphic canons we restrict ourselves to interval representations belonging to the set
$$H := \left\{h\colon \underline{k} \to \mathbb{Z}_{>0} \;\middle|\; \sum_{i=0}^{k-1} h(i) = n,\ h(0) \geq h(i),\ i \in \underline{k}\right\}.$$

We assume that we have found a vector $(h(0),\dots,h(i-1))$, $0 \le i < k-1$, so that the sets $S_{\tilde{L},h,j}$, $0 \le j \le i$ contain $(j+1)\cdot \ell$ elements. Then we determine the set of all possible values $h(i)$ in the following way:
Case 1. If $\sum_{j=0}^{i-1} h(j) < s-1$, then let

$$U_i := \left\{ t \;\middle|\; \sum_{j=0}^{i-1} h(j) < t < s-1,\ \left(t+\tilde{L}\right) \cap S_{\tilde{L},h,i} = \emptyset \right\},$$

$$t_0 := \min\left\{ t \ge s \mid t \notin S_{\tilde{L},h,i} \right\}$$

and

$$V_i := \begin{cases} \{t_0\} & \text{if } \left(t_0+\tilde{L}\right) \cap S_{\tilde{L},h,i} = \emptyset, \\ \emptyset & \text{else.} \end{cases}$$

If $i=0$, then the set of all possible values for $h(0)$ is given by

$$H_0 := \left\{ t \in U_0 \cup V_0 \;\middle|\; t \ge n/k \right\}.$$

Otherwise, the set of all possible values for $h(i)$ is given by

$$H_i := \left\{ t \in U_i \cup V_i \;\middle|\; t - \sum_{j=0}^{i-1} h(j) \le h(0) \right\}.$$

Case 2. If $\sum_{j=0}^{i-1} h(j) \ge s$, then the set of all possible values for $h(i)$ is

$$H_i := \left\{ t \in V_i \;\middle|\; t - \sum_{j=0}^{i-1} h(j) \le h(0) \right\}.$$

Thus, in case 2 the set H_i is either empty or contains exactly one element.

In order to extend the vector $(h(0),\dots,h(i-1))$ by appending the next entry, we choose $h(i)$ as the smallest element of H_i.

If we have determined $(h(0),\dots,h(k-2))$ then the value of $h(k-1)$ is also determined by $h(k-1) = n - \sum_{i=0}^{k-2} h(i)$. For that reason it is always enough to determine the sets $H_0,\dots,H_{k-2}$. If $h(k-1) > h(0)$ then h does not belong to H.

There are two situations in which it is impossible to extend the vector $(h(0), \ldots, h(i-1))$. Either the vector h has already full length, i.e. $i-1 = k-2$ (since then $h(k-1)$ is uniquely determined), or $H_i = \emptyset$. In both situations the backtracking algorithm tries to change the last entry of the vector $(h(0), \ldots, h(i-1))$ by replacing $h(i-1)$ by the smallest element in H_{i-1} which is greater than the present $h(i-1)$. If there is no successor of $h(i-1)$ in H_{i-1}, then the algorithm tries to find a successor of $h(i-2)$ and so on. Assume that a successor of $h(i-r)$ was found, then the algorithm tries to extend the new vector $(h(0), \ldots, h(i-r))$ by appending suitable values $h(i-r+1), h(i-r+2), \ldots$. The reader should realize that since we have changed the value of $h(i-r)$ also the sets H_{i-r+j} for $j \geq 1$ are changed.

The backtracking algorithm terminates when it cannot find a successor for $h(0)$.

As a matter of fact, this algorithm yields rhythmic tiling canons, not necessarily RCMC-canons. In certain situations it produces just one representative of an isomorphism class, in some other situations it can hit an isomorphism class several times. So we still have to check the output of this algorithm, delete all rhythmic tiling canons which are not RCMC and delete vectors yielding canons isomorphic to canons already listed. This can be done the following way: If $h(0) > h(i)$ for all $1 \leq i < k$, then h is acyclic and due to the construction of our algorithm it is the only representative of its orbit listed by this algorithm. If there exists some i, $1 \leq i < k$, so that $h(0) = h(i)$, then we have to check whether h has cyclic symmetries. If so, it is deleted from the list. When $h(0)$ occurs λ times in h and h is acyclic, then there are at most λ vectors in the output of the algorithm belonging to the orbit $C_k(h)$. In each of these vectors the value $h(0)$ occurs exactly λ times. In order to obtain nonisomorphic canons, separately for each $\lambda > 1$ we have to determine the canonical orbit representatives of those vectors in the algorithm's output which contain the value $h(0)$ in exactly λ positions.

For example, there exist only three different inner rhythms of Vuza-constructible RCMC-canons of length 108 with exactly 6 beats. The interval representation of these canons is given by $(57, 12, 12, 3, 12, 12)$, $(33, 24, 3, 21, 3, 24)$ and $(27, 21, 12, 15, 12, 21)$. If $\tilde{L}$ denotes any of these rhythms, the backtracking algorithm produces the following list of all possible outer rhythms $\hat{f}$ in interval representation so that $(\tilde{L}, \hat{f})$ is an RCMC-canon of length 108.

```
30 4 1 1 7 1 8 1 13 6 16 1 8 1 4 4 1 1
30 2 2 2 5 2 7 2 14 6 14 2 7 2 5 2 2 2
30 1 4 1 4 4 5 4 13 6 13 4 5 4 4 1 4 1
30 1 1 4 4 1 8 1 16 6 13 1 8 1 7 1 1 4
29 5 1 1 7 2 7 2 11 6 17 2 7 2 2 5 1 1
29 2 4 1 4 5 4 5 11 6 14 5 4 5 2 2 4 1
29 1 4 2 2 5 4 5 14 6 11 5 4 5 4 1 4 2
29 1 4 1 1 7 9 13 1 5 1 16 9 4 1 4 1 1
29 1 1 5 2 2 7 2 17 6 11 2 7 2 7 1 1 5
29 1 1 4 1 4 9 16 1 5 1 13 9 7 1 1 4 1
28 6 1 1 8 1 8 1 10 6 19 1 8 1 1 6 1 1
28 4 2 2 5 4 5 4 10 6 16 4 5 4 1 4 2 2
28 2 5 1 1 7 2 7 13 6 10 7 2 7 2 2 5 1
28 2 4 1 1 8 9 11 2 4 2 17 9 2 2 4 1 1
28 2 2 4 1 4 5 4 16 6 10 4 5 4 5 2 2 4
28 2 2 2 2 5 9 14 2 4 2 14 9 5 2 2 2 2
28 1 6 1 1 8 1 8 11 6 11 8 1 8 1 1 6 1
28 1 5 2 2 7 2 7 10 6 13 7 2 7 1 1 5 2
28 1 5 1 1 9 9 10 1 5 1 19 9 1 1 5 1 1
28 1 1 6 1 1 8 1 19 6 10 1 8 1 8 1 1 6
28 1 1 5 1 1 9 19 1 5 1 10 9 9 1 1 5 1
28 1 1 4 2 2 9 17 2 4 2 11 9 8 1 1 4 2
26 6 2 2 7 2 7 2 8 6 20 2 7 1 1 5 2 2
26 5 1 4 4 5 4 5 8 6 17 5 4 4 1 4 1 4
26 4 4 1 1 7 1 8 14 6 8 8 1 8 1 4 4 2
26 4 2 2 2 7 9 10 4 2 4 16 9 1 4 2 2 2
26 4 1 4 1 4 4 5 17 6 8 5 4 5 4 4 1 5
26 4 1 1 4 4 9 13 4 2 4 13 9 4 4 1 1 4
26 2 6 1 1 8 1 9 10 6 10 9 1 8 1 1 6 2
26 2 4 4 1 8 1 8 8 6 14 8 1 7 1 1 4 4
26 2 4 2 2 9 9 8 2 4 2 20 8 1 1 4 2 2
26 2 2 5 1 1 7 2 20 6 8 2 7 2 7 2 2 6
26 2 2 4 1 1 8 20 2 4 2 8 9 9 2 2 4 2
26 2 2 2 4 1 9 16 4 2 4 10 9 7 2 2 2 4
25 6 4 1 8 1 8 1 7 6 22 1 7 1 1 4 4 1
25 6 1 4 5 4 5 4 7 6 19 4 5 2 2 4 1 4
25 5 4 1 1 7 1 9 13 6 7 9 1 8 1 4 5 1
25 5 2 2 2 5 2 7 16 6 7 7 2 7 2 5 2 4
25 5 1 4 1 8 9 8 5 1 5 17 8 1 4 1 4 1
25 5 1 1 4 5 9 11 5 1 5 14 9 2 5 1 1 4
25 4 5 1 1 7 2 9 11 6 8 9 2 7 2 2 6 1
25 4 2 5 2 7 2 7 7 6 16 7 2 5 2 2 2 5
25 4 2 4 1 9 9 7 4 2 4 19 7 2 2 2 4 1
25 4 1 4 2 2 5 4 19 6 7 4 5 4 5 4 1 6
25 4 1 4 1 1 7 22 1 5 1 7 9 9 4 1 5 1
25 4 1 1 5 2 9 14 5 1 5 11 9 5 4 1 1 5
25 1 6 2 2 7 2 9 8 6 11 9 2 7 1 1 5 4
25 1 5 4 1 8 1 9 7 6 13 9 1 7 1 1 4 5
25 1 5 1 4 9 9 7 1 5 1 22 7 1 1 4 1 4
25 1 4 4 1 1 7 1 22 6 7 1 8 1 8 1 4 6
25 1 4 2 2 2 7 19 4 2 4 7 9 9 1 4 2 4
25 1 4 1 4 1 8 17 5 1 5 8 9 8 1 4 1 5
23 7 2 2 2 5 2 9 14 6 5 9 2 7 2 5 4 2
23 7 1 1 4 4 1 8 17 6 5 8 1 8 1 7 1 5
23 6 5 2 7 2 7 2 5 6 23 2 5 2 2 2 5 2
23 6 2 5 4 5 4 5 5 6 20 5 4 1 4 2 2 5
23 6 1 4 2 7 9 7 6 1 6 16 7 2 4 1 4 2
23 6 1 1 5 4 9 10 6 1 6 13 9 1 6 1 1 5
23 5 4 2 2 5 4 9 10 6 7 9 4 5 4 1 6 2
23 5 2 2 4 1 4 5 20 6 5 5 4 5 4 5 2 6
23 5 2 2 2 2 5 23 2 4 2 5 9 9 5 2 4 2
18 7 6 4 1 8 9 1 7 6 5 17 1 7 1 5 4 1
18 7 6 1 4 5 9 4 7 6 5 14 4 5 2 6 1 4
18 7 4 5 1 1 7 11 11 6 1 7 9 2 7 4 6 1
18 7 4 2 5 2 9 7 7 6 5 11 7 2 5 4 2 5
18 7 1 6 2 2 7 11 8 6 4 7 9 2 7 1 6 4
18 7 1 5 4 1 8 10 7 6 5 8 9 1 7 1 5 5
18 5 6 5 2 7 9 2 5 6 7 16 2 5 2 4 5 2
18 5 6 2 5 4 9 5 5 6 7 13 5 4 1 6 2 5
18 5 5 4 2 2 5 13 10 6 2 5 9 4 5 5 6 2
18 5 5 1 7 1 9 8 5 6 7 10 8 1 4 5 1 7
18 5 2 6 1 4 5 13 7 6 5 5 9 4 5 2 6 5
18 5 2 4 5 2 7 11 5 6 7 7 9 2 5 2 4 7
18 4 7 2 4 1 4 14 11 6 1 4 9 5 4 7 6 1
18 4 6 7 1 8 9 1 4 6 8 17 1 4 4 2 7 1
18 4 6 4 4 5 9 4 4 6 8 14 4 4 1 5 4 4
18 4 6 1 7 2 9 7 4 6 8 11 7 2 2 6 1 7
18 4 4 5 1 4 4 14 8 6 4 4 9 5 4 4 6 4
18 4 4 2 7 1 8 10 4 6 8 8 9 1 4 4 2 8
18 4 1 6 2 5 4 14 5 6 7 4 9 5 4 1 6 7
18 4 1 5 4 4 5 13 4 6 8 5 9 4 4 1 5 8
18 2 8 1 5 2 2 16 10 6 2 2 9 7 2 8 6 2
18 2 6 8 2 7 9 2 2 6 10 16 2 2 5 1 8 2
18 2 6 5 5 4 9 5 2 6 10 13 5 2 2 4 5 5
18 2 6 2 8 1 9 8 2 6 10 10 8 1 1 6 2 8
18 2 5 4 2 5 2 16 7 6 5 2 9 7 2 5 6 5
18 2 5 1 8 2 7 11 2 6 10 7 9 2 2 5 1 10
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18 2 2 4 5 5 4 14 2 6 10 4 9 5 2 2 4 10
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18 1 6 7 4 5 9 4 1 6 11 14 4 1 4 2 7 4
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18 1 6 1 9 1 8 10 1 6 11 8 9 1 1 6 1 10
18 1 4 5 1 7 1 17 5 6 7 1 9 8 1 4 6 7
18 1 4 2 7 4 5 13 1 6 11 5 9 4 1 4 2 11
18 1 1 6 2 8 1 17 2 6 10 1 9 8 1 1 6 10
18 1 1 5 4 7 2 16 1 6 11 2 9 7 1 1 5 11
17 13 4 1 1 7 9 1 13 5 1 16 1 8 5 4 1 1
17 13 1 4 1 4 9 4 13 5 1 13 4 5 8 1 4 1
17 11 2 5 1 1 9 7 13 5 1 10 7 2 9 2 5 1
17 9 4 4 1 1 7 9 14 5 1 8 8 1 8 5 4 2
17 9 4 1 4 1 4 9 17 5 1 8 5 4 5 8 1 5
17 9 2 6 1 1 9 9 10 6 1 9 9 1 9 1 6 2
17 8 5 4 1 1 7 10 13 5 1 7 9 1 8 5 5 1
17 8 1 5 4 1 9 9 7 6 4 9 9 1 7 2 4 5
17 6 7 4 2 7 9 1 6 7 6 16 1 6 2 5 4 2
17 6 7 1 5 4 9 4 6 7 6 13 4 5 1 7 1 5
17 6 5 2 6 1 9 7 6 7 6 10 7 2 4 5 2 6
17 6 2 5 4 2 7 10 6 7 6 7 9 1 6 2 5 6
17 5 8 1 4 1 4 13 13 5 1 4 9 4 5 8 5 1
17 5 4 2 7 1 9 9 4 6 7 9 9 1 4 5 1 8
17 5 1 7 1 5 4 13 6 7 6 4 9 4 5 1 7 6
17 2 9 2 5 1 1 16 13 5 1 1 9 7 2 11 5 1
17 2 6 1 9 1 9 9 1 6 10 9 9 1 1 6 2 9
17 2 4 5 8 1 8 8 1 5 14 9 7 1 1 4 4 9
17 2 4 5 2 6 1 16 6 7 6 1 9 7 2 4 7 6
17 1 5 11 2 7 9 1 1 5 13 16 1 1 5 2 9 2
17 1 5 8 5 4 9 4 1 5 13 13 4 1 4 1 8 5
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17 1 5 2 9 2 7 10 1 5 13 7 9 1 1 5 2 11
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23 5 1 5 2 9 9 5 5 1 5 20 5 4 1 1 5 2
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23 2 6 1 4 5 4 9 7 6 10 9 4 5 2 2 4 5
23 2 5 1 1 4 5 20 5 1 5 5 9 9 2 5 1 5
23 2 4 5 2 7 2 9 5 6 14 9 2 5 2 2 2 7
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22 8 1 4 1 4 4 9 13 6 4 9 4 5 4 4 5 1
22 8 1 1 4 4 1 9 16 6 4 9 1 8 1 7 2 4
22 7 2 4 1 4 5 9 11 6 5 9 5 4 5 2 6 1
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22 6 4 4 5 4 5 4 4 6 22 4 4 1 4 1 4 4
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22 6 2 2 4 5 9 8 6 2 6 14 8 1 5 2 2 4
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20 5 4 2 5 2 7 9 7 6 7 9 7 2 5 2 4 5
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20 2 4 5 5 4 5 9 2 6 14 9 5 2 2 4 1 9
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19 9 2 5 1 1 7 9 13 6 1 9 7 2 7 4 5 1
19 9 2 2 4 1 4 9 16 6 1 9 4 5 4 7 2 4
19 9 1 6 1 1 8 9 11 6 2 9 8 1 8 2 6 1
19 9 1 1 6 1 1 9 19 6 1 9 1 8 1 9 1 6
19 7 2 4 4 1 8 9 8 6 5 9 8 1 7 1 5 4
19 6 5 5 1 8 9 2 6 5 6 17 2 6 1 4 5 1
19 6 5 2 4 5 9 5 6 5 6 14 5 4 2 5 2 4
19 6 4 6 1 9 9 1 6 4 6 19 1 6 2 2 6 1
19 6 4 1 6 2 9 8 6 5 6 11 8 1 5 4 1 6
19 6 1 4 5 1 8 11 6 5 6 8 9 2 6 1 4 6
19 4 5 1 7 1 8 9 5 6 8 9 8 1 4 4 2 7
19 4 2 7 4 5 4 9 1 6 16 9 4 1 4 2 2 9
19 4 2 5 2 4 5 14 6 5 6 5 9 5 4 2 5 6
19 1 6 2 8 1 8 9 2 6 11 9 8 1 1 6 1 9
19 1 5 4 7 2 7 9 1 6 13 9 7 1 1 5 2 9
19 1 5 4 1 6 2 17 6 5 6 2 9 8 1 5 5 6
18 11 5 1 1 7 9 2 11 6 1 16 2 7 4 5 1 1
18 11 2 4 1 4 9 5 11 6 1 13 5 4 7 2 4 1
18 10 6 1 1 8 9 1 10 6 2 17 1 8 2 6 1 1
18 10 4 2 2 5 9 4 10 6 2 14 4 5 5 4 2 2
18 10 1 6 1 1 9 8 11 6 1 10 8 1 9 1 6 1
18 10 1 5 2 2 9 7 10 6 2 11 7 2 8 1 5 2
18 8 6 2 2 7 9 2 8 6 4 16 2 7 1 6 2 2
18 8 5 1 4 4 9 5 8 6 4 13 5 4 4 5 1 4
18 8 2 6 1 1 8 10 10 6 2 8 9 1 8 2 6 2
18 8 2 4 4 1 9 8 8 6 4 10 8 1 7 2 4 4

17 1 4 1 8 5 4 13 1 5 13 4 9 4 1 4 1 13
17 1 1 4 5 8 1 16 1 5 13 1 9 7 1 1 4 13
16 14 4 1 1 8 8 1 13 4 2 16 1 9 4 4 1 1
16 14 2 2 2 5 9 2 14 4 2 14 2 7 7 2 2 2
16 13 1 4 2 2 9 5 14 4 2 11 5 4 9 1 4 2
16 10 4 4 1 1 8 8 14 4 2 8 8 1 9 4 4 2
16 9 5 4 1 1 8 9 13 4 2 7 9 1 9 4 5 1
16 9 5 2 2 2 5 9 16 4 2 7 7 2 7 7 2 4
16 9 4 5 1 1 9 9 11 5 1 8 9 2 9 2 6 1
16 9 1 6 2 2 9 9 8 6 2 9 9 2 8 1 5 4
16 7 7 2 2 2 5 11 14 4 2 5 9 2 7 7 4 2
16 7 2 4 5 2 9 9 5 6 5 9 9 2 5 4 2 7
16 6 8 5 1 8 8 1 5 8 6 16 1 5 4 4 5 1
16 6 8 2 4 5 9 2 6 8 6 14 2 6 1 7 2 4
16 6 7 1 6 2 9 5 6 8 6 11 5 4 2 7 1 6
16 6 4 4 5 1 8 8 6 8 6 8 8 1 5 4 4 6
16 6 1 7 2 4 5 11 6 8 6 5 9 2 6 1 7 6
16 4 9 1 4 2 2 14 14 4 2 2 9 5 4 10 4 2
16 4 5 1 8 2 9 9 2 6 8 9 9 2 2 6 1 9
16 4 2 7 1 6 2 14 6 8 6 2 9 5 4 2 8 6
16 2 4 13 1 8 8 1 1 4 14 16 1 1 4 4 9 1
16 2 4 10 4 5 9 2 2 4 14 14 2 2 4 1 9 4
16 2 4 7 7 2 9 5 2 4 14 11 5 2 2 2 7 7
16 2 4 4 9 1 8 8 2 4 14 8 8 1 1 4 4 10
16 2 4 1 9 4 5 11 2 4 14 5 9 2 2 4 1 13
16 2 2 2 7 7 2 14 2 4 14 2 9 5 2 2 2 14
16 1 6 2 9 2 9 8 1 5 11 9 9 1 1 5 4 9
16 1 5 4 9 1 9 7 2 4 13 9 8 1 1 4 5 9
14 9 7 2 2 2 7 9 14 2 4 5 9 2 9 5 4 2
14 9 5 4 2 2 9 9 10 4 2 7 9 4 9 1 6 2
14 9 2 6 1 4 9 9 7 6 1 9 9 4 7 2 4 5
14 8 1 5 4 4 9 9 4 6 4 9 9 4 4 5 1 8
14 6 10 4 2 7 7 2 4 10 6 14 2 4 5 5 4 2
14 6 10 1 5 4 9 1 6 10 6 13 1 6 2 8 1 5
14 6 8 2 6 1 9 4 6 10 6 10 4 5 1 8 2 6
14 6 5 5 4 2 7 7 6 10 6 7 7 2 4 5 5 6
14 6 2 8 1 5 4 10 6 10 6 4 9 1 6 2 8 6
14 5 4 2 7 4 9 9 1 6 7 9 9 4 1 6 2 9
14 5 1 8 2 6 1 13 6 10 6 1 9 4 5 1 10 6
14 2 6 1 9 4 9 7 2 4 10 9 9 2 2 4 5 9
13 9 8 1 4 1 8 9 13 1 5 4 9 4 9 4 5 1
13 9 7 2 4 1 9 9 11 2 4 5 9 5 9 2 6 1
13 9 4 5 1 4 9 9 8 5 1 8 9 5 8 1 5 4
13 9 1 6 2 5 9 9 5 6 2 9 9 5 5 4 2 7
13 7 2 4 5 5 9 9 2 6 5 9 9 5 2 6 1 9
13 6 11 5 1 8 5 4 2 11 6 13 4 2 7 4 5 1
13 6 11 2 4 5 8 1 5 11 6 13 1 5 4 7 2 4
13 6 10 1 6 2 9 2 6 11 6 11 2 6 1 9 1 6
13 6 7 4 5 1 8 5 6 11 6 8 5 4 2 7 4 6
13 6 4 7 2 4 5 8 6 11 6 5 8 1 5 4 7 6
13 6 1 9 1 6 2 11 6 11 6 2 9 2 6 1 10 6
13 4 5 1 8 5 9 8 1 5 8 9 9 4 1 5 4 9
13 1 6 2 9 5 9 5 4 2 11 9 9 1 4 2 7 9
11 9 8 1 5 2 9 9 10 1 5 4 9 7 9 1 6 2
11 9 5 4 2 5 9 9 7 4 2 7 9 7 7 2 4 5
11 9 2 6 1 7 9 9 4 6 1 9 9 7 4 5 1 8
11 8 1 5 4 7 9 9 1 6 4 9 9 7 1 6 2 9
11 5 4 2 7 7 9 7 2 4 7 9 9 5 2 4 5 9
11 2 6 1 9 7 9 4 5 1 10 9 9 2 5 1 8 9
10 9 9 1 6 1 9 9 10 1 6 2 9 8 9 2 6 1
10 9 7 2 4 4 9 9 8 2 4 5 9 8 8 1 5 4
10 9 4 5 1 7 9 9 5 5 1 8 9 8 5 4 2 7
10 9 1 6 2 8 9 9 2 6 2 9 9 8 2 6 1 9
10 7 2 4 5 8 9 8 1 5 5 9 9 7 1 5 4 9
10 4 5 1 8 8 9 5 4 2 8 9 9 4 4 2 7 9

This list contains 252 outer rhythms $\hat{f}$. In [3] we have shown that for each of the 3 inner rhythms consisting of 6 beats, which were mentioned above, there are 180 different outer rhythms of Vuza-constructible canons. For instance, there are 48 interval representations $\hat{f}$ starting with 18. They do not occur in the list of all Vuza-constructible canons. Thus, together with $\tilde{L}$ they are RCMC canons which are not Vuza-constructible.

4 Complete Lists of RCMC-Canons

If (L, f) is a rhythmic tiling canon of length n, then the weights of L and f are divisors of n. For the complete classification of RCMC canons the following theorem is crucial.

Theorem 5. **(Sands** [7, Theorem 2]**)** *If G is a finite cyclic group, $G = A \oplus B$ and A has p^μ elements, where p is a prime, then either A or B is periodic.*

We obtain the following formulation for rhythmic tiling canons.

Corollary 6. *Assume that a rhythmic tiling canon is given by the pair (L, f). If $\mathrm{wt}(L)$ or $\mathrm{wt}(f)$ is a prime power, then f has cyclic symmetries. Thus, (L, f) is not an RCMC-canon.*

This theorem can be used in the following way to find complete lists of RCMC-canons of length $n = 72$ and $n = 108$. First we find all suitable decompositions of n as a product of two positive integers, $n = rs$. If both r and s are not powers of a prime we assume that $r \leq s$ and continue with the following construction. We determine all Lyndon words L of length n and weight r over $\{0, 1\}$. They serve as possible inner rhythms. Using the backtracking algorithm, we try to find all outer rhythms f so that (L, f) is an RCMC canon. In order to decrease the number of possible inner rhythms which must be input to the algorithm, we collect these Lyndon words into orbits under the action of the affine group. (From [10, Theorem 2.3] it is easy to deduce that the image of an RCMC canon under an affine transformations is again an RCMC canon.) From each orbit under the affine group we choose one representative L and determine all acyclic outer rhythms f, such that $L + f = Z_n$.

For $n = 72$ we have to consider the composition $72 = 6 \cdot 12$. There are 2 169 882 Lyndon words of length 72 and weight 6 over $\{0, 1\}$. There remain just 3 Lyndon words which can be extended to an RCMC-canon. For each Lyndon word there exist (the same 6) outer rhythms which can be used to determine an RCMC-canon. All these canons were already found by Vuza's algorithm.

For $n = 108$ we have to consider the composition $108 = 6 \cdot 18$. There are 17 717 859 Lyndon words of length 108 and weight 6 over $\{0, 1\}$. They are collected into 514 754 orbits under the action of the affine group. There remains only one orbit representative which can be extended to an RCMC-canon. (This orbit contains 3 different Lyndon words.) For each of these Lyndon words there exist (the same 252) outer rhythms which can be used to determine an RCMC-canon. They are listed at the end of the previous section. As already explained there, some of these RCMC-canons of length 108 are not Vuza-constructible.

In both situations these are the only compositions of n which possibly lead to RCMC-canons.

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Author's address:
Harald Fripertinger,
Institut für Mathematik,
Universität Graz,
Heinrichstr. 36/4,
A–8010 Graz, Austria,
e-mail: *fripert@uni-graz.at*