On a linear functional equation for formal power series

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Abstract

Let ρ be a primitive j_0-th complex root of 1, $\mathbb{C}[\![x]\!]$ the ring of formal power series in x over $\mathbb{C}$, and let $a(x),\ b(x) \in \mathbb{C}[\![x]\!]$. We study the two equations

$$\varphi(\rho x) = a(x)\varphi(x) + b(x) \tag{L}$$

and

$$\varphi(\rho x) = a(x)\varphi(x) \tag{L_h}$$

for $\varphi \in \mathbb{C}[\![x]\!]$, which occurred in connection with an interesting and important special case when dealing with the problem of a covariant embedding of (L) with respect to an iteration group. (See H. Fripertinger and L. Reich. On covariant embeddings of a linear functional equation with respect to an analytic iteration group. Accepted for publication in the *International Journal of Bifurcation and Chaos.*) We describe necessary and sufficient conditions for finding nontrivial solutions of (L_h) and for finding solutions of (L) in the form of "cyclic" functional equations for a and b. Then we describe the set of all solutions of these functional equations and present different representations of their general solutions.

1 Introduction

About linear functional equations there exists a rather rich literature. The main sources are [6] chapters 2, 8, 13, and [7] chapters 2, 3, and 4, where the general ideas for solving such equations, like iterating them, can be found. However, it seems that the setting of formal power series has not been studied so far in detail. For a foundation of the basic calculations with formal power series we refer the reader to [5] and to [2] or [3].

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We study the linear functional equation

$$\varphi(\rho x) = a(x)\varphi(x) + b(x) \tag{L}$$

for φ and its homogeneous form

$$\varphi(\rho x) = a(x)\varphi(x) \tag{L_h}$$

in $\mathbb{C}\llbracket x \rrbracket$, the ring of formal power series over $\mathbb{C}$. We always assume that $\operatorname{ord} a(x) = 0$, whence $a(x)$ has a reciprocal (i.e. a multiplicative inverse) in $\mathbb{C}\llbracket x \rrbracket$, and ρ is a complex primitive root of 1 of order j_0. This problem occurred in connection with an interesting and important special case when dealing with the problem of a covariant embedding of (L) with respect to an iteration group. (See [4].) First we determine necessary and sufficient conditions on $a(x)$ for the existence of non-trivial solutions of (L_h) and describe the set of all solutions in these situations. Then under the assumption that (L_h) has non-trivial solutions we investigate under which conditions (L) can be solved. The set of all solutions of (L) can easily be described then as the set of all series of the form $\varphi^{(0)}(x) + \psi(x)$, where $\varphi^{(0)}$ is a particular solution of (L) and $\psi(x)$ is an arbitrary solution of (L_h).

In Lemma 2 we determine necessary (and, as it finally turns out, sufficient) conditions on $a(x)$ (cf. (1)) for the existence of non-trivial solutions of (L_h). From Lemma 3 it follows that we can always assume that $a(x) = 1 + a_1 x + \ldots$. Then in Theorem 5 the general solution of (L_h) is presented, from which we derive projection formulae and other representations $(2, \ell_0)$ of the general solution in Lemma 7 and Theorem 8. Finally, in Theorem 9 we present a situation in which it is possible to describe the general solution of (L_h) by each of these different representations $(2, \ell_0)$. The previous results are summarized for the case when $a(x) = \rho^{k_0} + a_1(x) + \ldots$ in Theorem 10. Another form of the general solution of (L_h) is given in Theorem 12. At the end of Section 2, starting with Lemma 13, we describe necessary and sufficient conditions on $b(x)$ (cf. (7)) for the existence of solutions of (L). The general solution of (L) is presented in Theorem 15.

In Section 3 we apply the formal logarithm, which finally allows to describe the conditions (1) and (7) in more details (cf. Proposition 16 and Proposition 20). We also get another representation of the general solution of (L_h) in Theorem 18 and polynomial expressions for the coefficients of $a(x)$ and $b(x)$ in Proposition 16, Remark 19, Proposition 20, and Remark 32.

It is our main aim to work out the specific features of equations (L_h) and (L) in the setting of formal power series. Therefore, it seems important to present explicit formulas for the coefficients of the general solution, which is done in Section 4 in Theorem 22 and Theorem 23. These expressions for the coefficients imply also necessary and sufficient conditions (16) or (20) for the existence of a non-zero solution of (L_h), respectively of a solution of (L), in a form different from the compact equations (1) and (7).

The next two sections apply methods from linear algebra. In Section 5 we consider systems of (homogeneous) linear functional equations by replacing in (L_h) or (L) the variable x by $\rho x, \ldots, \rho^{j_0-1}x$. Then we derive the conditions (1) and (7) as rank conditions on certain matrices.

In Section 6 we introduce a direct decomposition of $\mathbb{C}[\![x]\!]$ into subspaces $(\mathbb{C}[\![x]\!])^{(k)}$ consisting of power series of the form $\gamma(x) = \sum_{n \equiv k \bmod j_0} \gamma_n x^n$. This also allows to apply methods from linear algebra, so that the conditions (1) and (7) can again be expressed as conditions on the rank of certain matrices (cf. Proposition 26). The solutions of (L_h) or (L) can be computed as solutions of systems of linear equations and are given in form of determinants (cf. Theorem 28 and Theorem 30). In Theorem 29 we derive an interesting identity by comparing these different representations of the condition on $a(x)$ for the existence of a non-trivial solution of (L_h). Finally, Theorem 31 describes a generalization of Theorem 12.

Then in Section 7 we investigate under which conditions solutions of (L_h), (L), (1), and (7) are holomorphic in a neighborhood of 0.

The results of this paper are derived for the substitution ρx into $\varphi(x)$. Corresponding results also hold when ρx is replaced by a formal power series of the form $p(x) = S^{-1}(\rho S(x))$, where $S(x) = x + s_2 x^2 \ldots \in \mathbb{C}[\![x]\!]$.

Theorem 1. *Let $p(x) = S^{-1}(\rho S(x))$ for $S(x) = x + s_2 x^2 \ldots \in \mathbb{C}[\![x]\!]$. The formal power series $\varphi(x)$ is a solution of*

$$\varphi(p(x)) = a(x)\varphi(x) + b(x) \tag{Lp}$$

if and only if $\tilde{\varphi} := \varphi \circ S^{-1}$ satisfies

$$\tilde{\varphi}(\rho y) = \tilde{a}(y)\tilde{\varphi}(y) + \tilde{b}(y), \tag{$\tilde{L}$}$$

where $\tilde{a} := a \circ S^{-1}$ and $\tilde{b} := b \circ S^{-1}$.

Proof. The formal series $\varphi(x)$ satisfies (Lp) if and only if

$$\varphi(S^{-1}(\rho S(x))) = a(x)\varphi(x) + b(x),$$

which is equivalent to

$$(\varphi \circ S^{-1})(\rho S(x)) = (a \circ S^{-1})(S(x))(\varphi \circ S^{-1})(S(x)) + (b \circ S^{-1})(S(x)),$$

which is equal to $(\tilde{L})$ after replacing $S(x)$ by y. $\square$

Theorem 1 allows to rewrite our results for the general form (Lp) of the linear functional equation, where $p(x) \in \mathbb{C}[\![x]\!]$ satisfies

$$p^{j_0}(x) = x, \quad p^k(x) \neq x \text{ for } 0 < k < j_0.$$

This condition is equivalent to the existence of an invertible series $S(x) = x + s_2 x^2 + \ldots$ and a primitive root ρ of order j_0, such that $p(x) = S^{-1}(\rho S(x))$. (Cf. [11] Theorem 1, page 248.) We will give some of the details at the very end of this paper in Section 8.

Whenever it is useful we write the series $\varphi(x)$, $a(x)$ and $b(x) \in \mathbb{C}[\![x]\!]$ in the form

$$\varphi(x) = \sum_{n \geq 0} \varphi_n x^n, \quad a(x) = \sum_{n \geq 0} a_n x^n, \quad b(x) = \sum_{n \geq 0} b_n x^n.$$

Most of the notation coincides with the notation used in [4]. For that reason, for instance, the order of ρ is denoted by j_0.

2 Iteration of the linear equation

Our first results are just derived from iterating the linear functional equation.

Lemma 2. *If $\varphi(x) \neq 0$ is a solution of (L_h), then*

$$\varphi(\rho^n x) = \prod_{\ell=0}^{n-1} a(\rho^\ell x)\varphi(x), \qquad n \geq 0$$

and

$$\prod_{\ell=0}^{j_0-1} a(\rho^\ell x) = 1. \tag{1}$$

Proof. The first statement is proved by induction over n. For $n = 0$ everything is clear. Assume that $n > 0$, then

$$\varphi(\rho^n x) = \varphi(\rho\rho^{n-1}x) = a(\rho^{n-1}x)\varphi(\rho^{n-1}x) = a(\rho^{n-1}x)\prod_{\ell=0}^{n-2} a(\rho^\ell x)\varphi(x) = \prod_{\ell=0}^{n-1} a(\rho^\ell x)\varphi(x).$$

For $n = j_0$ we get $\varphi(x) = \varphi(\rho^{j_0}x) = \prod_{\ell=0}^{j_0-1} a(\rho^\ell x)\varphi(x)$. Since $\varphi(x) \neq 0$ we get the second result. $\square$

The necessary condition (1) can also be found in [6] as formula (8.5) on page 182.

For $\psi(x) \in \mathbb{C}[\![x]\!]$ and $\ell \in \mathbb{N}_0$ let $\psi_\ell(x)$ be given by $\psi_\ell(x) := x^\ell\psi(x)$. (For this type of transformations of the unknown function see [1] page 59.)

Lemma 3. *The series $\psi(x)$ is a solution of (L_h) if and only if $\psi_\ell(x)$ is a solution of*

$$\varphi(\rho x) = \rho^\ell a(x)\varphi(x) \tag{L_h, ℓ}$$

for $\ell \in \mathbb{N}_0$.

Proof. First assume that $\psi(x)$ is a solution of (L_h). Then $\psi_\ell(\rho x) = \rho^\ell x^\ell a(x)\psi(x) = \rho^\ell a(x)\psi_\ell(x)$. Hence $\psi_\ell(x)$ is a solution of (L_h, ℓ). Conversely, assume that $\psi_\ell(x)$ is a solution of (L_h, ℓ). Then $\rho^\ell x^\ell\psi(\rho x) = \psi_\ell(\rho x) = \rho^\ell a(x)\psi_\ell(x) = \rho^\ell a(x)x^\ell\psi(x)$, whence $\psi(\rho x) = a(x)\psi(x)$. This means that $\psi(x)$ satisfies (L_h), and the proof is finished. $\square$

From Lemma 2 we deduce that when there exists a nontrivial solution of (L_h), then the coefficient a_0 of $a(x)$ is a complex j_0-th root of 1. Consequently, there exists an integer $\ell_0 \in \{0, \ldots, j_0 - 1\}$ such that $\tilde{a}(x) := \rho^{\ell_0}a(x) \equiv 1 \bmod x$. Assume that $\ell_0 \neq 0$. If $\tilde{\psi}(x)$ is a solution of (L_h, ℓ_0), then also $\tilde{\psi}_{j_0}(x)$ is a solution. From Lemma 3 it follows immediately that $\psi(x) := x^{j_0-\ell_0}\tilde{\psi}(x)$ is a solution of (L_h), since $\psi_{\ell_0}(x) = \tilde{\psi}_{j_0}(x)$ is a solution of (L_h, ℓ_0). Hence, without loss of generality we can always assume that $a_0 = 1$.

Lemma 4. *Assume that $a_0 = 1$ and that $\varphi(x) \neq 0$ is a solution of (L_h). Let $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$, then*

$$\varphi(x) = \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi_{tj_0} x^{tj_0}. \tag{2}$$

Proof. Computing the sum of $\varphi(\rho^n x)$ for n from 0 to $j_0 - 1$, we obtain

$$\sum_{n=0}^{j_0-1} \varphi(\rho^n x) = \sum_{n=0}^{j_0-1} \sum_{m\geq 0} \varphi_m \rho^{nm} x^m = \sum_{m\geq 0} \varphi_m \left(\sum_{n=0}^{j_0-1} \rho^{nm}\right) x^m = j_0 \sum_{t\geq 0} \varphi_{tj_0} x^{tj_0},$$

since

$$\sum_{n=0}^{j_0-1} \rho^{nm} = \begin{cases} 0, & \text{if } m \not\equiv 0 \bmod j_0 \\ j_0, & \text{if } m \equiv 0 \bmod j_0. \end{cases}$$

From Lemma 2 we deduce that

$$\sum_{n=0}^{j_0-1} \varphi(\rho^n x) = \sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \varphi(x).$$

Since $a_0 = 1$ and therefore $\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \equiv j_0 \bmod x$, it is possible to find the reciprocal of $\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x)$ in $\mathbb{C}\,[\![x]\!]$, whence $\varphi(x)$ is of the given form. □

Theorem 5. *If the series $a(x)$ satisfies (1) and $a_0 = 1$, then the general solution $\varphi(x)$ of (L_h) is given (similar to (2)) by*

$$\varphi(x) = \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0}, \tag{2'}$$

where $\sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0} \in \mathbb{C}\,[\![x]\!]$ is arbitrary. Furthermore, $\varphi_{tj_0} = \varphi^*_{tj_0}$ for $t \geq 0$.

Proof. Taking into account that $\rho^{j_0} = 1$ we derive from $(2')$ that $\varphi(\rho x)$ equals

$$\varphi(\rho x) = \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^{\ell+1} x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi^*_{tj_0} \rho^{tj_0} x^{tj_0} =$$

$$\left[\sum_{n=0}^{j_0-1} \prod_{\ell=1}^{n} a(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0} = a(x) \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n} a(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0} =$$

$$a(x) \left[\sum_{n=1}^{j_0} \prod_{\ell=0}^{n-1} a(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0}.$$

Applying (1) we continue with

$$a(x)\left[\sum_{n=1}^{j_0-1}\prod_{\ell=0}^{n-1}a(\rho^\ell x)+1\right]^{-1} j_0\sum_{t\geq 0}\varphi^*_{tj_0}x^{tj_0} = a(x)\left[\sum_{n=0}^{j_0-1}\prod_{\ell=0}^{n-1}a(\rho^\ell x)\right]^{-1} j_0\sum_{t\geq 0}\varphi^*_{tj_0}x^{tj_0}$$

which equals $a(x)\varphi(x)$. From Lemma 4 it follows now that the coefficients φ_n for $n \equiv 0 \bmod j_0$ are the prescribed complex numbers $\varphi^*_{tj_0}$ for $t \geq 0$. $\square$

For later use we just mention the formula

$$\left[\sum_{n=0}^{j_0-1}\prod_{\ell=0}^{n-1}a(\rho^{\ell+1}x)\right]^{-1} = a(x)\left[\sum_{n=0}^{j_0-1}\prod_{\ell=0}^{n-1}a(\rho^\ell x)\right]^{-1}. \tag{3}$$

As an immediate consequence of Theorem 5 we find

Remark 6. *If, under the hypotheses of Theorem 5, $\varphi(x) \neq 0$ is a solution of (L_h), then necessarily* $\operatorname{ord}\varphi(x) = t_0 j_0$ *for some $t_0 \in \mathbb{N}_0$.*

Proof. The series $\varphi(x) \neq 0$ is a solution of (L_h) if and only if $\sum_{t\geq 0}\varphi_{tj_0}x^{tj_0} \neq 0$. If $\sum_{t\geq 0}\varphi_{tj_0}x^{tj_0} \neq 0$, then $\sum_{t\geq 0}\varphi_{tj_0}x^{tj_0} = \sum_{t\geq t_0}\varphi_{tj_0}x^{tj_0}$ with $\varphi_{t_0j_0} \neq 0$, for some t_0. Then (2) yields the assertion, since

$$\operatorname{ord}\left[\sum_{n=0}^{j_0-1}\prod_{\ell=0}^{n-1}a(\rho^\ell x)\right]^{-1} = 0$$

and

$$\operatorname{ord}\left(\sum_{t\geq t_0}\varphi_{tj_0}x^{tj_0}\right) = t_0 j_0.$$

$\square$

For a solution $\varphi(x) = \sum_{n\geq 0}\varphi_n x^n$ of (L_h) where $a(x)$ satisfies (1) we get from (2) conversely that the partial series $\sum_{t\geq 0}\varphi_{tj_0}x^{tj_0+\ell_0}$ of $\varphi(x)$ is given by the so called projection formula

$$\sum_{t\geq 0}\varphi_{tj_0}x^{tj_0} = \frac{1}{j_0}\left(\sum_{n=0}^{j_0-1}\prod_{\ell=0}^{n-1}a(\rho^\ell x)\right)\varphi(x). \tag{4}$$

We may, similarly, consider the partial series $\sum_{t\geq 0}\varphi_{tj_0+\ell_0}x^{tj_0}$ of $\varphi(x)$ for $0 \leq \ell_0 < j_0$, and ask whether an analogous expression for these partial series holds. For $0 \leq \ell_0 < j_0$ let

$$A_{\ell_0}(x) := \sum_{n=0}^{j_0-1}\rho^{-n\ell_0}\prod_{\ell=0}^{n-1}a(\rho^\ell x). \tag{5}$$

We get

Lemma 7. *1. Let $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$ be a solution of (L_h) and assume that $a(x)$ satisfies (1) with $a(x) \equiv 1 \bmod x$. Then*

$$\sum_{t\geq 0} \varphi_{tj_0+\ell_0} x^{tj_0+\ell_0} = \frac{1}{j_0} A_{\ell_0}(x)\varphi(x) \tag{4, ℓ_0}$$

for $0 \leq \ell_0 < j_0$. If $A_{\ell_0}(x) \neq 0$, then

$$\varphi(x) = [A_{\ell_0}(x)]^{-1} j_0 \sum_{t\geq 0} \varphi_{tj_0+\ell_0} x^{tj_0+\ell_0}. \tag{2, ℓ_0}$$

2. If $a(x)$ is a solution of (1) with $a(x) \equiv 1 \bmod x$ and $A_{\ell_0}(x) \neq 0$, then

$$\operatorname{ord} A_{\ell_0}(x) \equiv \ell_0 \bmod j_0.$$

Proof. The proof of 1. is the same as the proof of Lemma 4. In order to prove 2., let $\varphi(x)$ be the solution of (L_h) such that $\sum_{t\geq 0} \varphi_{tj_0} x^{tj_0} = 1$. This solution exists by Theorem 5. Then by 1. either $A_{\ell_0}(x) = 0$ or its order is congruent ℓ_0 modulo j_0 since $\operatorname{ord} \varphi(x) = 0$. □

We also obtain in the same way as Theorem 5

Theorem 8. *Let $a(x)$ be a solution of (1), $a(x) \equiv 1 \bmod x$, and let $0 \leq \ell_0 < j_0$. Assume that $A_{\ell_0}(x) \neq 0$ and (by the second part of Lemma 7) $\operatorname{ord} A_{\ell_0}(x) = t_0 j_0 + \ell_0$ for some $t_0 \in \mathbb{N}_0$. Then the general solution $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$ of (L_h) is given by $(2, \ell_0)$ with an arbitrary series $\sum_{t\geq t_0} \varphi_{tj_0+\ell_0} x^{tj_0+\ell_0}$. The coefficients φ_n of $\varphi(x)$ with $n \equiv \ell_0 \bmod j_0$ and $n \geq t_0 j_0 + \ell_0$ are the prescribed complex numbers $\varphi_{tj_0+\ell_0}$ for $t \geq t_0$.*

If the coefficients $a_1, \ldots, a_{j_0-1}$ of $a(x)$, a solution of (1), are sufficiently general, then the solutions $\varphi(x)$ of (L_h) can be described by $(2, \ell_0)$ for any $\ell_0 \in \{0, \ldots, j_0 - 1\}$. We will see in Proposition 16 that there exist solutions $a(x)$ of (1) for which the coefficients $a_1, \ldots, a_{j_0-1}$ can be chosen arbitrarily.

Theorem 9. *Assume that $a(x) = 1 + a_1 x + \ldots$ is a solution of (1), where the coefficients $a_1, \ldots, a_{j_0-1}$ are algebraically independent over $\mathbb{Q}$, then $A_{\ell_0}(x) \neq 0$ for all $0 \leq \ell_0 < j_0$.*

Proof. From the definition of $A_{\ell_0}(x)$ it follows that

$$A_{\ell_0}(x) = \sum_{n=0}^{j_0-1} \rho^{-n\ell_0} \sum_{r\geq 0} \left(\sum_{\nu_0+\ldots+\nu_{n-1}=r} a_{\nu_0} \cdots a_{\nu_{n-1}} \rho^{\nu_1+2\nu_2+\ldots+(n-1)\nu_{n-1}} \right) x^r,$$

whence the coefficient $[A_{\ell_0}(x)]_{j_0-1}$ of x^{j_0-1} in $A_{\ell_0}(x)$ is of the form

$$[A_{\ell_0}(x)]_{j_0-1} = \sum_{n=0}^{j_0-1} \rho^{-n\ell_0} \sum_{\nu_0+\ldots+\nu_{n-1}=j_0-1} a_{\nu_0} \cdots a_{\nu_{n-1}} \rho^{\nu_1+2\nu_2+\ldots+(n-1)\nu_{n-1}}.$$

A summand of $[A_{\ell_0}(x)]_{j_0-1}$ consists only of powers of ρ and powers of a_1 if and only if $\nu_j \in \{0,1\}$ for $j = 0,\ldots,n-1$, since $a_0 = 1$. But since moreover we assume that the condition $\nu_0+\ldots+\nu_{n-1} = j_0-1$ is satisfied, $n = j_0-1$ and $\nu_j = 1$ for all j. Consequently, there exists exactly one summand of $[A_{\ell_0}(x)]_{j_0-1}$ consisting only of powers of ρ and a_1. It is given by

$$\rho^{-(j_0-1)\ell_0} a_1^{j_0-1} \rho^{1+2+\ldots+j_0-2} = a_1^{j_0-1} \rho^{(j_0-2)(j_0-1)/2-(j_0-1)\ell_0},$$

which we will abbreviate by $a_1^{j_0-1}\rho^{m(\ell_0)}$. Then

$$[A_{\ell_0}(x)]_{j_0-1} = a_1^{j_0-1}\rho^{m(\ell_0)} + R_{\ell_0}(\rho, a_1, \ldots, a_{j_0-1}),$$

where R_{ℓ_0} is a polynomial. As a polynomial in a_1 it is of degree less than j_0-1. From the fact that $a_1,\ldots,a_{j_0-1}$ are algebraically independent over $\mathbb{Q}$ we get that $[A_{\ell_0}(x)]_{j_0-1} \neq 0$ for $\ell_0 = 0,\ldots,j_0-1$. $\square$

We will use the partial series $\sum_{t\geq 0} \varphi_{tj_0+\ell_0} x^{tj_0+\ell_0}$ later systematically. (Cf. Section 6.)

We now give the representation of the general solution $\varphi(x)$ of (L_h) if $a(x)$ satisfies (1) and $a(x) \equiv \rho^{k_0} \bmod x$ with $0 \leq k_0 < j_0$.

Theorem 10. *In this case $A_{k_0}(x) \neq 0$, and the general solution $\varphi(x)$ of (L_h) is given by*

$$\varphi(x) = [A_{k_0}(x)]^{-1} j_0 \sum_{t\geq 0} \varphi_{tj_0+k_0} x^{tj_0+k_0},$$

where $\sum_{t\geq 0} \varphi_{tj_0+k_0} x^{tj_0+k_0}$ is arbitrary. The coefficients φ_n of $\varphi(x)$ with $n \equiv k_0 \bmod j_0$ are the prescribed complex numbers $\varphi_{tj_0+k_0}$ for $t \geq 0$.

Proof. Let $\tilde{a}(x) := \rho^{-k_0} a(x)$, then $\tilde{a}(x) \equiv 1 \bmod x$. According to Lemma 3, the series $\varphi(x)$ is a solution of (L_h) if and only if $\varphi(x) = x^{k_0}\tilde{\varphi}(x)$ where $\tilde{\varphi}(x) \in \mathbb{C}[\![x]\!]$ is a solution of

$$\tilde{\varphi}(\rho x) = \tilde{a}(x)\tilde{\varphi}(x). \qquad (L_h, \tilde{a})$$

Hence $\varphi_0 = \ldots = \varphi_{k_0-1} = 0$, i.e. $\operatorname{ord}\varphi(x) \geq k_0$, and $\varphi_{n+k_0} = \tilde{\varphi}_n$ for $n \geq 0$. In Theorem 5 the general solution $\tilde{\varphi}(x) = \sum_{n\geq 0} \tilde{\varphi}_n x^n$ of $(L_h, \tilde{a})$ was given as

$$\tilde{\varphi}(x) = \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} \tilde{a}(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \tilde{\varphi}_{tj_0} x^{tj_0},$$

where $\sum_{t\geq 0} \tilde{\varphi}_{tj_0} x^{tj_0}$ is arbitrary. Consequently,

$$\varphi(x) = \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} \rho^{-k_0} a(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi_{tj_0+k_0} x^{tj_0+k_0}$$

which yields the asserted form. $\square$

As in the case of $a(x) \equiv 1 \bmod x$ we find for $a(x) \equiv \rho^{k_0} \bmod x$, for each solution $\varphi(x)$ of (L_h) and for each $\ell_0 \in \{0, \ldots, j_0 - 1\}$ that

$$\sum_{t \geq 0} \varphi_{tj_0+\ell_0+k_0} x^{tj_0+\ell_0+k_0} = \frac{1}{j_0} \left(\sum_{n=0}^{j_0-1} \rho^{-n(\ell_0+k_0)} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \right) \varphi(x). \qquad (4, \ell_0, k_0)$$

Since there exist solutions of (L_h) with $\varphi(x) = x^{k_0} + \ldots$, we find that either

$$\sum_{n=0}^{j_0-1} \rho^{-n(\ell_0+k_0)} \prod_{\ell=0}^{n-1} a(\rho^\ell x) = 0 \qquad (6)$$

or

$$\operatorname{ord} \left(\sum_{n=0}^{j_0-1} \rho^{-n(\ell_0+k_0)} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \right) = t_0 j_0 + \ell_0$$

for some $t_0 \geq 0$. In the latter case we get for the general solution

$$\varphi(x) = \left[\sum_{n=0}^{j_0-1} \rho^{-n(\ell_0+k_0)} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \right]^{-1} j_0 \sum_{t \geq t_0} \varphi_{tj_0+\ell_0+k_0} x^{tj_0+\ell_0+k_0}, \qquad (2, \ell_0, k_0)$$

where $\sum_{t \geq t_0} \varphi_{tj_0+\ell_0+k_0} x^{tj_0+\ell_0+k_0}$ is arbitrary.

Remark 11. *Let $0 \leq \ell_0 < j_0$ and let $a(x)$ be a solution of (1) with $a(x) \equiv \rho^{k_0} \bmod x$. The series $a(x)$ satisfies (6) if and only if for all solutions $\varphi(x) = \sum_{n \geq 0} \varphi_n x^n$ of (L_h)*

$$\varphi_{tj_0+\ell_0+k_0} = 0 \qquad t \geq 0.$$

Proof. If (6) is satisfied, then from $(4, \ell_0, k_0)$ it follows that $\varphi_{tj_0+\ell_0+k_0} = 0$ for all $t \geq 0$. If (6) is not satisfied, then for all $\varphi(x) \neq 0$ it follows from $(4, \ell_0, k_0)$ that

$$\sum_{t \geq 0} \varphi_{tj_0+\ell_0+k_0} x^{tj_0+\ell_0+k_0} \neq 0.$$

□

There is still another way to describe the general solution of (L_h).

Theorem 12. *If $a(x)$ satisfies (1), then the general solution of (L_h) is given by*

$$\Gamma(x) := \sum_{k=0}^{j_0-1} \frac{\gamma(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)},$$

for an arbitrary series $\gamma(x) \in \mathbb{C}[\![x]\!]$.

We notice that the right hand side in this expression for $\Gamma(x)$ will also appear in (7) (as left hand side), which is the necessary and sufficient condition for (L) to have a solution.

Proof. The series $\Gamma(x)$ satisfies (L_h) since by an application of (1) we have

$$\Gamma(\rho x) = \sum_{k=0}^{j_0-1} \frac{\gamma(\rho^{k+1}x)}{\prod_{j=0}^{k} a(\rho^{j+1}x)} = \sum_{k=0}^{j_0-2} \frac{\gamma(\rho^{k+1}x)}{\prod_{j=1}^{k+1} a(\rho^{j}x)} + \frac{\gamma(x)}{(\prod_{j=1}^{j_0-1} a(\rho^{j}x))a(x)} =$$

$$\gamma(x) + \sum_{k=1}^{j_0-1} \frac{\gamma(\rho^{k}x)}{\prod_{j=1}^{k} a(\rho^{j}x)} = a(x)\left(\frac{\gamma(x)}{a(x)} + \sum_{k=1}^{j_0-1} \frac{\gamma(\rho^{k}x)}{\prod_{j=0}^{k} a(\rho^{j}x)}\right) = a(x)\Gamma(x).$$

If $a_0 = \rho^{k_0}$, then the general solution of (L_h) is given in Theorem 10 as

$$\varphi(x) = \left[\sum_{k=0}^{j_0-1} \rho^{-kk_0} \prod_{\ell=0}^{k-1} a(\rho^{\ell}x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi_{tj_0+k_0} x^{tj_0+k_0}.$$

In order to prove that the general solution can also be expressed in the form $\Gamma(x)$, we prove that for any choice of the coefficients $\Gamma_{tj_0+k_0}$ for $t \geq 0$ we can find a series $\gamma(x) \in \mathbb{C}[\![x]\!]$ such that

$$\sum_{k=0}^{j_0-1} \frac{\gamma(\rho^{k}x)}{\prod_{j=0}^{k} a(\rho^{j}x)} = \left[\sum_{k=0}^{j_0-1} \rho^{-kk_0} \prod_{\ell=0}^{k-1} a(\rho^{\ell}x)\right]^{-1} j_0 \sum_{t\geq 0} \Gamma_{tj_0+k_0} x^{tj_0+k_0} = \sum_{n\geq 0} \Gamma_n x^n. \qquad (*)$$

Since $a(x)$ satisfies (1) we have

$$\frac{\gamma(\rho^{k}x)}{\prod_{j=0}^{k} a(\rho^{j}x)} = \gamma(\rho^{k}x) \prod_{j=k+1}^{j_0-1} a(\rho^{j}x) = \gamma(\rho^{k}x)\left(a_0^{j_0-1-k} + \sum_{n\geq 1} P_n^{(k)}(a_0, \dots, a_n, \rho)x^n\right) =$$

$$\left(\sum_{n\geq 0} \gamma_n \rho^{kn} x^n\right)\left(a_0^{j_0-1-k} + \sum_{n\geq 1} P_n^{(k)}(a_0, \dots, a_n, \rho)x^n\right) =$$

$$a_0^{j_0-1-k} \sum_{n\geq 0} \gamma_n \rho^{kn} x^n + \sum_{n\geq 0}\left(\sum_{r=0}^{n-1} \gamma_r \rho^{kr} P_{n-r}^{(k)}(a_0, \dots, a_{n-r}, \rho)\right) x^n,$$

with universal polynomials $P_n^{(k)}(a_0, \dots, a_n, \rho)$. Moreover, $a_0^{j_0-1-k} = \rho^{k_0(j_0-1-k)}$. Hence

$$\sum_{k=0}^{j_0-1} \frac{\gamma(\rho^{k}x)}{\prod_{j=0}^{k} a(\rho^{j}x)} =$$

$$\sum_{n\geq 0}\left(\gamma_n \sum_{k=0}^{j_0-1} \rho^{kn}\rho^{k_0(j_0-1-k)} + \sum_{r=0}^{n-1} \gamma_r \sum_{k=0}^{j_0-1} \rho^{kr} P_{n-r}^{(k)}(a_0, \dots, a_{n-r}, \rho)\right) x^n =$$

$$\sum_{n\geq 0}\left(\gamma_n \rho^{-k_0} \sum_{k=0}^{j_0-1} (\rho^{n-k_0})^k + Q_n(a_0, \dots, a_n, \gamma_0, \dots, \gamma_{n-1}, \rho)\right) x^n,$$

with suitable polynomials $Q_n(a_0,\ldots,a_n,\gamma_0,\ldots,\gamma_{n-1},\rho)$. If $n\equiv k_0 \bmod j_0$, then

$$\sum_{k=0}^{j_0-1}(\rho^{n-k_0})^k = j_0.$$

When comparing the coefficients in $(*)$ we get for $n\equiv k_0 \bmod j_0$

$$j_0\gamma_n\rho^{-k_0}+Q_n(a_0,\ldots,a_n,\gamma_0,\ldots,\gamma_{n-1},\rho)=\Gamma_n,$$

which allows to determine γ_n in a unique way. □

Another proof of this theorem will be presented in Section 5, and an even stronger result will be proved in Theorem 31.

In the last part of this section we deal with the linear equation (L).

Lemma 13. *If $\varphi(x)$ is a solution of (L), then*

$$\varphi(\rho^n x)=\prod_{\ell=0}^{n-1}a(\rho^\ell x)\left(\varphi(x)+\sum_{k=0}^{n-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}\right),\qquad n\geq 0.$$

If moreover (1) is satisfied, then also

$$\sum_{k=0}^{j_0-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}=0 \tag{7}$$

holds.

Proof. For $n=0$ the formula is true. Let $n\geq 0$ and assume that the formula holds for n. Then $\varphi(\rho^{n+1}x)$ equals

$$\varphi(\rho^{n+1}x)=a(\rho^n x)\varphi(\rho^n x)+b(\rho^n x)=$$
$$a(\rho^n x)\prod_{\ell=0}^{n-1}a(\rho^\ell x)\left(\varphi(x)+\sum_{k=0}^{n-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}\right)+b(\rho^n x)=$$
$$\prod_{\ell=0}^{n}a(\rho^\ell x)\left(\varphi(x)+\sum_{k=0}^{n-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}+\frac{b(\rho^n x)}{\prod_{j=0}^{n}a(\rho^j x)}\right)=$$
$$\prod_{\ell=0}^{n}a(\rho^\ell x)\left(\varphi(x)+\sum_{k=0}^{n}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}\right)$$

which finishes the first part of the proof. For $n=j_0$ we get that

$$\varphi(x)=\varphi(\rho^{j_0}x)=\prod_{\ell=0}^{j_0-1}a(\rho^\ell x)\left(\varphi(x)+\sum_{k=0}^{j_0-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}\right).$$

If (1) is satisfied, then we get as an immediate consequence that (7) is also satisfied. □

From (7) it is clear that

$$\frac{b(x)}{a(x)} = -\sum_{k=1}^{j_0-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)},$$

whence

$$b(x) = -a(x) \sum_{k=1}^{j_0-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)}.$$

Lemma 14. *Assume that $a_0 = 1$ and $\varphi(x)$ is a solution of (L). Then*

$$\varphi(x) = [A_0(x)]^{-1} \left(j_0 \sum_{t\geq 0} \varphi_{tj_0} x^{tj_0} - \sum_{n=1}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \sum_{k=0}^{n-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} \right), \tag{8}$$

where $A_0(x)$ is defined by (5).

Proof. Computing the sum of $\varphi(\rho^n x)$ for n from 0 to $j_0 - 1$ we obtain as in the proof of Lemma 4 that

$$\sum_{n=0}^{j_0-1} \varphi(\rho^n x) = j_0 \sum_{t\geq 0} \varphi_{tj_0} x^{tj_0}.$$

From Lemma 13 we deduce that

$$\sum_{n=0}^{j_0-1} \varphi(\rho^n x) = \sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \left(\varphi(x) + \sum_{k=0}^{n-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} \right) =$$

$$A_0(x)\varphi(x) + \sum_{n=1}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \sum_{k=0}^{n-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)}.$$

Since $a_0 = 1$ it is possible to find the reciprocal of $A_0(x)$, whence $\varphi(x)$ is of the given form. $\square$

Theorem 15. *If the series $a(x)$ and $b(x)$ satisfy (1), (7), and $a_0 = 1$, then the general solution $\varphi(x)$ of (L) is given (similar to (8)) by*

$$\varphi(x) = [A_0(x)]^{-1} \left(j_0 \sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0} - \sum_{n=1}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \sum_{k=0}^{n-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} \right) \tag{8'}$$

where $\sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0} \in \mathbb{C}[\![x]\!]$ is arbitrary. Furthermore, $\varphi_{tj_0} = \varphi^*_{tj_0}$ for $t \geq 0$.

Proof. Similarly as in the proof of Theorem 5 we compute $\varphi(\rho x)$ as

$$[A_0(\rho x)]^{-1} \left(j_0 \sum_{t\geq 0} \varphi^*_{tj_0} \rho^{tj_0} x^{tj_0} - \sum_{n=1}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^{\ell+1} x) \sum_{k=0}^{n-1} \frac{b(\rho^{k+1} x)}{\prod_{j=0}^{k} a(\rho^{j+1} x)} \right) =$$

$$a(x)[A_0(x)]^{-1}\left(j_0\sum_{t\geq 0}\varphi^*_{tj_0}x^{tj_0}-\sum_{n=1}^{j_0-1}\prod_{\ell=1}^{n}a(\rho^\ell x)\sum_{k=1}^{n}\frac{b(\rho^k x)}{\prod_{j=1}^{k}a(\rho^j x)}\right) \qquad (*)$$

Now we want to evaluate the last term of this expression. Multiplying each summand by $a(x)/a(x)$ and replacing n by $m-1$ yields

$$\sum_{n=1}^{j_0-1}\prod_{\ell=1}^{n}a(\rho^\ell x)\sum_{k=1}^{n}\frac{b(\rho^k x)}{\prod_{j=1}^{k}a(\rho^j x)}=\sum_{m=2}^{j_0}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\sum_{k=1}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}=$$

$$\sum_{m=2}^{j_0}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\left(\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\frac{b(x)}{a(x)}\right)=$$

$$\sum_{m=2}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\left(\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\frac{b(x)}{a(x)}\right)+\prod_{\ell=0}^{j_0-1}a(\rho^\ell x)\left(\sum_{k=0}^{j_0-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\frac{b(x)}{a(x)}\right).$$

Using (1) and (7) we derive

$$\sum_{m=2}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\left(\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\frac{b(x)}{a(x)}\right)-\frac{b(x)}{a(x)}=$$

$$\sum_{m=1}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\left(\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\frac{b(x)}{a(x)}\right)-a(x)\left(\frac{b(x)}{a(x)}-\frac{b(x)}{a(x)}\right)-\frac{b(x)}{a(x)}=$$

$$\sum_{m=1}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\left(\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\frac{b(x)}{a(x)}\right)-\frac{b(x)}{a(x)}=$$

$$\sum_{m=1}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\sum_{m=1}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\frac{b(x)}{a(x)}-\frac{b(x)}{a(x)}=$$

$$\sum_{m=1}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}-\sum_{m=0}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\frac{b(x)}{a(x)}.$$

Inserting this into $(*)$ and using (5) we get

$$\varphi(\rho x)=a(x)[A_0(x)]^{-1}\left(j_0\sum_{t\geq 0}\varphi^*_{tj_0}x^{tj_0}-\sum_{m=1}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\sum_{k=0}^{m-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}\right)+$$

$$+a(x)\left[\sum_{n=0}^{j_0-1}\prod_{\ell=0}^{n-1}a(\rho^\ell x)\right]^{-1}\left(\sum_{m=0}^{j_0-1}\prod_{\ell=0}^{m-1}a(\rho^\ell x)\frac{b(x)}{a(x)}\right)$$

which equals $a(x)\varphi(x)+b(x)$. From Lemma 14 it follows now that the coefficients φ_n for $n\equiv 0 \bmod j_0$ are the prescribed complex numbers $\varphi^*_{tj_0}$ for $t\geq 0$. □

3 Cyclic functional equations

So far we gave necessary and sufficient conditions for (L_h) and (L) to be solvable, and we described the sets of solutions of these equations. By introducing the formal logarithm

$$\ln(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{n}$$

we can describe which coefficients of $a(x)$ and $b(x)$ can be chosen arbitrarily, and how the other coefficients depend on the previous ones.

Proposition 16. *The series $a(x)$ satisfies (1) if and only if*

$$a(x) = \xi \exp\left(\sum_{n\not\equiv 0 \bmod j_0} \gamma_n x^n\right), \qquad \gamma_n \in \mathbb{C}, \tag{9}$$

where $\xi^{j_0} = 1$. This is equivalent to

$$a(x) = \xi \left(1 + \sum_{n\not\equiv 0 \bmod j_0} a_n x^n + \sum_{\substack{n\geq 1\\ n\equiv 0 \bmod j_0}} P_n(a) x^n\right),$$

where a_n are arbitrary elements in $\mathbb{C}$ for $n \not\equiv 0 \bmod j_0$ and

$$P_n(a) = P_n(a_m \mid m < n,\ m \not\equiv 0 \bmod j_0)$$

are suitable universal polynomials in the coefficients a_m.

We notice that (1) is nothing else but the multiplicatively written cyclic equation for $a(x)$ in $\mathbb{C}[\![x]\!]$.

Proof. First assume that $a_0 = 1$. Let $\gamma(x) := \ln(a(x)) = \sum_{n\geq 1} \gamma_n x^n$, then each γ_n is a polynomial in the coefficients $a_1, \ldots, a_n$. The series $a(x)$ satisfies (1) if and only if $\gamma(x)$ satisfies

$$\gamma(x) + \gamma(\rho x) + \ldots + \gamma(\rho^{j_0-1}x) = 0.$$

Hence,

$$0 = \sum_{r=0}^{j_0-1} \gamma(\rho^r x) = \sum_{r=0}^{j_0-1}\sum_{n\geq 1} \gamma_n \rho^{rn} x^n = \sum_{n\geq 1} \gamma_n x^n \sum_{r=0}^{j_0-1} \rho^{rn} = j_0 \sum_{\substack{n\geq 1\\ n\equiv 0 \bmod j_0}} \gamma_n x^n,$$

which is equivalent to $\gamma_n = 0$ for $n \equiv 0 \bmod j_0$. Consequently (9) is proved for $a_0 = 1$ with $\xi = 1$.

In order to prove the second part assume again that $\xi = 1$. If

$$a(x) = \exp(\sum_{n\not\equiv 0 \bmod j_0} \gamma_n x^n),$$

then

$$a(x) = 1 + \sum_{n\geq 1} \frac{1}{n!}(\gamma_1 x + \gamma_2 x^2 + \ldots + \gamma_{j_0-1}x^{j_0-1} + \gamma_{j_0+1}x^{j_0+1} + \ldots)^n =$$

$$1 + \sum_{n\not\equiv 0 \bmod j_0} (\gamma_n + Q_n(\gamma_m \mid m < n))x^n + \sum_{\substack{n\geq 1 \\ n\equiv 0 \bmod j_0}} Q_n(\gamma_m \mid m < n)x^n,$$

where Q_n is a polynomial in γ_m for $m < n$ and $m \not\equiv 0 \bmod j_0$. For $n \not\equiv 0 \bmod j_0$ let $a_n := \gamma_n + Q_n(\gamma_m \mid m < n)$. Then $\gamma_n = a_n - Q_n(\gamma_m \mid m < n,\ m \not\equiv 0 \bmod j_0)$, whence

$$\gamma_n = a_n - R_n(a_m \mid m < n,\ m \not\equiv 0 \bmod j_0) \tag{10}$$

for a suitable polynomial R_n. Finally, for $n \equiv 0 \bmod j_0$ let a_n be the polynomial

$$Q_n(\gamma_m \mid m < n,\ m \not\equiv 0 \bmod j_0) =$$

$$Q_n\Big(a_m - R_m(a_k \mid k < m,\ k \not\equiv 0 \bmod j_0)\ \Big|\ m < n,\ m \not\equiv 0 \bmod j_0\Big)$$

which can also be expressed as a polynomial in a_m, whence

$$a_n = P_n(a_m \mid m < n,\ m \not\equiv 0 \bmod j_0). \tag{11}$$

So we end up with

$$a(x) = 1 + \sum_{n\not\equiv 0 \bmod j_0} a_n x^n + \sum_{\substack{n\geq 1 \\ n\equiv 0 \bmod j_0}} P_n(a_m \mid m < n,\ m \not\equiv 0 \bmod j_0)x^n.$$

Conversely, take any sequence $(a_n)_{n\not\equiv 0 \bmod j_0}$ and define γ_n for $n \not\equiv 0 \bmod j_0$ by (10). For $n \equiv 0 \bmod j_0$ let a_n be given by (11). Then

$$a(x) := 1 + \sum_{n\geq 1} a_n x^n = \exp(\sum_{n\not\equiv 0 \bmod j_0} \gamma_n x^n),$$

consequently it is of the form (9). If $a_0 \neq 1$, then $a_0 = \xi$, a complex root of 1, and $a(x) = \xi\tilde{a}(x)$, where $\tilde{a}(x) \equiv 1 \bmod x$ satisfies (1). Hence $\tilde{a}(x)$ is of the asserted form and the proof is finished. □

Remark 17. A different proof of Proposition 16 could be obtained by differentiating (1) (for $a(x) \equiv 1 \bmod x$) formally with respect to x and dividing by $a(x)\cdots a(\rho^{j_0-1}x) = 1$, which gives

$$\sum_{\ell=0}^{j_0-1} \rho^\ell \frac{\frac{da}{dx}(\rho^\ell x)}{a(\rho^\ell x)} = 0 \quad \text{or} \quad \sum_{\ell=0}^{j_0-1} \rho^\ell \tilde{a}(\rho^\ell x) = 0 \quad \text{for} \quad \tilde{a}(x) = \frac{\frac{da(x)}{dx}}{a(x)}.$$

Solving this functional equation for $\tilde{a}(x)$ and going back to $a(x)$ by solving the formal differential equation

$$\frac{da(x)}{dx} = \tilde{a}(x)a(x),$$

we find Proposition 16.

From Proposition 16 we derive still another representation of the general solution of (L_h).

Theorem 18. *If (9) is satisfied with $\xi = 1$, then the solutions of (L_h) are of the form*

$$\varphi(x) = \exp\left(\sum_{\substack{n\geq 1 \\ n\not\equiv 0 \bmod j_0}} \frac{\gamma_n}{\rho^n - 1} x^n \right) \sum_{t\geq 0} h_{tj_0} x^{tj_0},$$

where $\gamma(x) = \ln(a(x)) = \sum_{n\geq 1} \gamma_n x^n$ and $(h_{tj_0})_{t\geq 0}$ is an arbitrary sequence in $\mathbb{C}$.

Proof. The series $\varphi(x) = 1 + \sum_{n\geq 1} \varphi_n x^n$ is a solution of (L_h) if and only if $\psi(x) = \ln(\varphi(x))$ is a solution of

$$\psi(\rho x) = \gamma(x) + \psi(x),$$

where $\gamma(x) = \ln(a(x))$. Introducing coefficients ψ_n of ψ and γ_n of γ yields

$$\sum_{n\geq 1} \psi_n \rho^n x^n = \sum_{n\geq 1} \gamma_n x^n + \sum_{n\geq 1} \psi_n x^n,$$

or equivalently

$$\psi_n(\rho^n - 1) = \gamma_n, \qquad \forall n \geq 1.$$

If $n \not\equiv 0 \bmod j_0$, then ψ_n is uniquely determined as $\psi_n = \gamma_n/(\rho^n - 1)$. The coefficients ψ_{tj_0} can be chosen arbitrarily in $\mathbb{C}$. Hence

$$\varphi(x) = \exp\left(\sum_{n\not\equiv 0 \bmod j_0} \frac{\gamma_n}{\rho^n - 1} x^n + \sum_{t\geq 1} \psi_{tj_0} x^{tj_0} \right) =$$

$$\exp\left(\sum_{n\not\equiv 0 \bmod j_0} \frac{\gamma_n}{\rho^n - 1} x^n \right) \left(1 + \sum_{t\geq 1} h_{tj_0} x^{tj_0} \right).$$

Since $(\psi_{tj_0})_{t\geq 1}$ is an arbitrary sequence in $\mathbb{C}$, also h_{tj_0} for $t \geq 1$ can be chosen arbitrarily in $\mathbb{C}$. Finally, the general solution $\varphi(x)$ with φ_0 not necessarily equal to 1 is given by

$$\varphi(x) = \exp\left(\sum_{n\not\equiv 0 \bmod j_0} \frac{\gamma_n}{\rho^n - 1} x^n \right) \sum_{t\geq 0} h_{tj_0} x^{tj_0}.$$

□

Remark 19. *If $a(x)$ satisfies (9) with $\xi = 1$ and $b(x)$ satisfies (7), then $b_0 = 0$ and b_{tj_0} can be expressed as*

$$b_{tj_0} = S_{tj_0}(\rho, (a_n)_{n\geq 1}, b_m \mid m < tj_0,\ m \not\equiv 0 \bmod j_0), \qquad t \geq 1$$

where S_{tj_0} is a polynomial.

Proof. Since $a_0 = 1$, the reciprocal of $\prod_{j=0}^{k} a(\rho^j x)$ starts with the constant term 1 for $k \geq 0$, thus from (7) it follows immediately that $b_0 = 0$.

Indicating the reciprocal of $\prod_{j=0}^{k} a(\rho^j x)$ by $1 + \sum_{n\geq 1} a_n^{(k)} x^n$, then from (7) we get

$$0 = \sum_{k=0}^{j_0-1} b(\rho^k x)\left(1 + \sum_{n\geq 1} a_n^{(k)} x^n\right) = \sum_{k=0}^{j_0-1}\left(\sum_{n\geq 1} b_n \rho^{kn} x^n\right)\left(1 + \sum_{n\geq 1} a_n^{(k)} x^n\right) =$$

$$\sum_{n\geq 1} b_n x^n \left(1 + \sum_{n\geq 1} a_n^{(0)} x^n\right) + \sum_{k=1}^{j_0-1}\left(\sum_{\substack{n\geq 1\\ n\not\equiv 0 \bmod j_0}} b_n \rho^{kn} x^n + \sum_{t\geq 1} b_{tj_0} x^{tj_0}\right)\left(1 + \sum_{n\geq 1} a_n^{(k)} x^n\right).$$

Hence, the coefficient of x^{tj_0} for $t \geq 1$ satisfies

$$j_0 b_{tj_0} + R_{tj_0}(\rho, (a_n)_{n\geq 1}, b_m \mid m < tj_0) = 0,$$

where R_{tj_0} is a suitable polynomial. Consequently

$$b_{tj_0} = -\frac{1}{j_0} R_{tj_0}(\rho, (a_n)_{n\geq 1}, b_m \mid m < tj_0).$$

By induction we prove that $b_{tj_0} = S_{tj_0}(\rho, (a_n)_{n\geq 1}, b_m \mid m < tj_0,\ m \not\equiv 0 \bmod j_0)$, with a suitable polynomial S_{tj_0}. $\square$

Now we will characterize the solutions $b(x)$ of (7) if $a(x)$ satisfies (1) and $a_0 = 1$, i.e. those $b(x)$ for which (L) has a solution.

Proposition 20. *If $a(x)$ satisfies (9) with $\xi = 1$, then (L) has a solution if and only if $b(x)$ is of the form*

$$b(x) = \sum_{\substack{n\geq 1\\ n\not\equiv 0 \bmod j_0}} \psi_n x^n + \sum_{\substack{n\geq 1\\ n\equiv 0 \bmod j_0}} M_n((a_r)_{r\geq 1}, \psi_m \mid m < n,\ m \not\equiv 0 \bmod j_0) x^n, \tag{12}$$

for arbitrary $\psi_n \in \mathbb{C}$ ($n \not\equiv 0 \bmod j_0$) and for suitable polynomials M_n ($n \equiv 0 \bmod j_0$).

Proof. First assume that (L) has a solution $\varphi(x)$. We may assume $\varphi_0 = 1$ and $\varphi_{tj_0} = 0$ for $t \geq 1$, since it is possible to add a suitable solution of (L_h) (c.f. Theorem 18) in order to determine a solution with these properties. Hence

$$\varphi(x) = 1 + \sum_{n\not\equiv 0 \bmod j_0} \varphi_n x^n.$$

As a consequence of (L) we derive

$$b(x) = \left(1 + \sum_{n\not\equiv 0 \bmod j_0} \varphi_n \rho^n x^n\right) - \left(1 + \sum_{n\not\equiv 0 \bmod j_0} \varphi_n x^n\right)\left(1 + \sum_{n\geq 1} a_n x^n\right) =$$

$$\sum_{n\not\equiv 0 \bmod j_0} \left[(\rho^n - 1)\varphi_n + L_n((a_r)_{r\geq 1}, \varphi_m \mid m < n,\ m \not\equiv 0 \bmod j_0)\right] x^n +$$
$$+ \sum_{\substack{n\geq 1\\ n\equiv 0 \bmod j_0}} L_n((a_r)_{r\geq 1}, \varphi_m \mid m < n,\ m \not\equiv 0 \bmod j_0) x^n,$$
where L_n are suitable polynomials.

For $n \not\equiv 0 \bmod j_0$ denote $(\rho^n - 1)\varphi_n + L_n((a_r)_{r\geq 1}, \varphi_m \mid m < n,\ m \not\equiv 0 \bmod j_0)$ by ψ_n. Hence for $n \not\equiv 0 \bmod j_0$

$$\varphi_n = \frac{\psi_n - L_n((a_r)_{r\geq 1}, \varphi_m \mid m < n,\ m \not\equiv 0 \bmod j_0)}{\rho^n - 1} =$$
$$K_n(\rho, (a_r)_{r\geq 1}, \psi_n, \varphi_m \mid m < n,\ m \not\equiv 0 \bmod j_0),$$

where K_n is a suitable polynomial. By induction we get

$$\varphi_n = \tilde{K}_n(\rho, (a_r)_{r\geq 1}, \psi_m \mid m \leq n,\ m \not\equiv 0 \bmod j_0) \tag{13}$$

with suitable polynomials $\tilde{K}_n$. For $n \equiv 0 \bmod j_0$ replace φ_m in L_n by (13). Then $b(x)$ is of the form (12).

Conversely, assume that $(\psi_n)_{n\not\equiv 0 \bmod j_0}$ is a sequence in $\mathbb{C}$, and let $b(x)$ be given by (12). Then there exists a unique sequence $(\varphi_n)_{n\geq 1}$ such that $\varphi_n = 0$ for $n \equiv 0 \bmod j_0$ and φ_n given by (13) for $n \not\equiv 0 \bmod j_0$. According to the computation above, $b(x)$ satisfies $b(x) = \varphi(\rho x) - a(x)\varphi(x)$ for $\varphi(x) = 1 + \sum_{n\not\equiv 0 \bmod j_0} \varphi_n x^n$. Hence, φ is a solution of (L). $\square$

Assume that $a(x)$ satisfies (9) with $\xi = 1$. From the last lemma we deduce that (L) can be solved if $b(x)$ is of the form

$$b(x) = \sum_{\substack{n\geq 1\\ n\not\equiv 0 \bmod j_0}} b_n x^n + \sum_{\substack{n\geq 1\\ n\equiv 0 \bmod j_0}} M_n((a_r)_{r\geq 1}, b_m \mid m < n,\ m \not\equiv 0 \bmod j_0)$$

with arbitrary $(b_n)_{n\not\equiv 0 \bmod j_0}$ in $\mathbb{C}$. Determining φ_n for $n \not\equiv 0 \bmod j_0$ by (13) (where we have to replace ψ_m by b_m) and setting $\varphi_0 = 1$ and $\varphi_{sj_0} = 0$ for $s \geq 1$, we compute a particular solution $\varphi(x) = \sum_{n\geq 0} \varphi_n x^n$ of (L). All the other solutions of (L) can be found by adding series as given in Theorem 18.

Another characterization of those $b(x) \in \mathbb{C}\llbracket x \rrbracket$ for which (L) has a solution is given in Remark 32.

4 Explicit formulas for the coefficients of the solutions

Our next approach allows to compute (more or less) explicitly the coefficients of the solutions φ of (L_h) or (L) respectively. The series φ is a solution of (L_h) if and only if

$$\sum_{n\geq 0} \varphi_n \rho^n x^n = \sum_{n\geq 0} \left(\sum_{r=0}^{n} a_r \varphi_{n-r} \right) x^n.$$

This is equivalent to

$$\varphi_n(\rho^n - a_0) = \sum_{r=1}^{n} a_r \varphi_{n-r}, \qquad \forall n \geq 0. \tag{14}$$

If $a_0^{j_0} = 1$, then a_0 is of the form ρ^n for suitable $n \in \mathbb{Z}$. Let k_0 be the minimum of $\{n \in \mathbb{N}_0 \mid \rho^n = a_0\}$, and let K denote the set

$$K := \{k_0 + nj_0 \mid n \in \mathbb{N}_0\}.$$

The proof of the next lemma, which uses (14) and induction, is left to the reader.

Lemma 21. *If φ is a solution of (L_h), then $\varphi_n = 0$ for $0 \leq n < k_0$. If $\varphi \neq 0$ is a solution of (L_h), then* $\min\{k \in \mathbb{N}_0 \mid \varphi_k \neq 0\} = k_0 + rj_0$ *for some $r \in \mathbb{N}_0$.*

As a consequence of Lemma 3 we derive that φ is a solution of (L_h) if and only if $x^{rj_0}\varphi$ is a solution of (L_h) for $r \in \mathbb{N}_0$. If $\varphi \neq 0$ is a solution of (L_h), then there exists an index $k \in \mathbb{N}_0$ such that $\varphi_k \neq 0$. From the last lemma we deduce that $\min\{k \in \mathbb{N}_0 \mid \varphi_k \neq 0\} = k_0 + rj_0$ for a suitable $r \in \mathbb{N}_0$. Without loss of generality we assume that $r = 0$.

In combinatorics an ordered partition of the integer $n \geq 0$ is an ordered tuple $(r_1, \ldots, r_\ell)$ of integers $r_i > 0$ such that $\sum_{i=1}^{\ell} r_i = n$. For $n = 0$ there exists only one ordered partition, the empty tuple $()$. In the context of the present article we are rather interested in the finite sequences $\sigma = (\sigma_1, \ldots, \sigma_\ell)$, $\sigma_i = \sum_{j=1}^{i} r_j$, corresponding to an ordered partition $(r_1, \ldots, r_\ell)$. Let Σ_n indicate the set of all those sequences σ corresponding to ordered partition of n such that $\sigma_i \not\equiv 0 \bmod j_0$ for all $i = 1, \ldots, \ell$. Then $\Sigma_0 = \{()\}$, and for $n > 0$ we have

$$\Sigma_n = \{(\sigma_1, \ldots, \sigma_\ell) \mid \sigma_i \in \mathbb{N},\ \sigma_i \not\equiv 0 \bmod j_0,\ 1 \leq i \leq \ell,\ \sigma_j < \sigma_{j+1},\ 1 \leq j < \ell,\ \sigma_\ell = n\}.$$

The length of $\sigma = (\sigma_1, \ldots, \sigma_\ell) \in \Sigma_n$ is ℓ, which will also be indicated as $\ell(\sigma)$. Moreover, by $a(\sigma)$ we denote the product

$$a(\sigma) = \prod_{i=1}^{\ell} a_{\sigma_i - \sigma_{i-1}},$$

where a_n are the coefficients of the series $a(x)$, and where we assume that $\sigma_0 = 0$. Finally $N(\sigma)$ stands for

$$N(\sigma) = \prod_{i=1}^{\ell} (\rho^{\sigma_i} - 1).$$

Theorem 22. *1. If φ is a non-trivial solution of (L_h), then the coefficients of φ satisfy*

$$\varphi_n = \sum_{t=0}^{\left[\frac{n-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{\sigma \in \Sigma_{n-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)}, \tag{15}$$

and for all $s \geq 0$

$$\sum_{r=1}^{sj_0} a_r \sum_{\sigma\in\Sigma_{sj_0-r}} \frac{a(\sigma)}{a_0^{\ell(\sigma)}N(\sigma)} = 0 \tag{16}$$

holds.

2. *If (16) is satisfied for all $s \geq 0$, then each series φ with coefficients given by (15) and any choice of the coefficients $(\varphi_{k_0+tj_0})_{t\geq 0}$ is a solution of (L_h).*

Proof. 1. Let $\varphi(x) = \sum_{n\geq 0}\varphi_n x^n$ be a non-trivial solution of (L_h). In Lemma 21 it was shown that $\varphi_n = 0$ for $0 \leq n < k_0$. This coincides with (15), since in this case the first sum is empty. If n is of the form k_0+sj_0 for $s \in \mathbb{N}_0$, then the right hand side of (15) is just $\varphi_{k_0+sj_0}$, since $\Sigma_{n-k_0-tj_0} = \emptyset$ for $0 \leq t < s$, and $\Sigma_{n-k_0-sj_0} = \Sigma_0 = \{()\}$. Finally, for $n > k_0$ and $n \notin K$ we will use induction to prove the theorem. Setting $n = k_0 + 1$ in (14) we get $\varphi_{k_0+1}(\rho^{k_0+1} - a_0) = a_1\varphi_{k_0}$, hence

$$\varphi_{k_0+1} = \frac{a_1\varphi_{k_0}}{a_0(\rho-1)} = \varphi_{k_0}\sum_{\sigma\in\Sigma_1}\frac{a(\sigma)}{a_0^{\ell(\sigma)}N(\sigma)}.$$

Let $n > k_0 + 1$, $n \notin K$, and assume that φ_j is given by (15) for $j < n$. From (14) we deduce

$$\varphi_n = \frac{1}{a_0(\rho^{n-k_0}-1)}\sum_{r=1}^{n-k_0} a_r\varphi_{n-r} =$$

$$\frac{1}{a_0(\rho^{n-k_0}-1)}\sum_{r=1}^{n-k_0} a_r \sum_{t=0}^{\left[\frac{n-r-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{\sigma\in\Sigma_{n-r-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)}N(\sigma)} =$$

$$\sum_{t=0}^{\left[\frac{n-1-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{r=1}^{n-k_0-tj_0} \frac{a_r}{a_0(\rho^{n-k_0}-1)} \sum_{\sigma\in\Sigma_{n-r-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)}N(\sigma)} =$$

$$\sum_{t=0}^{\left[\frac{n-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{\sigma\in\Sigma_{n-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)}N(\sigma)}.$$

For doing these computations we used that if $n \notin K$, then the two values $\left[\frac{n-1-k_0}{j_0}\right]$ and $\left[\frac{n-k_0}{j_0}\right]$ coincide. Moreover

$$\Sigma_{n-k_0-tj_0} = \dot{\bigcup_{r=1}^{n-k_0-tj_0}} \left\{(\sigma_1,\ldots,\sigma_{\ell(\sigma)}, n-k_0-tj_0) \mid \sigma \in \Sigma_{n-k_0-tj_0-r}\right\}.$$

If $\tilde\sigma$ denotes a sequence $\tilde\sigma = (\sigma_1,\ldots,\sigma_{\ell(\sigma)}, n-k_0-tj_0) \in \Sigma_{n-k_0-tj_0}$, where $\sigma \in \Sigma_{n-k_0-tj_0-r}$ for some $r \in \{1,\ldots,n-k_0-tj_0\}$, then $\ell(\tilde\sigma) = \ell(\sigma)+1$, $a(\tilde\sigma) = a(\sigma)a_r$ and $N(\tilde\sigma) = N(\sigma)(\rho^{n-k_0-tj_0}-1) = N(\sigma)(\rho^{n-k_0}-1)$.

So far we proved that φ_n is of the form (15). We still have to show that (16) holds for all $s \geq 0$. For $s = 0$ this is clear. Assume that $s > 0$, then by induction we get

$$\varphi_{k_0+sj_0}(\rho^{k_0+sj_0} - a_0) = 0 = \sum_{r=1}^{k_0+sj_0} a_r \varphi_{k_0+sj_0-r} =$$

$$\sum_{r=1}^{k_0+sj_0} a_r \sum_{t=0}^{\left[\frac{k_0+sj_0-r-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{\sigma \in \Sigma_{k_0+sj_0-r-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)} =$$

$$\sum_{t=0}^{s-1} \varphi_{k_0+tj_0} \sum_{r=1}^{(s-t)j_0} a_r \sum_{\sigma \in \Sigma_{(s-t)j_0-r}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)} =$$

$$\varphi_{k_0} \sum_{r=1}^{sj_0} a_r \sum_{\sigma \in \Sigma_{sj_0-r}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)} + \sum_{t=1}^{s-1} \varphi_{k_0+tj_0} \underbrace{\sum_{r=1}^{(s-t)j_0} a_r \sum_{\sigma \in \Sigma_{(s-t)j_0-r}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)}}_{=0} =$$

$$\varphi_{k_0} \sum_{r=1}^{sj_0} a_r \sum_{\sigma \in \Sigma_{sj_0-r}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)}.$$

From $\varphi_{k_0} \neq 0$ it follows that (16) holds also for s.

2. Assume that $(\varphi_{k_0+tj_0})_{t\geq 0}$ is an arbitrary sequence in $\mathbb{C}$, and let φ_n be given by (15) for all $n \geq 0$. In order to prove that φ satisfies (L_h) if (16) is satisfied for all s, we prove that (14) holds. If $n \notin K$, then φ_n is computed by (15) which was in the first part of this proof deduced from

$$\varphi_n = \frac{1}{\rho^n - a_0} \sum_{r=1}^{n} a_r \varphi_{n-r}.$$

Hence (14) is satisfied. If $n \in K$, then $n = k_0 + sj_0$ for a suitable $s \geq 0$. In this situation (14) reduces to

$$0 = \sum_{r=1}^{k_0+sj_0} a_r \varphi_{k_0+sj_0-r}.$$

In the first part of this proof this sum was computed as

$$\sum_{t=0}^{s-1} \varphi_{k_0+tj_0} \sum_{r=1}^{(s-t)j_0} a_r \sum_{\sigma \in \Sigma_{(s-t)j_0-r}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)}$$

which equals 0, since (16) is satisfied. $\square$

In order to deal with the equation (L) we introduce some further notation: For integers m and j let

$$\bar{\Sigma}_{m,j} := \{\sigma \in \Sigma_m \mid \sigma_{i+j} \notin K,\ 1 \le i \le \ell(\sigma)\},$$

and finally for $\sigma \in \Sigma_m$ let

$$\bar{N}(j,\sigma) := \prod_{i=1}^{\ell(\sigma)} (\rho^{\sigma_i + j} - a_0) = a_0^{\ell(\sigma)} \prod_{i=1}^{\ell(\sigma)} (\rho^{\sigma_i + j - k_0} - 1).$$

The series φ is a solution of (L) if and only if

$$\sum_{n \ge 0} \varphi_n \rho^n x^n = \sum_{n \ge 0} \left(\sum_{r=0}^{n} a_r \varphi_{n-r} + b_n \right) x^n.$$

This is equivalent to

$$\varphi_n(\rho^n - a_0) = \sum_{r=1}^{n} a_r \varphi_{n-r} + b_n, \qquad \forall n \ge 0. \tag{17}$$

Theorem 23. *1. If φ is a solution of (L), then the coefficients of φ satisfy*

$$\varphi_n = \sum_{t=0}^{\left[\frac{n-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{\sigma \in \Sigma_{n-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)} + \sum_{\substack{j=0 \\ j \notin K}}^{n} \frac{b_j}{\rho^j - a_0} \sum_{\sigma \in \bar{\Sigma}_{n-j,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)}, \tag{18}$$

and for all $n \in K$ (assume $n = k_0 + sj_0$)

$$\sum_{t=0}^{s-1} \varphi_{k_0+tj_0} \sum_{r=1}^{(s-t)j_0} a_r \sum_{\sigma \in \Sigma_{(s-t)j_0 - r}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)} + \sum_{\substack{j=0 \\ j \notin K}}^{n-1} \frac{b_j}{\rho^j - a_0} \sum_{r=1}^{n-j} a_r \sum_{\sigma \in \bar{\Sigma}_{n-j-r,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)} + b_n = 0 \tag{19}$$

holds.

2. If there exist non-trivial solutions of (L_h) and if

$$\sum_{\substack{j=0 \\ j \notin K}}^{n-1} \frac{b_j}{\rho^j - a_0} \sum_{r=1}^{n-j} a_r \sum_{\sigma \in \bar{\Sigma}_{n-j-r,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)} + b_n = 0 \tag{20}$$

holds for all $n \in K$, then each choice of $(\varphi_{k_0+tj_0})_{t \ge 0}$ in $\mathbb{C}$ yields via (18) a solution of (L).

Proof. 1. Let φ be a solution of (L). For $n = k_0 + sj_0 \in K$ the coefficient φ_n is of the form (18) since the first sum reduces to $\varphi_{k_0+sj_0}$ as shown in the proof of Theorem 22, and the second sum yields 0. The last fact is true since for $n \in K$ by definition the set $\bar{\Sigma}_{n-j,j} = \emptyset$ for all $j \in \{0, \ldots, n\}$. Now we will apply induction to prove (18) for all n. If $k_0 = 0$, then (18) holds for $n = 0$. If $k_0 \neq 0$, then $a_0 \neq 1$, and from (17) we deduce that $\varphi_0(1 - a_0) = b_0$ which yields

$$\varphi_0 = \frac{b_0}{\rho^0 - a_0}$$

in accordance with (18). Now assume that $n \notin K$ and that φ_j is given by (18) for $j < n$. From (17) we deduce

$$\varphi_n(\rho^n - a_0) = \sum_{r=1}^{n} a_r \varphi_{n-r} + b_n =$$

$$\sum_{r=1}^{n} a_r \left(\sum_{t=0}^{\left[\frac{n-r-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{\sigma \in \Sigma_{n-r-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)} + \sum_{\substack{j=0 \\ j \notin K}}^{n-r} \frac{b_j}{\rho^j - a_0} \sum_{\sigma \in \bar{\Sigma}_{n-r-j,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)} \right) + b_n =$$

$$\sum_{r=1}^{n} a_r \sum_{t=0}^{\left[\frac{n-r-k_0}{j_0}\right]} \varphi_{k_0+tj_0} \sum_{\sigma \in \Sigma_{n-r-k_0-tj_0}} \frac{a(\sigma)}{a_0^{\ell(\sigma)} N(\sigma)} + \sum_{r=1}^{n} a_r \sum_{\substack{j=0 \\ j \notin K}}^{n-r} \frac{b_j}{\rho^j - a_0} \sum_{\sigma \in \bar{\Sigma}_{n-r-j,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)} + b_n.$$

In order to determine φ_n, we have to divide this formula by $\rho^n - a_0$. With the first expression of the last line we dealt already in the proof of Theorem 22, so we only have to compute

$$\frac{1}{\rho^n - a_0} \left(\sum_{r=1}^{n} a_r \sum_{\substack{j=0 \\ j \notin K}}^{n-r} \frac{b_j}{\rho^j - a_0} \sum_{\sigma \in \bar{\Sigma}_{n-r-j,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)} + b_n \right).$$

Changing the order of summation yields

$$\sum_{\substack{j=0 \\ j \notin K}}^{n-1} \frac{b_j}{\rho^j - a_0} \sum_{r=1}^{n-j} \frac{a_r}{\rho^n - a_0} \sum_{\sigma \in \bar{\Sigma}_{n-r-j,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)} + \frac{b_n}{\rho^n - a_0} =$$

$$\sum_{\substack{j=0 \\ j \notin K}}^{n} \frac{b_j}{\rho^j - a_0} \sum_{\sigma \in \bar{\Sigma}_{n-j,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)}.$$

Similar as in the proof of Theorem 22 we used the fact that for $j < n$

$$\bar{\Sigma}_{n-j,j} = \dot{\bigcup_{r=1}^{n-j}} \left\{ (\sigma_1, \ldots, \sigma_{\ell(\sigma)}, n-j) \mid \sigma \in \bar{\Sigma}_{n-r-j,j} \right\}.$$

Let $\tilde{\sigma} = (\sigma_1, \ldots, \sigma_{\ell(\sigma)}, n-j) \in \bar{\Sigma}_{n-j,j}$ such that $\sigma \in \bar{\Sigma}_{n-r-j,j}$ for some $r \in \{1, \ldots, n-j\}$, then $\bar{N}(j,\tilde{\sigma}) = \bar{N}(j,\sigma)(\rho^n - a_0)$. Combining this result with the result from Theorem 22 proves (18).

Let $n = k_0 + sj_0$ for $s \in \mathbb{N}_0$ be an element of K, then from (17) we deduce that

$$\varphi_n(\rho^n - a_0) = 0 = \sum_{r=1}^{n} a_r \varphi_{n-r} + b_n.$$

Expressing φ_{n-r} by (18) and changing the sequence of summation yields (19).

2. If there exist non-trivial solutions of (L_h), then (16) is satisfied, hence (19) reduces for $n \in K$ to (20). If this equation is satisfied for all $n \in K$, then a particular solution of (L) is given by $\psi(x)$ with coefficients $\psi_{k_0+tj_0} = 0$ for $t \geq 0$ and

$$\psi_n = \sum_{\substack{j=0 \\ j \notin K}}^{n} \frac{b_j}{\rho^j - a_0} \sum_{\sigma \in \bar{\Sigma}_{n-j,j}} \frac{a(\sigma)}{\bar{N}(j,\sigma)}$$

for $n \notin K$. It remains to show that ψ_n satisfies (17). If $n \notin K$, we computed in 1. that

$$\sum_{r=1}^{n} a_r \psi_{n-r} + b_n = (\rho^n - a_0)\psi_n.$$

If $n \in K$, then (20) implies that

$$\sum_{r=1}^{n} a_r \psi_{n-r} + b_n = 0 = (\rho^n - a_0)\psi_n.$$

Hence ψ is a particular solution of (L). All solutions of (L) are of the form $\psi(x) + \varphi(x)$, where $\varphi(x)$ is a solution of (L_h), described in Theorem 22. $\square$

5 Representation of the solutions by means of basic linear algebra

In the next two approaches some methods of linear algebra will be useful. We describe the necessary and sufficient conditions for the existence of non-trivial solutions of (L_h) and of solutions of (L) as conditions on the rank of certain matrices. Moreover, we present another proof for the form of the general solution of (L_h) given in Theorem 12.

Replacing in the homogeneous linear functional equation (L_h) the variable x by $\rho x, \rho^2 x, \ldots, \rho^{j_0-1}x$ and writing $\varphi_n(x)$ for $\varphi(\rho^n x)$ for $n = 0, \ldots, j_0 - 1$, we get the system of homogeneous linear equations

$$\begin{aligned}
-a(x)\varphi_0(x) + \varphi_1(x) &= 0 \\
-a(\rho x)\varphi_1(x) + \varphi_2(x) &= 0 \\
&\cdots \\
-a(\rho^{j_0-2}x)\varphi_{j_0-2}(x) + \varphi_{j_0-1}(x) &= 0 \\
-a(\rho^{j_0-1}x)\varphi_{j_0-1}(x) + \varphi_0(x) &= 0
\end{aligned}$$

which can be written in matrix form as

$$A(x)\begin{pmatrix}\varphi_0(x)\\ \vdots\\ \varphi_{j_0-1}(x)\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

with

$$A(x):=\begin{pmatrix}-a(x) & 1 & 0 & \dots & 0\\ 0 & -a(\rho x) & 1 & & 0\\ \vdots & & \ddots & \ddots & \vdots\\ 0 & \dots & \dots & -a(\rho^{j_0-2}x) & 1\\ 1 & 0 & \dots & 0 & -a(\rho^{j_0-1}x)\end{pmatrix}.$$

If (L_h) has non-trivial solutions, then this system has non-trivial solutions, whence the determinant of the coefficient matrix, which we will call $A(x)$, vanishes. Hence $\det A(x)=0$. Developing this determinant with respect to the first column, we immediately get that (1) is satisfied, and that $A(x)$ is of rank j_0-1. Assuming that (1) is satisfied, we can apply a method described in [6] pages 267–271, originating from [9] and [10], which expresses the general solution of this system in the form

$$\begin{pmatrix}\varphi_0(x)\\ \varphi_1(x)\\ \vdots\\ \varphi_{j_0-2}(x)\\ \varphi_{j_0-1}(x)\end{pmatrix}=B(x)\begin{pmatrix}\gamma(x)\\ \gamma(\rho x)\\ \vdots\\ \gamma(\rho^{j_0-2}x)\\ \gamma(\rho^{j_0-1}x)\end{pmatrix}$$

with a suitable matrix $B(x)$ and an arbitrary series $\gamma(x)\in\mathbb{C}[\![x]\!]$. In general the expressions obtained for $\varphi(\rho^k x)$ in this way are contradictory. However, when introducing matrices $M_k=(m_{ij}^k)_{0\le i,j<j_0}$ given by

$$m_{ij}^k=\begin{cases}1, & \text{if } i+k\equiv j \bmod j_0\\ 0, & \text{otherwise,}\end{cases}$$

and if $B(x)$ satisfies $B(x)=M_kB(x)M_k^{-1}$ for all $k=0,\dots,j_0-1$, then they are not. In this situation $B(x)$ is called a compatible matrix.

In order to determine $B(x)$, the main task is to find a matrix $B_1(x)$ which is compatible and which satisfies

$$A(x)B_1(x)A(x)+A(x)=0. \tag{21}$$

Then we put $B(x):=B_1(x)A(x)+I$, where I is the unit matrix. First we determine matrices $P(x)$, $Q(x)$, and $D(x)$ such that $D(x)$ is a diagonal matrix of the same rank as $A(x)$ and such that $A(x)=P(x)D(x)Q(x)$. In the present situation we have

$$P(x)=\begin{pmatrix}1 & 0 & \dots & 0 & 0\\ 0 & 1 & & 0 & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & \dots & 1 & 0\\ \frac{-1}{a(x)} & \frac{-1}{a(x)a(\rho x)} & \dots & \frac{-1}{\prod_{j=0}^{j_0-2}a(\rho^j x)} & 1\end{pmatrix},\quad D(x)=\begin{pmatrix}1 & 0 & \dots & 0 & 0\\ 0 & 1 & & 0 & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & \dots & 1 & 0\\ 0 & 0 & \dots & 0 & 0\end{pmatrix}$$

and

$$Q(x) = \begin{pmatrix} -a(x) & 1 & 0 & \dots & 0 \\ 0 & -a(\rho x) & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & -a(\rho^{j_0-2}x) & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then we define a matrix $B_0(x) := -Q^{-1}(x)D(x)P^{-1}(x)$ which satisfies (21). The inverse matrices are given by

$$P^{-1}(x) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ \frac{1}{a(x)} & \frac{1}{a(x)a(\rho x)} & \dots & \frac{1}{\prod_{j=0}^{j_0-2} a(\rho^j x)} & 1 \end{pmatrix}$$

and

$$Q^{-1}(x) = \begin{pmatrix} \frac{-1}{a(x)} & \frac{-1}{a(x)a(\rho x)} & \dots & \frac{-1}{\prod_{j=0}^{j_0-2} a(\rho^j x)} & \frac{1}{\prod_{j=0}^{j_0-2} a(\rho^j x)} \\ 0 & \frac{-1}{a(\rho x)} & \dots & \frac{-1}{\prod_{j=1}^{j_0-2} a(\rho^j x)} & \frac{1}{\prod_{j=1}^{j_0-2} a(\rho^j x)} \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{-1}{a(\rho^{j_0-2}x)} & \frac{1}{a(\rho^{j_0-2}x)} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Hence $B_0(x)$ is of the form

$$B_0(x) = \begin{pmatrix} \frac{1}{a(x)} & \frac{1}{a(x)a(\rho x)} & \dots & \frac{1}{\prod_{j=0}^{j_0-2} a(\rho^j x)} & 0 \\ 0 & \frac{1}{a(\rho x)} & \dots & \frac{1}{\prod_{j=1}^{j_0-2} a(\rho^j x)} & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{a(\rho^{j_0-2}x)} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Finally, let

$$B_1(x) := \frac{1}{j_0} \sum_{k=0}^{j_0-1} M_k^{-1} B_0(\rho^k x) M_k,$$

then $B_1(x)$ is compatible and satisfies also (21). Consequently,

$$B(x) := B_1(x)A(x) + I = \frac{1}{j_0} \begin{pmatrix} 1 & \frac{1}{a(x)} & \dots & \frac{1}{\prod_{j=0}^{j_0-2} a(\rho^j x)} \\ a(x) & \frac{a(x)}{a(x)} & \dots & \frac{a(x)}{\prod_{j=0}^{j_0-2} a(\rho^j x)} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=0}^{j_0-2} a(\rho^j x) & \frac{\prod_{j=0}^{j_0-2} a(\rho^j x)}{a(x)} & \dots & \frac{\prod_{j=0}^{j_0-2} a(\rho^j x)}{\prod_{j=0}^{j_0-2} a(\rho^j x)} \end{pmatrix}$$

is also compatible and can be used to determine

$$\varphi(x)=\varphi_0(x)=\frac{1}{j_0}\sum_{k=0}^{j_0-1}\frac{\gamma(\rho^k x)}{\prod_{j=0}^{k-1}a(\rho^j x)}, \tag{22}$$

a solution of (L_h).

Summarizing, we proved

Theorem 24. *A necessary and sufficient condition for the existence of non-trivial solutions of (L_h) is given by (1), and the general solution of (L_h) can be determined by (22), where $\gamma(x)$ is an arbitrary series in $\mathbb{C}[\![x]\!]$.*

This way we also gave a second proof for Theorem 12. We only have to replace $\gamma(x)$ by $\gamma(x)/a(x)$ and multiply $\varphi(x)$ by j_0 in order to get $\Gamma(x)$ from Theorem 12. An even stronger result will be proved in Theorem 31.

Turning our attention to (L), we assume that (1) is satisfied. If we replace in (L) the variable x by $\rho x, \rho^2 x, \ldots, \rho^{j_0-1}x$, we obtain a system of j_0 inhomogeneous linear equations

$$A(x)\begin{pmatrix}\varphi_0(x)\\ \vdots\\ \varphi_{j_0-1}(x)\end{pmatrix}=\begin{pmatrix}b(x)\\ \vdots\\ b(\rho^{j_0-1}x)\end{pmatrix}=:\mathbf{b}(x)$$

with $A(x)$ introduced at the beginning of Section 5. A necessary and sufficient condition for the existence of solutions $(\varphi_0(x),\ldots,\varphi_{j_0-1}(x))$ is that the rank of $A(x)$ and the rank of the enlarged matrix $(A(x),\mathbf{b}(x))$ coincide. The matrix $P^{-1}(x)A(x)$ is an upper triangular matrix, where the last row consists of zeroes only. Since

$$P^{-1}(x)\mathbf{b}(x)=\begin{pmatrix}b(x)\\ b(\rho x)\\ \vdots\\ b(\rho^{j_0-2}x)\\ \sum_{k=0}^{j_0-1}\frac{b(\rho^k x)}{\prod_{j=0}^{k}a(\rho^j x)}\end{pmatrix},$$

we derive again that (7) is a necessary and sufficient condition for the existence of solutions of (L).

6 Representation of the solutions by means of determinants

The methods applied in this part lead to a representation of the general solutions of (L_h) and (L) in terms of some determinants. Also the necessary and sufficient conditions for the existence of non-zero solutions of (L_h) or the existence of a solution of (L) can be formulated by means of determinants. We will compare these conditions with the form we found before. (Cf. Section 2 and Section 5.) Moreover, in Theorem 31 we present a generalization of Theorem 12 and Theorem 24. Finally, in Remark 32 another characterization of those $b(x) \in \mathbb{C}[\![x]\!]$ is given, for which (L) has a solution.

For $0 \leq k < j_0$ put

$$(\mathbb{C}[\![x]\!])^{(k)} := \left\{ \gamma(x) \in \mathbb{C}[\![x]\!] \mid \gamma(x) = \sum_{n \equiv k \bmod j_0} \gamma_n x^n \right\},$$

then clearly $(\mathbb{C}[\![x]\!])^{(k)}$ is a subspace of the $\mathbb{C}$-vector space $\mathbb{C}[\![x]\!]$, and

$$\mathbb{C}[\![x]\!] = \bigoplus_{k=0}^{j_0-1} (\mathbb{C}[\![x]\!])^{(k)}$$

is a direct decomposition of $\mathbb{C}[\![x]\!]$. Hence each series $\gamma(x) \in \mathbb{C}[\![x]\!]$ can uniquely be decomposed into

$$\gamma(x) = \sum_{k=0}^{j_0-1} \gamma^{(k)}(x), \qquad \gamma^{(k)}(x) \in (\mathbb{C}[\![x]\!])^{(k)}.$$

The series $\gamma(x)$ belongs to $(\mathbb{C}[\![x]\!])^{(k)}$ if and only if $\gamma(\rho x) = \rho^k \gamma(x)$. Furthermore, if $\gamma(x) \in (\mathbb{C}[\![x]\!])^{(k)}$, then $\gamma(\rho^\ell x) = \rho^{\ell k} \gamma(x)$.

Obviously the following lemma holds.

Lemma 25. *1. If $f(x) \in (\mathbb{C}[\![x]\!])^{(i)}$ and $g(x) \in (\mathbb{C}[\![x]\!])^{(j)}$, then $(fg)(x) \in (\mathbb{C}[\![x]\!])^{(k)}$ for $k \equiv i + j \bmod j_0$ and $k \in \{0, \dots, j_0 - 1\}$.*

2. If $f(x) \in (\mathbb{C}[\![x]\!])^{(i)}$, $g(x) \in (\mathbb{C}[\![x]\!])^{(j)}$ and $(fg)(x)$ is a series of order 0, then $i = j = 0$.

Using this notation (L_h) is equivalent to

$$\sum_{k=0}^{j_0-1} \rho^k \varphi^{(k)}(x) = \left(\sum_{k=0}^{j_0-1} a^{(k)}(x) \right) \left(\sum_{k=0}^{j_0-1} \varphi^{(k)}(x) \right),$$

since $\varphi^{(k)}(\rho x) = \rho^k \varphi^{(k)}(x)$. This yields the system of linear equations

$$\begin{array}{ccccccc}
a^{(0)}(x)\varphi^{(0)}(x) & + & a^{(j_0-1)}(x)\varphi^{(1)}(x) & + \dots + & a^{(1)}(x)\varphi^{(j_0-1)}(x) & = & \varphi^{(0)}(x) \\
a^{(1)}(x)\varphi^{(0)}(x) & + & a^{(0)}(x)\varphi^{(1)}(x) & + \dots + & a^{(2)}(x)\varphi^{(j_0-1)}(x) & = & \rho\varphi^{(1)}(x) \\
 & & & \dots & & & \\
a^{(j_0-1)}(x)\varphi^{(0)}(x) & + & a^{(j_0-2)}(x)\varphi^{(1)}(x) & + \dots + & a^{(0)}(x)\varphi^{(j_0-1)}(x) & = & \rho^{j_0-1}\varphi^{(j_0-1)}(x)
\end{array}$$

for the unknown functions $\varphi^{(k)}(x) \in (\mathbb{C}[\![x]\!])^{(k)}$. It can be written as a homogeneous system of linear equations

$$A(x) \begin{pmatrix} \varphi^{(0)}(x) \\ \vdots \\ \varphi^{(j_0-1)}(x) \end{pmatrix} = 0 \tag{23}$$

with coefficient matrix

$$A(x) = \begin{pmatrix} a^{(0)}(x) - 1 & a^{(j_0-1)}(x) & \dots & a^{(1)}(x) \\ a^{(1)}(x) & a^{(0)}(x) - \rho & \dots & a^{(2)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a^{(j_0-1)}(x) & a^{(j_0-2)}(x) & \dots & a^{(0)}(x) - \rho^{j_0-1} \end{pmatrix}. \tag{24}$$

It is convenient to write $A(x)$ also in the form $A(x) = (\alpha_{ij})_{0\leq i,j<j_0}$, where

$$\alpha_{ij} = a^{(k)}(x) - \rho^i \delta_{ij}, \qquad k \equiv i - j \bmod j_0$$

and δ_{ij} is the Kronecker δ-function. Then a necessary condition for the existence of non-trivial solutions of (L_h) is $\det A(x) = 0$.

Proposition 26. *If (L_h) has non-trivial solutions, then the rank of the matrix $A(x)$ is equal to $j_0 - 1$.*

Proof. Since $\det A(x) = 0$, the rank of $A(x)$ is smaller than j_0. Moreover, we know that the coefficient a_0 satisfies $a_0^{j_0} = 1$, thus there exists exactly one integer $k_0 \in \{0, \dots, j_0 - 1\}$ such that $a_0 = \rho^{k_0}$. Deleting the $(k_0 + 1)$-th row and $(k_0 + 1)$-th column of $A(x)$ we get the matrix $A'(x)$, given by

$$\begin{pmatrix} a^{(0)}(x) - 1 & a^{(j_0-1)}(x) & \dots & a^{(j_0-k_0+1)}(x) & a^{(j_0-k_0-1)}(x) & \dots & a^{(1)}(x) \\ a^{(1)}(x) & a^{(0)}(x) - \rho & \dots & a^{(j_0-k_0+2)}(x) & a^{(j_0-k_0)}(x) & \dots & a^{(2)}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ a^{(k_0-1)}(x) & a^{(k_0-2)}(x) & \dots & a^{(0)}(x) - \rho^{k_0-1} & a^{(j_0-2)}(x) & \dots & a^{(k_0)}(x) \\ a^{(k_0+1)}(x) & a^{(k_0)}(x) & \dots & a^{(2)}(x) & a^{(0)}(x) - \rho^{k_0+1} & \dots & a^{(k_0+2)}(x) \\ \vdots & \vdots & \dots & \vdots & \vdots & \ddots & \vdots \\ a^{(j_0-1)}(x) & a^{(j_0-2)}(x) & \dots & a^{(j_0-k_0)}(x) & a^{(j_0-k_0-2)}(x) & \dots & a^{(0)}(x) - \rho^{j_0-1} \end{pmatrix}.$$

Then

$$\det A'(x) = \sum_{\substack{\pi \in S_{j_0} \\ \pi(k_0)=k_0}} \operatorname{sgn} \pi \prod_{\substack{i=0 \\ i \neq k_0}}^{j_0-1} \alpha_{i\pi(i)}$$

where S_{j_0} is the group of all permutations of $\{0, 1, \dots, j_0 - 1\}$, which is isomorphic to the symmetric group of degree j_0. According to Lemma 25 each summand of $\det A'(x)$ belongs to $(\mathbb{C}\,[\![x]\!])^{(k)}$ for

$$k \equiv \sum_{\substack{j=0 \\ j \neq k_0}}^{j_0-1} (j - \pi(j)) \equiv 0 \bmod j_0,$$

and the only summand which could be of order 0 is the summand for $\pi = \mathrm{id}$. This summand indeed yields the non-zero constant term

$$(a_0 - 1) \cdot \ldots \cdot (a_0 - \rho^{k_0-1}) \cdot (a_0 - \rho^{k_0+1}) \cdot \ldots \cdot (a_0 - \rho^{j_0-1}).$$

Hence, $\det A'(x) \neq 0$ and $A'(x)$ is of rank $j_0 - 1$. Since $A'(x)$ is a submatrix of $A(x)$, the rank of $A(x)$ is at least $j_0 - 1$, actually it is $j_0 - 1$. $\square$

Solving (23) is equivalent to solving the inhomogeneous system of linear equations

$$A'(x)\begin{pmatrix} \varphi^{(0)}(x) \\ \vdots \\ \varphi^{(k_0-1)}(x) \\ \varphi^{(k_0+1)}(x) \\ \vdots \\ \varphi^{(j_0-1)}(x) \end{pmatrix} = \begin{pmatrix} -a^{(j_0-k_0)}(x)\varphi^{(k_0)}(x) \\ \vdots \\ -a^{(j_0-1)}(x)\varphi^{(k_0)}(x) \\ -a^{(1)}(x)\varphi^{(k_0)}(x) \\ \vdots \\ -a^{(j_0-k_0-1)}(x)\varphi^{(k_0)}(x) \end{pmatrix}. \tag{25}$$

For each choice of $\varphi^{(k_0)}(x)$ in $\mathbb{C}[\![x]\!]$ there exists exactly one solution $(\zeta_0, \ldots, \zeta_{k_0-1}, \zeta_{k_0+1}, \ldots, \zeta_{j_0-1})$ of (25). In general these ζ_j are Laurent series.

Lemma 27. *If $\varphi^{(k_0)}(x) \in (\mathbb{C}[\![x]\!])^{(k_0)}$, then each ζ_k is a formal power series belonging to $(\mathbb{C}[\![x]\!])^{(k)}$ for $k \in \{0, \ldots, k_0-1, k_0+1, \ldots, j_0-1\}$.*

Proof. Applying Cramer's rule ζ_k can be computed as

$$\zeta_k = \frac{\det A'_{[k]}(x)}{\det A'(x)}, \tag{26}$$

where $A'_{[k]}(x)$ is constructed from $A'(x)$ by replacing one column of $A'(x)$ by the right hand side of (25) in an obvious way. We already know from the proof of Proposition 26 that $\operatorname{ord}\det A'(x) = 0$, whence $1/\det A'(x)$ is a formal power series, and $\det A'(x) \in (\mathbb{C}[\![x]\!])^{(0)}$, thus also $1/\det A'(x) \in (\mathbb{C}[\![x]\!])^{(0)}$. This implies that all $\zeta_k \in \mathbb{C}[\![x]\!]$. The components $\alpha_{ij}^{[k]}$ of $A'_{[k]}(x)$ for $i, j \in \{0, \ldots, k_0-1, k_0+1, \ldots, j_0-1\}$ are

$$\alpha_{ij}^{[k]} = \begin{cases} \alpha_{ij}, & \text{if } j \neq k \\ -a^{(i-k_0)}(x)\varphi^{(k_0)}(x), & \text{if } j = k \text{ and } i > k_0 \\ -a^{(i+j_0-k_0)}(x)\varphi^{(k_0)}(x), & \text{if } j = k \text{ and } i < k_0, \end{cases}$$

hence

$$\det A'_{[k]}(x) = \sum_{\substack{\pi \in S_{j_0} \\ \pi(k_0)=k_0}} \operatorname{sgn}\pi \prod_{\substack{i=0 \\ i \neq k_0}}^{j_0-1} \alpha_{i\pi(i)}^{[k]}.$$

In each summand there is exactly one term, actually the term $\alpha_{\pi^{-1}(k)k}^{[k]}$, which does not belong to $A'(x)$. Since this term belongs to $(\mathbb{C}[\![x]\!])^{(\pi^{-1}(k))}$, it follows from Lemma 25 that $\prod_{\substack{i=0 \\ i\neq k_0}}^{j_0-1} \alpha_{i\pi(i)}^{[k]}$ belongs to $(\mathbb{C}[\![x]\!])^{(\ell)}$ for

$$\ell \equiv \sum_{\substack{i=0 \\ i\neq k_0,\ i\neq\pi^{-1}(k)}}^{j_0-1} (i-\pi(i)) + \pi^{-1}(k) = \sum_{\substack{i=0 \\ i\neq k_0}}^{j_0-1}(i-\pi(i)) - (\pi^{-1}(k)-k) + \pi^{-1}(k) \equiv k \bmod j_0.$$

Consequently $\det A'_{[k]}(x) \in (\mathbb{C}[\![x]\!])^{(k)}$ and finally $\zeta_k \in (\mathbb{C}[\![x]\!])^{(k)}$ which implies that ζ_k is of the form $\varphi^{(k)}(x)$ for $\varphi(x) \in \mathbb{C}[\![x]\!]$. □

Summarizing, we obtain a representation of the general solution of (L_h) by means of determinants.

Theorem 28. *The homogeneous linear functional equation (L_h) has a non-trivial solution if and only if $\det A(x) = 0$, where $A(x)$ is given in (24). In this case, the rank of $A(x)$ equals $j_0 - 1$, and $a_0 = \rho^{k_0}$ for some $k_0 \in \{0, \dots, j_0 - 1\}$. The matrix $A'(x)$ constructed from $A(x)$ by deleting the k_0-th row and column is of full rank. For arbitrary $\varphi^{(k_0)}(x) \in (\mathbb{C}[\![x]\!])^{(k_0)}$ the unique solution of (25) given by (26) yields a solution $\varphi(x) = \sum_{\ell=0}^{j_0-1} \varphi^{(\ell)}(x)$ of (L_h).*

Proof. If (L_h) has a non-trivial solution everything was shown in Proposition 26 and Lemma 27. If $\det A(x) = 0$, then there exists a $k_0 \in \{0, \dots, j_0 - 1\}$ such that $a_0 = \rho^{k_0}$. Assume in contrary that $a_0 \neq \rho^k$ for $k \in \mathbb{Z}$, then similarly as in the proof of Proposition 26 we get that

$$\det A(x) = \sum_{\pi \in S_{j_0}} \operatorname{sgn} \pi \prod_{i=0}^{j_0-1} \alpha_{i\pi(i)}$$

belongs to $(\mathbb{C}[\![x]\!])^{(0)}$ and the only summand which could be of order 0 is the summand for $\pi = \mathrm{id}$. This summand however yields the non-zero constant term

$$\prod_{k=0}^{j_0-1} (a_0 - \rho^k)$$

in contradiction to $\det A(x) = 0$. Hence, also in this situation Proposition 26 and Lemma 27 can be applied and the proof is finished. □

We now want to compare the necessary and sufficient condition (1) for the existence of a non-trivial solution of (L_h) with the necessary and sufficient condition $\det A(x) = 0$ of Theorem 28. Using the decomposition $a(x) = \sum_{k=0}^{j_0-1} a^{(k)}(x)$ with $a^{(k)}(x) \in (\mathbb{C}[\![x]\!])^{(k)}$, we get from (1) the identity

$$0 = \prod_{\ell=0}^{j_0-1} a(\rho^\ell x) - 1 = \prod_{\ell=0}^{j_0-1} \sum_{k=0}^{j_0-1} \rho^{\ell k} a^{(k)}(x) =$$

$$[a^{(0)}(x)]^{j_0} + \sum_{r=0}^{j_0-1} P_r(a^{(1)}(x), \dots, a^{(j_0-1)}(x))[a^{(0)}(x)]^r - 1,$$

where each P_r is a polynomial over $\mathbb{Q}(\rho)$. We know from Proposition 16 that for given $a^{(k)}(x) \in (\mathbb{C}[\![x]\!])^{(k)}$ for $k = 1, \dots, j_0 - 1$ the equation

$$[a^{(0)}(x)]^{j_0} + \sum_{r=0}^{j_0-1} P_r(a^{(1)}(x), \dots, a^{(j_0-1)}(x))[a^{(0)}(x)]^r - 1 = 0 \tag{27}$$

has a unique solution $a^{(0)}(x) \in (\mathbb{C}\llbracket x \rrbracket)^{(0)}$ with $a^{(0)}(x) \equiv 1 \bmod x$.

For each solution $\tilde{a}(x)$ of (1) with $\tilde{a}(x) \equiv 1 \bmod x$ and each ρ^m for $0 \le m < j_0$, also $\rho^m \tilde{a}(x)$ is a solution of (1) with $\rho^m \tilde{a}(x) \equiv \rho^m \bmod x$, with the decomposition

$$\rho^m \tilde{a}(x) = \sum_{k=0}^{j_0-1} \rho^m \tilde{a}^{(k)}(x).$$

For given $a^{(1)}(x), \dots, a^{(j_0-1)}(x)$ we define for $k = 1, \dots, j_0 - 1$

$$\tilde{a}^{(k),m}(x) := \rho^{-m} a^{(k)}(x).$$

Then, by our remark above, the equation

$$X^{j_0} + \sum_{r=0}^{j_0-1} P_r(\tilde{a}^{(1),m}(x), \dots, \tilde{a}^{(j_0-1),m}(x)) X^r - 1 = 0$$

has a unique solution $\tilde{a}^{(0),m}(x) \in (\mathbb{C}\llbracket x \rrbracket)^{(0)}$ with $\tilde{a}^{(0),m}(x) \equiv 1 \bmod x$. Hence

$$\alpha_m(x) := \rho^m \left(\sum_{k=0}^{j_0-1} \tilde{a}^{(k),m}(x) \right) = \rho^m \tilde{a}^{(0),m}(x) + \sum_{k=1}^{j_0-1} a^{(k)}(x)$$

is a solution of (1) with $\alpha_m(x) \equiv \rho^m \bmod x$, and $\rho^m \tilde{a}^{(0),m}(x)$ satisfies (27). Also, $\alpha_m(x) \neq \alpha_n(x)$ for $n \neq m$, $0 \le n, m < j_0$. So we constructed j_0 different solutions of (27) in $(\mathbb{C}\llbracket x \rrbracket)^{(0)}$.

Let us develop the expression $\det A(x)$ according to powers of $a^{(0)}(x)$ with coefficients which are polynomials over $\mathbb{Q}(\rho)$ in $a^{(1)}(x), \dots, a^{(j_0-1)}(x)$. This way we get a monic polynomial of degree j_0. It has the j_0 different zeros α_m for $0 \le m < j_0$, because for $a(x) = \alpha_m(x)$ the equation (L_h) has non-trivial solutions. Hence we get

Theorem 29. *Let $a(x) \in \mathbb{C}\llbracket x \rrbracket$ denote a series of order 0 and $a(x) = \sum_{k=0}^{j_0-1} a^{(k)}(x)$ with $a^{(k)}(x) \in (\mathbb{C}\llbracket x \rrbracket)^{(k)}$. If $A(x)$ is given by (24), then*

$$\det A(x) = \prod_{\ell=0}^{j_0-1} a(\rho^\ell x) - 1 = \prod_{\ell=0}^{j_0-1} \sum_{k=0}^{j_0-1} \rho^{\ell k} a^{(k)}(x) - 1,$$

which is an identity in $\mathbb{C}\llbracket x \rrbracket$.

Now we come back to the inhomogeneous linear functional equation (L), and we assume that (L_h) has non-trivial solutions, i.e. $\det A(x) = 0$, where $A(x)$ is given by (24). If we also decompose $b(x)$ in the form $\sum_{k=0}^{j_0-1} b^{(k)}(x)$ with $b^{(k)}(x) \in (\mathbb{C}\llbracket x \rrbracket)^{(k)}$, then (L) has a solution if and only if the inhomogeneous system of linear equations

$$A(x) \begin{pmatrix} \varphi^{(0)}(x) \\ \vdots \\ \varphi^{(j_0-1)}(x) \end{pmatrix} = \begin{pmatrix} b^{(0)}(x) \\ \vdots \\ b^{(j_0-1)}(x) \end{pmatrix} =: \mathbf{b}(x) \tag{28}$$

has a solution with $\varphi^{(k)}(x) \in (\mathbb{C}[\![x]\!])^{(k)}$ for $k = 0, \ldots, j_0 - 1$. This system has a solution without any extra condition on $\varphi^{(k)}(x)$ if and only if the rank of $A(x)$ coincides with the rank of the enlarged matrix $(A(x), \mathbf{b}(x))$. If $A_{(k)}(x)$ denotes the matrix derived from $A(x)$ by deleting the $(k+1)$-th column, then (28) has a solution without any extra condition on $\varphi^{(k)}(x)$ if and only if $\det(A_{(k)}(x), \mathbf{b}(x)) = 0$ for $k = 0, \ldots, j_0 - 1$.

Theorem 30. *Assume that* $\det A(x) = 0$ *and that* $a(x) \equiv \rho^{k_0} \bmod x$ *for some* $k_0 \in \{0, \ldots, j_0 - 1\}$. *Then* (L) *has a solution if and only if* $\det(A_{(k_0)}(x), \mathbf{b}(x)) = 0$.

Proof. If (L) has a solution, then the assertion is obvious. Now we assume that $\det(A_{(k_0)}(x), \mathbf{b}(x)) = 0$, which implies that $\mathbf{b}(x)$ is linearly dependent on the column vectors of $A_{(k_0)}(x)$. From the proof of Proposition 26 we already know that $A_{(k_0)}$ is a matrix of rank $j_0 - 1$ and that all the column vectors of $A(x)$ are linear combinations of the column vectors of $A_{(k_0)}(x)$. Hence the rank of $(A(x), \mathbf{b}(x))$ equals the rank of $A(x)$, which is $j_0 - 1$. Consequently (28) has a solution. Also from the proof of Proposition 26 we know that the (k_0+1)-th row of $A(x)$, thus also of $(A(x), \mathbf{b}(x))$, is linearly dependent on the other rows, so that it can be omitted. Finally (28) can be rewritten in the form

$$A'(x) \begin{pmatrix} \varphi^{(0)}(x) \\ \vdots \\ \varphi^{(k_0-1)}(x) \\ \varphi^{(k_0+1)}(x) \\ \vdots \\ \varphi^{(j_0-1)}(x) \end{pmatrix} = \begin{pmatrix} b^{(0)}(x) \\ \vdots \\ b^{(k_0-1)}(x) \\ b^{(k_0+1)}(x) \\ \vdots \\ b^{(j_0-1)}(x) \end{pmatrix} + \begin{pmatrix} -a^{(j_0-k_0)}(x)\varphi^{(k_0)}(x) \\ \vdots \\ -a^{(j_0-1)}(x)\varphi^{(k_0)}(x) \\ -a^{(1)}(x)\varphi^{(k_0)}(x) \\ \vdots \\ -a^{(j_0-k_0-1)}(x)\varphi^{(k_0)}(x) \end{pmatrix}, \tag{29}$$

where $\varphi^{(k_0)}(x)$ can be chosen arbitrarily in $(\mathbb{C}[\![x]\!])^{(k_0)}$ and $A'(x)$ is defined in the proof of Proposition 26. Again we apply Cramer's rule in order to solve this system of equations. Similar to the proof of Lemma 27 we derive that the components $\varphi^{(k)}(x)$ of the unique solution of (29) belong to $(\mathbb{C}[\![x]\!])^{(k)}$ for $k \in \{0, \ldots, k_0 - 1, k_0 + 1, \ldots, j_0 - 1\}$, since the components $b^{(k)}(x) - a^{(k-k_0)}(x)\varphi^{(k_0)}(x)$ of the right hand side of (29) belong to $(\mathbb{C}[\![x]\!])^{(k)}$. □

From the representation of the general solution of (L_h) under the assumption (1) given in Theorem 12 or Theorem 24, it is easy to obtain still another form, which is very close to (2) or (2′), but not identical. Let us restrict to the case that $a(x) = 1 + a_1 x + \ldots$, and consider the decomposition

$$\gamma(x) = \sum_{\ell=0}^{j_0-1} \gamma^{(\ell)}(x)$$

with $\gamma^{(\ell)}(x) \in (\mathbb{C}[\![x]\!])^{(\ell)}$. Then we easily find that

$$\Gamma(x) = \sum_{k=0}^{j_0-1} \frac{\gamma(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} = \sum_{k=0}^{j_0-1} \sum_{\ell=0}^{j_0-1} \frac{\gamma^{(\ell)}(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)} = \sum_{\ell=0}^{j_0-1} \left(\sum_{k=0}^{j_0-1} \frac{\rho^{k\ell}}{\prod_{j=0}^{k} a(\rho^j x)} \right) \gamma^{(\ell)}(x) =$$

$$\left(\sum_{k=0}^{j_0-1} \frac{1}{\prod_{j=0}^{k} a(\rho^j x)}\right) \gamma^{(0)}(x) + \sum_{\ell=1}^{j_0-1} \left(\sum_{k=0}^{j_0-1} \frac{\rho^{k\ell}}{\prod_{j=0}^{k} a(\rho^j x)}\right) \gamma^{(\ell)}(x).$$

It follows from Theorem 12 that for each $\gamma^{(0)}(x) \in (\mathbb{C}\,[\![x]\!])^{(0)}$

$$\left(\sum_{k=0}^{j_0-1} \frac{1}{\prod_{j=0}^{k} a(\rho^j x)}\right) \gamma^{(0)}(x) \tag{30}$$

is a solution of (L_h). We claim

Theorem 31. *If (1) holds, then the general solution of* (L_h) *is given by (30) with* $\gamma^{(0)}(x) \in (\mathbb{C}\,[\![x]\!])^{(0)}$.

Proof. By Theorem 5 we know that to each $\gamma^{(0)}(x) \in (\mathbb{C}\,[\![x]\!])^{(0)}$ there exists a series $\sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0}$ such that

$$\left(\sum_{k=0}^{j_0-1} \frac{1}{\prod_{j=0}^{k} a(\rho^j x)}\right) \gamma^{(0)}(x) = \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x)\right]^{-1} j_0 \sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0}. \tag{31}$$

Hence, it is easily seen that

$$\left(\sum_{k=0}^{j_0-1} \frac{1}{\prod_{j=0}^{k} a(\rho^j x)}\right) \left(\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x)\right) \in (\mathbb{C}\,[\![x]\!])^{(0)}, \tag{32}$$

and that its reciprocal also belongs to $(\mathbb{C}\,[\![x]\!])^{(0)}$. If now $\sum_{t\geq 0} \varphi^*_{tj_0} x^{tj_0}$ is given, which determines by $(2')$ the general solution of (L_h), we determine $\gamma^{(0)}(x)$ from (31), and consequently (30) can be any solution of (L_h). □

Similar representations like (30) hold in each case $a(x) = \rho^k + a_1 x + \ldots$, and again (cf. Theorem 9), if the coefficients $a_1, \ldots, a_{j_0-1}$ of $a(x)$ are algebraically independent over $\mathbb{Q}$, then we get simultaneously j_0 representations for the general solution of (L_h) in the form

$$\left(\sum_{k=0}^{j_0-1} \frac{\rho^{k\ell}}{\prod_{j=0}^{k} a(\rho^j x)}\right) \gamma^{(\ell)}(x)$$

for $\ell = 0, \ldots, j_0 - 1$.

If we denote

$$\left(\sum_{k=0}^{j_0-1} \frac{1}{\prod_{j=0}^{k} a(\rho^j x)}\right) \left(\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x)\right)$$

by $\Phi(x)$, then it follows from (32) that $\Phi(\rho x) = \Phi(x)$. This together with (3) implies that

$$\sum_{k=0}^{j_0-1} \frac{1}{\prod_{j=0}^{k} a(\rho^j \rho x)} = a(x) \sum_{k=0}^{j_0-1} \frac{1}{\prod_{j=0}^{k} a(\rho^j x)}.$$

Moreover, it is easy to prove that

$$\Phi(x) = j_0 + \sum_{k=0}^{j_0-1} \sum_{\ell=1}^{j_0-1} \prod_{j=1}^{\ell} a(\rho^{k+j}x).$$

In Proposition 20 we already characterized those $b(x) \in \mathbb{C}\llbracket x \rrbracket$ for which (L) has a solution. Using the decomposition of $\mathbb{C}\llbracket x \rrbracket$ into subspaces $(\mathbb{C}\llbracket x \rrbracket)^{(k)}$, the following gives another method for solving the functional equation (7), i.e. of characterizing those series $b(x)$, for which (L) has a solution.

Remark 32. Assume that $a(x)$ is a solution of (1) with $a(x) \equiv 1 \bmod x$. Now we decompose $\mathbb{C}\llbracket x \rrbracket$ in the form

$$\mathbb{C}\llbracket x \rrbracket = (\mathbb{C}\llbracket x \rrbracket)^{(0)} \oplus \left(\bigoplus_{k=1}^{j_0-1} (\mathbb{C}\llbracket x \rrbracket)^{(k)} \right).$$

By $\Phi = \Phi_a$ we denote the $\mathbb{C}$-linear mapping

$$\gamma(x) \mapsto \Phi(\gamma)(x) := \gamma(\rho x) - a(x)\gamma(x)$$

from $\mathbb{C}\llbracket x \rrbracket$ to $\mathbb{C}\llbracket x \rrbracket$, associated with (L_h). Clearly $\varphi(x) \in \operatorname{Ker}\Phi$ if and only if $\varphi(x)$ is a solution of (L_h). From Theorem 5 we know that

$$\operatorname{Ker}\Phi = \underbrace{j_0 \left[\sum_{n=0}^{j_0-1} \prod_{\ell=0}^{n-1} a(\rho^\ell x) \right]^{-1}}_{=j_0[A_0(x)]^{-1}=:A(x)} (\mathbb{C}\llbracket x \rrbracket)^{(0)}.$$

Since, under our hypotheses, the $\mathbb{C}$-linear mapping $\tau \colon \mathbb{C}\llbracket x \rrbracket \to \mathbb{C}\llbracket x \rrbracket$

$$\gamma(x) \mapsto \tau(\gamma)(x) := A(x)\gamma(x)$$

is a vector space automorphism of $\mathbb{C}\llbracket x \rrbracket$, we get the decomposition

$$\mathbb{C}\llbracket x \rrbracket = A(x)(\mathbb{C}\llbracket x \rrbracket)^{(0)} \oplus A(x) \left(\bigoplus_{k=1}^{j_0-1} (\mathbb{C}\llbracket x \rrbracket)^{(k)} \right) = \operatorname{Ker}\Phi \oplus A(x) \left(\bigoplus_{k=1}^{j_0-1} (\mathbb{C}\llbracket x \rrbracket)^{(k)} \right). \quad (33)$$

From this we see immediately that $b(x) \in \operatorname{Im}(\Phi)$ if and only if (L) (written with $a(x)$ and $b(x)$) has a solution in $\mathbb{C}\llbracket x \rrbracket$. Hence, in order to characterize the solutions $b(x)$ of (7), we describe $\operatorname{Im}(\Phi)$. According to (33), each $\gamma(x) \in \mathbb{C}\llbracket x \rrbracket$ has a unique decomposition $\gamma(x) = \gamma_0(x) + \gamma_1(x)$ with $\gamma_0(x) \in \operatorname{Ker}(\Phi)$ and $\gamma_1(x) \in A(x) \left(\bigoplus_{k=1}^{j_0-1} (\mathbb{C}\llbracket x \rrbracket)^{(k)} \right)$. Hence $\Phi(\gamma)(x) = \Phi(\gamma_1)(x)$. With $\gamma_1(x) = A(x)\psi(x)$ for $\psi(x) = \sum_{n \not\equiv 0 \bmod j_0} \psi_n x^n \in \bigoplus_{k=1}^{j_0-1} (\mathbb{C}\llbracket x \rrbracket)^{(k)}$ we calculate by using (3) (which means that $A(\rho x) = a(x)A(x)$)

$$\Phi(\gamma)(x) = \Phi(\gamma_1)(x) = \gamma_1(\rho x) - a(x)\gamma_1(x) = A(\rho x)\psi(\rho x) - a(x)A(x)\psi(x) =$$

$$a(x)A(x)\left(\psi(\rho x)-\psi(x)\right)=a(x)A(x)\sum_{n\not\equiv 0 \bmod j_0}(\rho^n-1)\psi_n x^n.$$

By similar calculations and considerations as before and by using the facts that the product $a(x)A(x)\equiv 1 \bmod x$ and $\rho^n-1\neq 0$ for $n\not\equiv 0 \bmod j_0$ we see that

$$b(x)=\Phi(\gamma)(x)=\sum_{\substack{n\geq 1\\ n\not\equiv 0 \bmod j_0}}\psi_n x^n+\sum_{\substack{n\geq 1\\ n\equiv 0 \bmod j_0}}M_n((a_r)_{r\geq 1},\psi_m \mid m<n,\ m\not\equiv 0 \bmod j_0)x^n,$$

i.e. (12) holds.

7 Analytic solutions

We are now looking for the solutions of (L), (L_h), (1), and (7) which are holomorphic in a neighborhood of $x=0$. In this part we always assume that $a(x)\equiv 1 \bmod x$.

Remark 33. *Assume that $a(x)$ and $b(x)$ are holomorphic in a neighborhood of $x=0$. If $a(x)\equiv 1 \bmod x$, then from $(2')$ or $(8')$ we derive that a solution $\varphi(x)$ of (L_h) or (L), respectively, is convergent in a neighborhood of $x=0$ if and only if the projection $\sum_{t\geq 0}\varphi_{tj_0}x^{tj_0}$ to $(\mathbb{C}\llbracket x\rrbracket)^{(0)}$ is holomorphic.*

Corresponding results can be obtained for projections into $(\mathbb{C}\llbracket x\rrbracket)^{(\ell_0)}$ from $(2,\ell_0)$ and for $a(x)\equiv\rho^{k_0} \bmod x$ from $(2,\ell_0,k_0)$.

More interesting are the solutions of the other two equations: Let us start with the "cyclic" equation (1) which describes the case when (L_h) has a non-zero solution.

Proposition 34. *1. A solution $a(x)$ of (1) is holomorphic at $x=0$ if and only if in the representation (9) of $a(x)$ the series $\sum_{n\not\equiv 0 \bmod j_0}\gamma_n x^n$ is convergent for $|x|<r$, with some $r>0$.*

2. A solution $a(x)$ of (1) is holomorphic at $x=0$ if and only if in the decomposition $a(x)=\sum_{k=0}^{j_0-1}a^{(k)}(x)$ with $a^{(k)}(x)\in(\mathbb{C}\llbracket x\rrbracket)^{(k)}$ the series $a^{(1)}(x),\dots,a^{(j_0-1)}(x)$ are convergent.

Proof. 1. is immediately clear.

2. If $a(x)$ is convergent for $|x|<r$, then it is absolutely convergent for $|x|<r$, and so are the partial series $a^{(k)}(x)$. Assume conversely that for $|x|<r$ the series $a^{(1)}(x),\dots,a^{(j_0-1)}(x)$ are convergent, i.e. holomorphic functions at $x=0$. Then all possible $a^{(0),m}(x)$, $0\leq m<j_0$, for which $a(x)=a^{(0),m}(x)+\sum_{k=1}^{j_0-1}a^{(k)}(x)$ are (formal) solutions of (1), have the form

$$a^{(0),m}(x)=\rho^m\left(1+\sum_{n\equiv 0 \bmod j_0}R_n^{(m)}(a_k \mid k<n,\ k\not\equiv 0 \bmod j_0)x^n\right)$$

for $0 \le m < j_0$, with polynomials $R_n^{(m)}$. They are exactly the solutions of

$$X^{j_0} + \sum_{r=0}^{j_0-1} P_r(a^{(1)}(x), \ldots, a^{(j_0-1)}(x))X^r - 1 = 0 \tag{34}$$

derived from (27). From the latter it follows by the theorem of Puiseux (cf. [12] pages 50 – 55, or [8] pages 98 – 104) that the series $a^{(0),m}(x)$ are all convergent, since the coefficients $P_r(a^{(1)}(x), \ldots, a^{(j_0-1)}(x))$ are convergent power series. Another way of proving this, is to observe that

$$X^{j_0} + \sum_{r=0}^{j_0-1} P_r(0, \ldots, 0)X^r - 1 = 0$$

(derived from (34) by setting $a^{(1)}(x) = \ldots = a^{(j_0-1)}(x) = 0$) has exactly j_0 different solutions 1, ρ, …, ρ^{j_0-1}. Therefore, by the implicit function theorem, there exist exactly j_0 different at $x = 0$ holomorphic solutions $w^{[m]}(x)$ of (34) such that $w^{[m]}(0) = \rho^m$ for $0 \le m < j_0$. These solutions have Taylor expansions

$$w^{[m]}(x) = \rho^m + \sum_{n \ge 1} w_n^{[m]} x^n,$$

which, considered as formal series, are also j_0 distinct solutions of (34) and hence of (1) in the field of formal Laurent series. Hence they must coincide with the solutions constructed before, i.e.

$$a^{(0),m}(x) = w^{[m]}(x) \in \mathbb{C}\,[\![x]\!].$$

Consequently, each $a^{(0),m}(x)$ is convergent, and so each $a(x)$ of the form

$$a^{(0),m}(x) + \sum_{k=1}^{j_0-1} a^{(k)}(x)$$

for $0 \le m < j_0$ is convergent. $\square$

Now we turn to (7) characterizing the existence of solutions of (L). We assume here that $a(x) \equiv 1 \bmod x$ is a solution of (1) holomorphic at $x = 0$.

Theorem 35. *Under the previous assumptions the solution $b(x)$ of (7) is convergent if and only if $b^{(1)}(x), \ldots, b^{(j_0-1)}(x)$ are convergent power series.*

Proof. Replacing $b(x)$ in (7) by its decomposition $\sum_{k=0}^{j_0-1} b^{(k)}(x)$ we get

$$0 = \sum_{\ell=0}^{j_0-1} \frac{b(\rho^\ell x)}{\prod_{j=0}^{\ell} a(\rho^j x)} = \sum_{\ell=0}^{j_0-1} \frac{\sum_{k=0}^{j_0-1} \rho^{\ell k} b^{(k)}(x)}{\prod_{j=0}^{\ell} a(\rho^j x)} = \sum_{k=0}^{j_0-1} \left(\sum_{\ell=0}^{j_0-1} \frac{\rho^{\ell k}}{\prod_{j=0}^{\ell} a(\rho^j x)} \right) b^{(k)}(x) =$$

$$\left(\sum_{\ell=0}^{j_0-1}\frac{1}{\prod_{j=0}^{\ell}a(\rho^j x)}\right)b^{(0)}(x)+\sum_{k=1}^{j_0-1}\left(\sum_{\ell=0}^{j_0-1}\frac{\rho^{\ell k}}{\prod_{j=0}^{\ell}a(\rho^j x)}\right)b^{(k)}(x).$$

Since the coefficient $a_0 = 1$, the reciprocal of $a(\rho^j x)$ exists for $j = 0, \ldots, j_0 - 1$ in $\mathbb{C}[\![x]\!]$. These series $1/a(\rho^j x)$ are convergent in a neighborhood of $x = 0$, hence also

$$\alpha(x) := \sum_{\ell=0}^{j_0-1}\frac{1}{\prod_{j=0}^{\ell}a(\rho^j x)}$$

is a convergent series. Moreover, the reciprocal of $\alpha(x)$ belongs to $\mathbb{C}[\![x]\!]$, since $\alpha(x) \equiv j_0 \bmod x$, and it is also a convergent power series. □

Remark 36. Another proof of Theorem 35 can be derived from Theorem 30. Here we assume more generally that $a(x)$ is holomorphic at $x = 0$ and $a(x) \equiv \rho^{k_0} \bmod x$ for some $k_0 \in \{0, \ldots, j_0 - 1\}$, and we prove that a solution $b(x)$ of (7) is convergent if and only if $b^{(k)}(x)$ is convergent for all $k \neq k_0$. Expanding $\det(A_{(k_0)}(x), \mathbf{b}(x)) = 0$ with respect to the last column, we derive that

$$\pm\left(\det A'(x)b^{(k_0)}(x)+\sum_{\substack{k=0\\k\neq k_0}}^{j_0-1}P_k(a^{(0)}(x),\ldots,a^{(j_0-1)}(x),\rho)b^{(k)}(x)\right)=0,$$

where P_k are polynomials in $a^{(0)}(x)$, …, $a^{(j_0-1)}(x)$ and ρ. Hence

$$b^{(k_0)}(x) = -[\det A'(x)]^{-1}\sum_{\substack{k=0\\k\neq k_0}}^{j_0-1}P_k(a^{(0)}(x),\ldots,a^{(j_0-1)}(x),\rho)b^{(k)}(x)$$

is convergent, since the reciprocal of $\det A'(x)$ is a convergent power series.

8 The general linear functional equation $(\boldsymbol{Lp})$

Let $S(x) = x + s_2x^2 + \ldots$ be a formal power series as in Theorem 1 such that $p(x) = S^{-1}(\rho S(x))$. The proof of the next lemma is straight forward, hence it is omitted.

Lemma 37. *For all $k \in \mathbb{N}$ the equality*

$$S^{-1}(\rho^k S(x)) = p^k(x)$$

holds.

All formal power series $\gamma(x)$ belonging to $(\mathbb{C}[\![x]\!])^{(k)}$ for some $k \in \mathbb{Z}$ satisfy $\gamma(\rho x) = \rho^k\gamma(x)$. In the general situation of substituting $p(x)$ instead of ρx we derive

Lemma 38. *Let $\gamma(x) \in \mathbb{C}[\![x]\!]$. If k denotes an integer such that $0 \leq k < j_0$, then $\gamma(p(x)) = \rho^k \gamma(x)$ if and only if*

$$\gamma(x) = \sum_{t \geq 0} \tilde{\gamma}_{k+tj_0} [S(x)]^{k+tj_0}, \tag{35}$$

where $\tilde{\gamma}_n$ are the coefficients of the series $\tilde{\gamma} := \gamma \circ S^{-1}$.

Proof. From $\gamma(p(x)) = \rho^k \gamma(x)$ it follows that $\gamma(S^{-1}(\rho S(x)) = \rho^k \gamma(x)$. When putting $y = S(x)$ and $\tilde{\gamma} = \gamma \circ S^{-1}$ we get $\tilde{\gamma}(\rho y) = \rho^k \tilde{\gamma}(y)$, which is equivalent to $\tilde{\gamma}(y) = \sum_{t \geq 0} \tilde{\gamma}_{k+tj_0} y^{k+tj_0}$. Hence, $(\gamma \circ S^{-1})(S(x)) = \sum_{t \geq 0} \tilde{\gamma}_{k+tj_0} [S(x)]^{k+tj_0}$ which is the same as (35). □

The necessary and sufficient condition for the existence of non-trivial solutions of the homogeneous linear functional equation and its general solution are described in

Theorem 39. *The homogeneous linear functional equation*

$$\varphi(p(x)) = a(x)\varphi(x) \tag{Lp_h}$$

has non-trivial solutions if and only if

$$\prod_{\ell=0}^{j_0-1} a(p^\ell(x)) = 1. \tag{36}$$

If $a_0 = \rho^{k_0}$ and if (36) is satisfied, then the general solution $\varphi(x)$ of (Lp_h) is given by

$$\varphi(x) = \left[\sum_{n=0}^{j_0-1} \rho^{-nk_0} \prod_{\ell=0}^{n-1} a(p^\ell(x)) \right]^{-1} \gamma(x), \tag{37}$$

where $\gamma(x)$ is an arbitrary solution of $\gamma(p(x)) = \rho^{k_0}\gamma(x)$, hence of the form (35).

Proof. From Theorem 1 it follows that (Lp_h) has non-trivial solutions if and only if

$$\tilde{\varphi}(\rho y) = \tilde{a}(y)\tilde{\varphi}(y)$$

has non-trivial solutions, where $\tilde{\varphi} = \varphi \circ S^{-1}$ and $y = S(x)$. According to (1) the necessary and sufficient condition for the existence of non-trivial solutions $\tilde{\varphi}(y)$ is given by

$$\prod_{\ell=0}^{j_0-1} \tilde{a}(\rho^\ell y) = 1$$

which is the same as

$$\prod_{\ell=0}^{j_0-1} (a \circ S^{-1})(\rho^\ell S(x)) = 1,$$

whence

$$\prod_{\ell=0}^{j_0-1} a(S^{-1}(\rho^\ell S(x)) = 1.$$

Since $S^{-1}(\rho^k S(x)) = p^k(x)$, we derive

$$\prod_{\ell=0}^{j_0-1} a(p^\ell(x)) = 1.$$

The general solution of the transformed homogeneous linear functional equation where $\tilde{a}_0 = \rho^{k_0}$ (which is equivalent to $a_0 = \rho^{k_0}$) is given by Theorem 10 as

$$\tilde{\varphi}(y) = \left[\sum_{n=0}^{j_0-1} \rho^{-nk_0} \prod_{\ell=0}^{n-1} \tilde{a}(\rho^\ell y)\right]^{-1} j_0 \sum_{t\geq 0} \tilde{\varphi}_{tj_0+k_0} y^{tj_0+k_0}.$$

Replacing y by $S(x)$, $\tilde{\varphi}$ by $\varphi \circ S^{-1}$ and $\tilde{a}$ by $a \circ S^{-1}$ we get (37). □

Applying similar methods as in the last proof, it is possible to show that the next theorem holds.

Theorem 40. *Assume that* (Lp_h) *has non-trivial solutions. The inhomogeneous linear functional equation* (Lp) *has solutions if and only if*

$$\sum_{k=0}^{j_0-1} \frac{b(p^k(x))}{\prod_{j=0}^{k} a(p^j(x))} = 0. \tag{38}$$

If $a_0 = \rho^{k_0}$ *and if (36) and (38) are satisfied, then the general solution* $\varphi(x)$ *of* (Lp) *is given by*

$$\varphi(x) = \left[\sum_{n=0}^{j_0-1} \rho^{-nk_0} \prod_{\ell=0}^{n-1} a(p^\ell(x))\right]^{-1} \left(\gamma(x) - \sum_{n=1}^{j_0-1} \rho^{-nk_0} \prod_{\ell=0}^{n-1} a(p^\ell(x)) \sum_{k=0}^{n-1} \frac{b(p^k(x))}{\prod_{j=0}^{k} a(p^j(x))}\right) \tag{39}$$

where $\gamma(x)$ *is an arbitrary solution of* $\gamma(p(x)) = \rho^{k_0}\gamma(x)$*, hence of the form (35).*

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