Enumeration of mosaics

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Abstract

Mosaics are orbits of partitions arising from music theoretical investigations. Various theorems from the field of "enumeration under finite group actions" are applied for enumerating mosaics. In other words, it is demonstrated how to enumerate G-orbits of partitions of given size, block-type or stabilizer-type.

Key words: Combinatorics under group actions, applications in music theory, enumeration of partition patterns.

1 Preliminaries

Applying methods from *Pólya Theory* it is possible to enumerate various kinds of musical objects as *intervals, chords, scales, tone-rows, motives* and so on. See for instance [7], [18], [4,5], [16,17]. Usually these results are given for an *n-scale*, which means that there are exactly n tones within one octave. Collecting all tones, which are any number of octaves apart, into a *pitch class*, there are exactly n pitch classes in an n-scale. These pitch classes can be considered as elements of the *residue class group* $(Z_n, +)$ of $\mathbb{Z}$ modulo $n\mathbb{Z}$. The musical operator of *transposing* by one pitch class can be interpreted as a *permutation* of Z_n

$$T: Z_n \to Z_n, \qquad i \mapsto T(i) := i + 1.$$

Inversion at pitch class 0 is the following permutation

$$I: Z_n \to Z_n, \qquad i \mapsto I(i) := -i.$$

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The *group* of all possibilities to transpose is the *cyclic group* $\langle T \rangle = C_n$ of order n. The *permutation group* generated by T and I is the *dihedral group* D_n of order $2n$ (for $n \geq 3$). Sometimes in *twelve tone music* a further operator, the so called *quart circle symmetry*, is used, which is given by

$$Q\colon Z_{12} \to Z_{12}, \qquad i \mapsto Q(i) := 5i.$$

Generalising this concept to n tone music the *affine group*

$$\mathrm{Aff}(1, Z_n) := \{(a,b) \mid a \in Z_n^*,\ b \in Z_n\}$$

(the set of all *unit elements* in the *ring* Z_n is indicated by Z_n^*) acts on Z_n by $(a,b)(i) := ai + b$.

These permutation groups induce *group actions* on the sets of musical objects. (For basic definitions and notions in *enumeration under finite group actions* see [12].) Let me explain these group actions by introducing the so called *mosaics* in Z_n (see chapters 2 and 3 of [1]). In [11] it is stated that the enumeration of mosaics is an open research problem communicated by R. Morris. A *partition* π of Z_n is a collection of subsets of Z_n, such that the empty set is not an element of π and such that for each $i \in Z_n$ there is exactly one $P \in \pi$ with $i \in P$. If π consists of exactly k subsets, then π is called a *partition of size* k. Let Π_n denote the set of all partitions of Z_n, and let $\Pi_{n,k}$ be the set of all partitions of Z_n of size k. A permutation group G of Z_n induces the following group action of G on Π_n:

$$G \times \Pi_n \to \Pi_n, \qquad (g, \pi) \mapsto g\pi := \{gP \mid P \in \pi\},$$

where $gP := \{gi \mid i \in P\}$. This action can be restricted to an action of G on $\Pi_{n,k}$. The G-*orbits* on Π_n are called G-*mosaics*. (This is a slight generalisation of the definition given in [11].) Correspondingly the G-orbits on $\Pi_{n,k}$ are called G-*mosaics of size* k.

It is well known [2,3] how to enumerate G-mosaics (G-orbits of partitions) by identifying them with $G \times S_{\underline{n}}$-orbits on the set of all functions from Z_n to $\underline{n} := \{1, \ldots, n\}$. (The *symmetric group* of the set $\underline{n}$ is denoted by $S_{\underline{n}}$.) Furthermore G-mosaics of size k correspond to $G \times S_{\underline{k}}$-orbits on the set of all *surjective* functions from Z_n to $\underline{k}$. We want to express the number of G-mosaics using the cycle index notion: The *cycle index* of a finite group G acting on a finite set X is the following polynomial $Z(G, X)$ in the indeterminates $x_1, x_2, \ldots, x_{|X|}$ over $\mathbb{Q}$, the set of rationals, defined by

$$Z(G, X) := \frac{1}{|G|} \sum_{g \in G} \prod_{i=1}^{|X|} x_i^{a_i(g)},$$

where $(a_1(g),\ldots,a_{|X|}(g))$ is the cycle type of the permutation induced by the action of $g \in G$ on X. This means that the induced permutation decomposes into $a_i(g)$ disjoint cycles of length i for $i = 1,\ldots,|X|$. Furthermore $Z(G, X \mid x_i = f(i))$ means that the variables x_i in $Z(G, X)$ must be replaced by the expression $f(i)$. Now we can apply a theorem from [2] to compute the number of orbits of all (surjective) functions under the group action of $G \times S_{\underline{n}}$ (or $G \times S_{\underline{k}}$ respectively).

Theorem 1 *For $1 \leq k \leq n$ let*

$$M_k := Z(G, Z_n \mid x_i = \frac{\partial}{\partial x_i}) Z(S_{\underline{k}}, \underline{k} \mid x_i = e^{s_i})\Big|_{x_i=0},$$

where $s_i := i(x_i + x_{2i} + \ldots + x_{i[\frac{n}{i}]})$ and where $[\frac{n}{i}]$ is the greatest integer less than or equal to $\frac{n}{i}$. This cycle index expression indicates that the operators of the first cycle index must be applied to the polynomial given by the substitution into the second cycle index and finally all indeterminates that have not yet vanished must be set to 0. The number of G-mosaics in Z_n is given by M_n, and the number of G-mosaics of size k is given by $M_k - M_{k-1}$, where $M_0 := 0$.

The Cauchy-Frobenius-Lemma *[12] computes M_k as*

$$M_k = \frac{1}{|G|\,|S_{\underline{k}}|} \sum_{(g,\sigma)\in G\times S_{\underline{k}}} \prod_{i=1}^{n} a_1(\sigma^i)^{a_i(g)},$$

where $a_i(g)$ (or $a_i(\sigma)$) are the numbers of i-cycles in the cycle decomposition of g (or σ respectively).

Finally the number of G-mosaics of size k could be derived by the Cauchy-Frobenius-Lemma *[12] as*

$$\frac{1}{|G|\,|S_{\underline{k}}|} \sum_{(g,\sigma)\in G\times S_{\underline{k}}} \sum_{\ell=1}^{c(\sigma)} (-1)^{c(\sigma)-\ell} \sum_{a} \prod_{i=1}^{k} \binom{a_i(\sigma)}{a_i} \prod_{j=1}^{n} \left(\sum_{d|j} d \cdot a_d \right)^{a_j(g)},$$

where the inner sum is taken over the sequences $a = (a_1,\ldots,a_k)$ of nonnegative integers a_i such that $\sum_{i=1}^{k} a_i = \ell$, and where $c(\sigma)$ is the number of all cycles in the cycle decomposition of σ.

In order to apply this Theorem one has to know the cycle indices of G and $S_{\underline{k}}$. The formulae for the cycle indices of C_n, D_n and $S_{\underline{k}}$ are well known. (See [2,12].) The cycle index of $\mathrm{Aff}(1, Z_n)$ is computed in [20]. All these cycle index methods are implemented in SYMMETRICA [19], a computer algebra system devoted to combinatorics and representation theory of the symmetric group and of related groups. Using this program system for twelve tone music the following numbers of C_{12}, D_{12} and $\mathrm{Aff}(1, Z_{12})$-mosaics of size k were computed. (See table 1.)

Table 1
Number of mosaics in twelve tone music.

$G \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	179	7254	51075	115100	110462	52376	13299	1873	147	6	1
D_{12}	1	121	3838	26148	58400	56079	26696	6907	1014	96	6	1
$\mathrm{Aff}(1, Z_{12})$	1	87	2155	13730	30121	28867	13835	3667	571	63	5	1

In conclusion there are 351773 C_{12}-mosaics, 179307 D_{12}-mosaics and 93103 $\mathrm{Aff}(1, Z_{12})$-mosaics in twelve tone music.

2 Enumeration by block-type

If $\pi \in \Pi_n$ consists of λ_i blocks of size i for $i \in \underline{n}$ then π is said to be of *block-type* $\lambda = (\lambda_1, \ldots, \lambda_n)$. From the definition it is obvious that $\sum_i i\lambda_i = n$, which will be indicated by $\lambda \vdash\!\dashv n$. Furthermore it is clear that π is a partition of size $\sum_i \lambda_i$. In this section the number of G-mosaics of type λ (i. e. G-patterns of partitions of block-type λ) will be derived. For doing that let $\bar{\lambda}$ be any partition of type λ. (For instance $\bar{\lambda}$ can be defined such that the blocks of $\bar{\lambda}$ of size 1 are given by $\{1\}$, $\{2\}$, …, $\{\lambda_1\}$, the blocks of $\bar{\lambda}$ of size 2 are given by $\{\lambda_1 + 1, \lambda_1 + 2\}$ $\{\lambda_1 + 3, \lambda_1 + 4\}, \ldots$, $\{\lambda_1 + 2\lambda_2 - 1, \lambda_1 + 2\lambda_2\}$, and so on.) According to [12] the *stabilizer* H_λ of $\bar{\lambda}$ in the symmetric group $S_{\underline{n}}$ (H_λ is the set of all permutations $\sigma \in S_{\underline{n}}$ such that $\sigma\bar{\lambda} = \bar{\lambda}$) is *similar* to the *direct sum*

$$\bigoplus_{i=1}^{n} S_{\underline{\lambda_i}}[S_{\underline{i}}]$$

of *compositions* of symmetric groups, which is a permutation representation of the *direct product*

$$\mathop{\times}_{i=1}^{n} S_{\underline{i}} \wr S_{\underline{\lambda_i}}$$

of *wreath products* of symmetric groups. The wreath product is defined by

$$S_{\underline{i}} \wr S_{\underline{j}} = \Big\{(\psi, \sigma) \mid \sigma \in S_{\underline{j}}, \psi \colon \underline{j} \to S_{\underline{i}}\Big\}$$

together with the multiplication

$$(\psi, \sigma)(\psi', \sigma') = (\psi\psi'_\sigma, \sigma\sigma'),$$

where $\psi\psi'_\sigma(i) = \psi(i)\psi'_\sigma(i)$ and $\psi'_\sigma(i) = \psi'(\sigma^{-1}i)$. The composition $S_{\underline{\lambda_i}}[S_{\underline{i}}]$ is the following permutation representation of $S_{\underline{i}} \wr S_{\underline{\lambda_i}}$ on the set $\underline{i} \times \underline{\lambda_i}$

$$((\psi, \sigma), (r, s)) \mapsto (\psi(\sigma(s))r, \sigma(s)).$$

In other words H_λ is the set of all permutations $\sigma \in S_{\underline{n}}$, which map each block of the partition $\bar{\lambda}$ again onto a block (of the same size) of the partition. It is well known that the cycle index of the composition of two groups can be determined from the cycle indices of the two groups ([12]), so we know how to compute the cycle index of H_λ.

All partitions of Z_n of type λ can be derived in the form $\{f^{-1}(P) \mid P \in \bar{\lambda}\}$, where f runs through the set of all bijective mappings from Z_n to $\underline{n}$. Two such bijections f and f' define the same partition of Z_n, if and only if $f' = f \circ \sigma$, where σ is an element of the stabilizer H_λ of $\bar{\lambda}$. This induces a group action of H_λ on the set of all bijections from Z_n to $\underline{n}$

$$H_\lambda \times \underline{n}_{\text{bij}}^{Z_n} \to \underline{n}_{\text{bij}}^{Z_n}, \qquad (\sigma, f) \mapsto f \circ \sigma^{-1},$$

such that the H_λ-orbits correspond to the partitions of Z_n of type λ.

Since the action of G on Π_n can be restricted to an action of G on Π_λ, the set of all partitions of type λ, we have the following action on the set of all bijections from Z_n to $\underline{n}$:

$$G \times \underline{n}_{\text{bij}}^{Z_n} \to \underline{n}_{\text{bij}}^{Z_n}, \qquad (g, f) \mapsto g \circ f.$$

Two bijections f and f' from Z_n to $\underline{n}$ define G-equivalent partitions of type λ, if and only if there is some $g \in G$ such that $g \circ f$ and f' define the same partition, which implies that there is some $\sigma \in H_\lambda$ such that $g \circ f \circ \sigma^{-1} = f'$. In conclusion we can say that G-mosaics of type λ correspond to $G \times H_\lambda$-orbits of bijections from Z_n to $\underline{n}$ under the following group action:

$$(G \times H_\lambda) \times \underline{n}_{\text{bij}}^{Z_n} \to \underline{n}_{\text{bij}}^{Z_n}, \qquad ((g, \sigma), f) \mapsto g \circ f \circ \sigma^{-1}.$$

It is well known how to enumerate these orbits. When interpreting the bijections from Z_n to $\underline{n}$ as permutations of the n-set $\underline{n}$ then G-mosaics of type λ correspond to *double cosets* ([12]) of the form

$$G \backslash S_{\underline{n}} / H_\lambda.$$

Theorem 2 *The number of G-mosaics of type λ can be derived with the fol-*

lowing formula due to N. G. de Bruijn [2]

$$Z(G, Z_n \mid x_i = \frac{\partial}{\partial x_i}) Z(H_\lambda, \mid x_i = i x_i)|_{x_i=0}.$$

The Cauchy-Frobenius-Lemma *[12] determines the number of orbits of bijections from Z_n to $\underline{n}$ under the action of $G \times H_\lambda$ by*

$$\frac{1}{|G|\,|H_\lambda|} \sum_{\substack{(g,\sigma)\in G\times H_\lambda \\ z(g)=z(\sigma)}} \prod_{i=1}^{n} a_i(\sigma)! i^{a_i(g)},$$

where $z(g)$ and $z(\sigma)$ are the cycle types *of the permutations induced by the actions of g on the set Z_n and of σ on $\underline{n}$ respectively, given in the form $(a_i(g))_{i\in\underline{n}}$ or $(a_i(\sigma))_{i\in\underline{n}}$. In other words we are summing over those pairs (g, σ) such that g and σ determine permutations of the same cycle type.*

The double coset approach leads to an application of the Redfield cap-operator *[12] or to* Read's $N(.*.)$ operator *[14,15], and the number of G-mosaics of type λ is given by*

$$Z(G, Z_n) \cap Z(H_\lambda, \underline{n}) \quad \text{or} \quad N(Z(G, Z_n) * Z(H_\lambda, \underline{n})).$$

Table 2 gives the numbers of D_{12}-mosaics of type λ, (in this table the type $(\lambda_1, \ldots, \lambda_n)$ of a partition is written in the form $(1^{\lambda_1}, 2^{\lambda_2}, \ldots)$, where all terms with $\lambda_i = 0$ are omitted) which were computed by using SYMMETRICA routines for the Redfield-cap operator.

If n is even then a mosaic consisting of two blocks of size $n/2$ corresponds to a *trope* introduced by J. M. Hauer in [9,10]. By applying the *power group enumeration theorem* ([8]) an explicit formula for the number of all orbits of tropes under a group action was determined in [4].

3 Enumeration by stabilizer type

Let $U \leq G$ be a subgroup of G, then a partition $\pi \in \Pi_n$ is called *U-invariant* if $g\pi = \pi$ for all $g \in U$. The set of all U-invariant partitions will be denoted by $(\Pi_n)_U$. The *stabilizer* of a partition π is the subgroup $U := \{g \in G \mid g\pi = \pi\}$ of G. The stabilizers of all partitions in the orbit $G(\pi)$ of π lie in the *conjugacy class* $\widetilde{U}$ of the stabilizer U of π. So the orbit $G(\pi)$ is called of *stabilizer type* $\widetilde{U}$. The set of all orbits of type $\widetilde{U}$ is also called the *U-stratum* and it will be indicated as $\widetilde{U}\backslash\backslash\Pi_n$. The *Lemma of Burnside* provides a formula which

Table 2
Number of D_{12}-mosaics in twelve tone music of type λ.

λ		λ		λ		λ	
(12)	1	$(1,11)$	1	$(2,10)$	6	$(1^2,10)$	6
$(3,9)$	12	$(1,2,9)$	30	$(1^3,9)$	12	$(4,8)$	29
$(1,3,8)$	85	$(2^2,8)$	84	$(1^2,2,8)$	140	$(1^4,8)$	29
$(5,7)$	38	$(1,4,7)$	170	$(2,3,7)$	340	$(1^2,3,7)$	340
$(1,2^2,7)$	510	$(1^3,2,7)$	340	$(1^5,7)$	38	(6^2)	35
$(1,5,6)$	236	$(2,4,6)$	610	$(1^2,4,6)$	610	$(3^2,6)$	424
$(1,2,3,6)$	2320	$(1^3,3,6)$	781	$(2^3,6)$	645	$(1^2,2^2,6)$	1820
$(1^4,2,6)$	610	$(1^6,6)$	50	$(2,5^2)$	386	$(1^2,5^2)$	386
$(3,4,5)$	1170	$(1,2,4,5)$	3480	$(1^3,4,5)$	1170	$(1,3^2,5)$	2330
$(2^2,3,5)$	3510	$(1^2,2,3,5)$	6960	$(1^4,3,5)$	1170	$(1,2^3,5)$	3500
$(1^3,2^2,5)$	3510	$(1^5,2,5)$	708	$(1^7,5)$	38	(4^3)	297
$(1,3,4^2)$	2915	$(2^2,4^2)$	2347	$(1^2,2,4^2)$	4470	$(1^4,4^2)$	792
$(2,3^2,4)$	5890	$(1^2,3^2,4)$	5890	$(1,2^2,3,4)$	17370	$(1^3,2,3,4)$	11580
$(1^5,3,4)$	1170	$(2^4,4)$	2325	$(1^2,2^3,4)$	8860	$(1^4,2^2,4)$	4463
$(1^6,2,4)$	610	$(1^8,4)$	29	(3^4)	713	$(1,2,3^3)$	7740
$(1^3,3^3)$	2610	$(2^3,3^2)$	6005	$(1^2,2^2,3^2)$	17630	$(1^4,2,3^2)$	5890
$(1^6,3^2)$	424	$(1,2^4,3)$	8725	$(1^3,2^3,3)$	11623	$(1^5,2^2,3)$	3510
$(1^7,2,3)$	340	$(1^9,3)$	12	(2^6)	554	$(1^2,2^5)$	2792
$(1^4,2^4)$	2325	$(1^6,2^3)$	645	$(1^8,2^2)$	84	$(1^{10},2)$	6
(1^{12})	1						

allows the numbers of G-orbits of type $\widetilde{U}$ to be computed from the number of V-invariant partitions for $U, V \leq G$. In [12] it is formulated in the following way: Let $\widetilde{U}_1, \ldots, \widetilde{U}_d$ be the conjugacy classes of subgroups of G then

$$\begin{pmatrix} \vdots \\ \left|\widetilde{U}_i \backslash\!\backslash\!\backslash \Pi_n\right| \\ \vdots \end{pmatrix} = B(G) \begin{pmatrix} \vdots \\ |(\Pi_n)_U| \\ \vdots \end{pmatrix},$$

where $B(G) := (b_{ij})_{1\le i,j\le d}$ is the *Burnside matrix* of G given by

$$b_{ij} = \frac{\left|\widetilde{U}_i\right|}{|G/U_i|} \sum_{V\in\widetilde{U}_j} \mu(U_i, V),$$

where μ is the *Moebius function* in the *incidence algebra* over the *subgroup lattice* $L(G)$ of G. So, for computing the number of U-strata, we have to determine the number of V-invariant partitions. In [21] the following formula is proved:

Theorem 3 *Let $\mathcal{T}$ be a system of representatives of the G-orbits on X and let $\mathcal{H}$ be a system of representatives of the conjugacy classes of G (e. g. $\mathcal{H} = \{U_1, \ldots, U_d\}$). Then the number of G-invariant partitions of X is given by*

$$\sum_{\delta\in\Pi_\mathcal{T}} \prod_{A\in\delta} \sum_{U\in\mathcal{H}} \frac{1}{m_U(U)} \prod_{t\in A} m_U(G_t),$$

where $\Pi_\mathcal{T}$ denotes the set of all partitions δ of $\mathcal{T}$. The blocks of the partition δ are indicated as A. For $t \in X$ the stabilizer of t in G is denoted by G_t. Finally for subgroups U, V of G

$$m_U(V) = \frac{|N_G(U)|}{|U|} \sum_{W\in\widetilde{U}} \zeta(V, W)$$

is the mark *of U at V, where ζ is the* zeta-*function in the incidence algebra over $L(G)$.*

In the case that $U \in \widetilde{U}_j$ and $V \in \widetilde{U}_i$ then

$$m_U(V) = m_{ij} := \frac{|N_G(U_j)|}{|U_j|} \sum_{W\in\widetilde{U}_j} \zeta(U_i, W).$$

The matrix $M(G) := (m_{ij})_{1\le i,j\le d}$ is called the *table of marks* of G and it is the inverse of the Burnside matrix $B(G)$.

The conjugacy classes of subgroups of C_n are well known. In [5] it is shown that a system of representatives of the conjugacy classes of subgroups of D_n (for $n \in \mathbb{N}$) is given as a disjoint union

$$\dot{\bigcup_{d|n}} \mathcal{R}(d),$$

where

$$\mathcal{R}(d) := \begin{cases} \{\langle T^d\rangle, \langle T^d, I\rangle, \langle T^d, TI\rangle\} & \text{if } d \equiv 0 \bmod 2 \\ \{\langle T^d\rangle, \langle T^d, I\rangle\} & \text{if } d \equiv 1 \bmod 2. \end{cases}$$

A system of representatives of the conjugacy classes of subgroups of D_{12} together with the numbers of U-invariant partitions and the numbers of D_{12}-orbits of stabilizer type $\widetilde{U}$ is given in the table 3: T stands for the permutation $(0, 1, 2, \ldots, 11)$ and I stands for the permutation $(0)(1, 11)(2, 10)(3, 9)(4, 8)$ $(5, 7)(6)$. In table 4 a list of all the 54 conjugacy classes of subgroups of $\mathrm{Aff}(1, Z_{12})$ together with the number of U-invariant partitions and the number of U-strata is given. (The subgroup lattice of $\mathrm{Aff}(1, Z_{12})$ was computed with GAP [6].) The permutation Q is given by $(0)(1, 5)(2, 10)(3)(4, 8)(6)(7, 11)(9)$.

Table 3
U-invariant partitions and U-strata for D_{12}-mosaics

group	order	$\lvert\widetilde{U}\rvert$	$\lvert(\Pi_{12})_U\rvert$	$\lvert\widetilde{U}\backslash\backslash\backslash\Pi_{12}\rvert$
$U_1 = \langle 1\rangle$	1	1	4213597	172037
$U_2 = \langle T^6\rangle$	2	1	6841	416
$U_3 = \langle I\rangle$	2	6	6841	3227
$U_4 = \langle TI\rangle$	2	6	6841	3242
$U_5 = \langle T^4\rangle$	3	1	268	11
$U_6 = \langle T^3\rangle$	4	1	111	2
$U_7 = \langle T^6, I\rangle$	4	3	349	150
$U_8 = \langle T^6, TI\rangle$	4	3	319	136
$U_9 = \langle T^2\rangle$	6	1	28	0
$U_{10} = \langle T^4, I\rangle$	6	2	56	19
$U_{11} = \langle T^4, TI\rangle$	6	2	54	19
$U_{12} = \langle T^3, I\rangle$	8	3	37	31
$U_{13} = \langle T^2, I\rangle$	12	1	18	6
$U_{14} = \langle T^2, TI\rangle$	12	1	16	5
$U_{15} = \langle T\rangle$	12	1	6	0
$U_{16} = \langle T, I\rangle$	24	1	6	6

The number of all U_1-invariant partitions is just the number of all partitions of $\underline{12}$ which is the *Bell-number* $B(12)$. The Burnside matrix of D_{12} was derived by inverting the table of marks of D_{12} computed with the computer algebra system GAP.

Table 4
U-invariant partitions and U-strata for $\mathrm{Aff}(1, Z_{12})$-mosaics

group	order	$\lvert\widetilde{U}\rvert$	$\lvert(\Pi_{12})_U\rvert$	$\lvert\widetilde{U}\backslash\backslash\backslash\Pi_{12}\rvert$	group	order	$\lvert\widetilde{U}\rvert$	$\lvert(\Pi_{12})_U\rvert$	$\lvert\widetilde{U}\backslash\backslash\backslash\Pi_{12}\rvert$
$\langle 1\rangle$	1	1	4213597	83267	$\langle T^3, IQ\rangle$	8	1	81	1
$\langle T^6\rangle$	2	1	6841	140	$\langle T^6, I, Q\rangle$	8	3	245	102
$\langle IQ\rangle$	2	2	43693	3109	$\langle T^6, T^2Q, T^3IQ\rangle$	8	3	91	29
$\langle T^3IQ\rangle$	2	2	6841	395	$\langle T^3, T^2Q\rangle$	8	3	31	1
$\langle Q\rangle$	2	3	14325	1407	$\langle T^3, I\rangle$	8	3	37	4
$\langle T^2Q\rangle$	2	3	6841	592	$\langle T^3Q, I\rangle$	8	3	37	4
$\langle I\rangle$	2	6	6841	1244	$\langle TQ, TI\rangle$	8	3	81	25
$\langle TI\rangle$	2	6	6841	1474	$\langle TQ, T^2\rangle$	12	1	8	0
$\langle T^4\rangle$	3	1	268	3	$\langle T\rangle$	12	1	6	0
$\langle T^3\rangle$	4	1	111	0	$\langle T^2IQ, T^6\rangle$	12	1	20	0
$\langle T^6, IQ\rangle$	4	1	1913	88	$\langle T^2, I\rangle$	12	1	18	0
$\langle T^6, T^3IQ\rangle$	4	1	319	3	$\langle T^2, TI\rangle$	12	1	16	1
$\langle TQ\rangle$	4	3	111	5	$\langle TIQ, T^6\rangle$	12	1	10	0
$\langle T^6, TI\rangle$	4	3	319	41	$\langle T^2, Q\rangle$	12	1	22	0
$\langle T^6, Q\rangle$	4	3	469	40	$\langle TI, Q\rangle$	12	2	22	6
$\langle T^6, I\rangle$	4	3	349	22	$\langle TI, T^2Q\rangle$	12	2	16	3
$\langle T^4Q, TI\rangle$	4	6	469	183	$\langle T^2I, Q\rangle$	12	2	34	8
$\langle T^{10}Q, TI\rangle$	4	6	319	111	$\langle T^2Q, I\rangle$	12	2	28	5
$\langle I, Q\rangle$	4	6	1159	449	$\langle T^{10}Q, TI, IQ\rangle$	16	3	29	23
$\langle IQ, T^2Q\rangle$	4	6	835	290	$\langle T^6, T^2Q, I\rangle$	24	1	18	6
$\langle T^4, Q\rangle$	6	1	94	2	$\langle T^3Q, TI\rangle$	24	1	8	1
$\langle T^4, T^2Q\rangle$	6	1	54	0	$\langle T^3, TIQ\rangle$	24	1	6	0
$\langle T^2\rangle$	6	1	28	0	$\langle T^6, T^2Q, TI\rangle$	24	1	10	2
$\langle TIQ\rangle$	6	2	28	0	$\langle TQ, I\rangle$	24	1	6	0
$\langle T^4, TI\rangle$	6	2	54	5	$\langle T, Q\rangle$	24	1	6	0
$\langle T^4, I\rangle$	6	2	56	3	$\langle T, I\rangle$	24	1	6	0
$\langle T^2IQ\rangle$	6	2	58	3	$\langle T, I, Q\rangle$	48	1	6	6

$$B(D_{12}) = \frac{1}{24}\begin{pmatrix}
1 & -1 & -6 & -6 & -1 & . & 6 & 6 & 1 & 6 & 6 & . & -6 & -6 & . & . \\
. & 2 & . & . & . & -2 & -6 & -6 & -2 & . & . & 12 & 6 & 6 & 2 & -12 \\
. & . & 12 & . & . & . & -12 & . & . & -12 & . & . & 12 & . & . & . \\
. & . & . & 12 & . & . & . & -12 & . & . & -12 & . & . & 12 & . & . \\
. & . & . & . & 3 & . & . & . & -3 & -6 & -6 & . & 6 & 6 & . & . \\
. & . & . & . & . & 4 & . & . & . & . & . & -12 & . & . & -4 & 12 \\
. & . & . & . & . & . & 12 & . & . & . & . & -12 & -12 & . & . & 12 \\
. & . & . & . & . & . & . & 12 & . & . & . & -12 & . & -12 & . & 12 \\
. & . & . & . & . & . & . & . & 6 & . & . & . & -6 & -6 & -6 & 12 \\
. & . & . & . & . & . & . & . & . & 12 & . & . & -12 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 12 & . & . & -12 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 24 & . & . & . & -24 \\
. & . & . & . & . & . & . & . & . & . & . & . & 12 & . & . & -12 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 12 & . & -12 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 12 & -12 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 24
\end{pmatrix}$$

In the same way the table of marks and the Burnside matrix of $\mathrm{Aff}(1, Z_{12})$ can be computed.

From the previous section we know that partitions of block-type $\lambda \vDash n$ correspond to the right cosets $H_\lambda f$ in $H_\lambda \backslash S_{\underline{n}}$. The partition $H_\lambda f$ is U-invariant, if and only if $H_\lambda f U = H_\lambda f$. This implies

$$\begin{aligned} H_\lambda f u &= H_\lambda f \text{ for all } u \in U \\ H_\lambda f u f^{-1} &= H_\lambda \text{ for all } u \in U \\ f U f^{-1} &\leq H_\lambda . \end{aligned}$$

So the number of U-invariant partitions of block-type λ is

$$\left|\left\{ H_\lambda f \in H_\lambda \backslash S_{\underline{n}} \mid f U f^{-1} \leq H_\lambda \right\}\right| = \frac{1}{|H_\lambda|} \left|\left\{ f \in \underline{n}^{Z_n}_{\mathrm{bij}} \mid f U f^{-1} \leq H_\lambda \right\}\right| .$$

And the number of all U-invariant partitions is given by

$$\sum_{\lambda \vDash n} \left|\left\{ H_\lambda f \in H_\lambda \backslash S_{\underline{n}} \mid f U f^{-1} \leq H_\lambda \right\}\right| .$$

This formula can be found in [13].

In the case when $U = \langle u \rangle$ and u is of cycle type $(a_1, \ldots, a_n)$ then these formulae can be expressed using the *Redfield-cap operator*:

$$\prod_{i=1}^{n} x_i^{a_i} \cap Z(H_\lambda, \underline{n}) \qquad \text{or} \qquad \prod_{i=1}^{n} x_i^{a_i} \cap \left(\sum_{\lambda \vDash n} Z(H_\lambda, \underline{n}) \right) .$$

Finally we will enumerate G-mosaics according to their block-type λ and stabilizer type $\widetilde{U}$. The formula of D. E. White and S. G. Williamson can be extended to a *weighted formula* in the following way. Define a *weight function* $w \colon \Pi_n \to \mathbb{Q}[x_1, \ldots, x_n]$ such that the weight of all partitions π of block-type λ is equal to $\prod_{i=1}^{n} x_i^{\lambda_i}$.

Theorem 4 *Then the sum of the weights of all G-invariant partitions is derived by:*

$$\sum_{\delta \in \Pi_{\mathcal{T}}} \prod_{A \in \delta} \sum_{U \in \mathcal{H}} \left(\frac{1}{m_U(U)} \prod_{t \in A} m_U(G_t) \right) x_\ell^{|G|/|U|},$$

where

$$\ell = \sum_{t \in A} |U| \,/\, |G_t|.$$

PROOF. From the proof in [21] it is obvious that $\frac{1}{m_U(U)} \prod_{t\in A} m_U(G_t)$ counts partitions of $\underline{n}$ consisting of $|G| / |U|$ blocks P of size $\sum_{t\in A} |U| / |G_t|$ each, since each of these blocks P is the disjoint union of sets of size $|U| / |G_t|$ for $t \in A$. $\square$

The numbers of D_{12}-mosaics of block-type $\lambda \vdash 12$ and stabilizer type $\widetilde{U}_j$ for $1 \le j \le 16$ are the coefficients of $\prod_{i=1}^{12} x_i^{\lambda_i}$ in the following list:

$\widetilde{U}_1:$ $320x_1^3x_2x_7 + 678x_1^5x_2x_5 + 760x_1^3x_3x_6 + 1140x_1^5x_3x_4 + 546x_1^4x_2x_6 + 195x_4^3 + 546x_2x_4x_6 + 2860x_1x_3x_4^2 + 1140x_3x_4x_5 + 1648x_1^2x_2^2x_6 + 1140x_1^4x_3x_5 + 11480x_1^3x_2^3x_3 + 28x_5x_7 + 14x_4x_8 + 7x_3x_9 + 349x_3^2x_6 + 309x_2x_5^2 + 44x_2^2x_8 + 2004x_2^2x_4^2 + 516x_2^3x_6 + 5554x_2^3x_3^2 + 2022x_2^4x_4 + 309x_1^2x_5^2 + 2412x_1^2x_2^5 + 7x_1^3x_9 + 2527x_1^3x_3^3 + 14x_1^4x_8 + 660x_1^4x_4^2 + 2022x_1^4x_2^4 + 28x_1^5x_7 + 29x_1^6x_6 + 349x_1^6x_3^2 + 516x_1^6x_2^3 + 28x_1^7x_5 + 14x_1^8x_4 + 44x_1^8x_2^2 + 7x_1^9x_3 + 320x_1^7x_2x_3 + 3450x_1x_2x_4x_5 + 2300x_1x_2x_3x_6 + 17280x_1x_2^2x_3x_4 + 6900x_1^2x_2x_3x_5 + 11520x_1^3x_2x_3x_4 + 581x_3^4 + 8x_6^2 + 337x_2^6 + 320x_2x_3x_7 + 5662x_2x_3^2x_4 + 3420x_2^2x_3x_5 + 226x_1x_5x_6 + 160x_1x_4x_7 + 80x_1x_3x_8 + 2290x_1x_3^2x_5 + 25x_1x_2x_9 + 7660x_1x_2x_3^3 + 480x_1x_2^2x_7 + 3430x_1x_2^3x_5 + 8600x_1x_2^4x_3 + 546x_1^2x_4x_6 + 320x_1^2x_3x_7 + 5662x_1^2x_3^2x_4 + 108x_1^2x_2x_8 + 4196x_1^2x_2x_4^2 + 17026x_1^2x_2^2x_3^2 + 8468x_1^2x_2^3x_4 + 1140x_1^3x_4x_5 + 3420x_1^3x_2^2x_5 + 5662x_1^4x_2x_3^2 + 4208x_1^4x_2^2x_4 + 3420x_1^5x_2^2x_3 + 546x_1^6x_2x_4,$

$\widetilde{U}_2:$ $4x_1^4x_2x_6 + 6x_4^3 + 4x_2x_4x_6 + 12x_1^2x_2^2x_6 + 5x_3^2x_6 + 6x_2x_5^2 + 23x_2^2x_4^2 + 9x_2^3x_6 + 41x_2^3x_3^2 + 18x_2^4x_4 + 6x_1^2x_5^2 + 34x_1^2x_2^5 + 7x_1^4x_4^2 + 18x_1^4x_2^4 + x_1^6x_6 + 5x_1^6x_3^2 + 9x_1^6x_2^3 + 8x_3^4 + x_6^2 + 11x_2^6 + 18x_2x_3^2x_4 + 4x_1^2x_4x_6 + 18x_1^2x_3^2x_4 + 2x_1^2x_2x_8 + 24x_1^2x_2x_4^2 + 54x_1^2x_2^2x_3^2 + 32x_1^2x_2^3x_4 + 18x_1^4x_2x_3^2 + 14x_1^4x_2^2x_4 + 4x_1^6x_2x_4,$

$\widetilde{U}_3:$ $20x_1^3x_2x_7+30x_1^5x_2x_5+19x_1^3x_3x_6+30x_1^5x_3x_4+28x_1^4x_2x_6+23x_4^3+28x_2x_4x_6+55x_1x_3x_4^2 + 30x_3x_4x_5 + 64x_1^2x_2^2x_6 + 30x_1^4x_3x_5 + 137x_1^3x_2^3x_3 + 10x_5x_7 + 6x_4x_8 + 4x_3x_9+26x_3^2x_6+2x_2x_{10}+19x_2x_5^2+14x_2^2x_8+66x_2^2x_4^2+43x_2^3x_6+117x_2^3x_3^2+88x_2^4x_4+x_1x_{11}+2x_1^2x_{10}+19x_1^2x_5^2+90x_1^2x_2^5+4x_1^3x_9+76x_1^3x_3^3+6x_1^4x_8+38x_1^4x_4^2+88x_1^4x_2^4+10x_1^5x_7+8x_1^6x_6+26x_1^6x_3^2+43x_1^6x_2^3+10x_1^7x_5+6x_1^8x_4+14x_1^8x_2^2+4x_1^9x_3+20x_1^7x_2x_3+30x_1x_2x_4x_5 + 20x_1x_2x_3x_6 + 90x_1x_2^2x_3x_4 + 60x_1^2x_2x_3x_5 + 60x_1^3x_2x_3x_4 + 2x_1^{10}x_2 + 33x_3^4 + 4x_6^2 + 33x_2^6 + 20x_2x_3x_7 + 86x_2x_3^2x_4 + 90x_2^2x_3x_5 + 10x_1x_5x_6 + 10x_1x_4x_7 + 5x_1x_3x_8+40x_1x_3^2x_5+5x_1x_2x_9+80x_1x_2x_3^3+30x_1x_2^2x_7+70x_1x_2^3x_5+125x_1x_2^4x_3+28x_1^2x_4x_6 + 20x_1^2x_3x_7 + 86x_1^2x_3^2x_4 + 14x_1^2x_2x_8 + 78x_1^2x_2x_4^2 + 178x_1^2x_2^2x_3^2 + 144x_1^2x_2^3x_4+30x_1^3x_4x_5+90x_1^3x_2^2x_5+86x_1^4x_2x_3^2+98x_1^4x_2^2x_4+90x_1^5x_2^2x_3+28x_1^6x_2x_4,$

$\widetilde{U}_4:$ $30x_1^4x_2x_6 + 55x_4^3 + 30x_2x_4x_6 + 90x_1^2x_2^2x_6 + 6x_4x_8 + 39x_3^2x_6 + 3x_2x_{10} + 48x_2x_5^2 + 18x_2^2x_8 + 216x_2^2x_4^2 + 67x_2^3x_6 + 277x_2^3x_3^2 + 168x_2^4x_4 + 3x_1^2x_{10} + 48x_1^2x_5^2 + 243x_1^2x_2^5+6x_1^4x_8+72x_1^4x_4^2+168x_1^4x_2^4+9x_1^6x_6+39x_1^6x_3^2+67x_1^6x_2^3+6x_1^8x_4+18x_1^8x_2^2+3x_1^{10}x_2 + 72x_3^4 + 15x_6^2 + 130x_2^6 + 120x_2x_3^2x_4 + 30x_1^2x_4x_6 + 120x_1^2x_3^2x_4 + 15x_1^2x_2x_8 + 165x_1^2x_2x_4^2 + 360x_1^2x_2^2x_3^2 + 210x_1^2x_2^3x_4 + 120x_1^4x_2x_3^2 + 126x_1^4x_2^2x_4 + 30x_1^6x_2x_4,$

$\widetilde{U}_5:$ $x_1^3x_3x_6 + x_4^3 + 3x_1^3x_2^3x_3 + 3x_1^3x_3^3 + 3x_3^4,$

$\widetilde{U}_6:\ x_3^4+x_1^4x_2^2x_4,$

$\widetilde{U}_7:\ 2x_1^4x_2x_6+2x_4^3+2x_2x_4x_6+6x_1^2x_2^2x_6+x_4x_8+3x_3^2x_6+x_2x_{10}+4x_2x_5^2+3x_2^2x_8+5x_2^2x_4^2+5x_2^3x_6+11x_2^3x_3^2+7x_2^4x_4+x_1^2x_{10}+4x_1^2x_5^2+13x_1^2x_2^5+x_1^4x_8+3x_1^4x_4^2+7x_1^4x_2^4+x_1^6x_6+3x_1^6x_3^2+5x_1^6x_2^3+x_1^8x_4+3x_1^8x_2^2+x_1^{10}x_2+2x_3^4+4x_2^6+4x_2x_3^2x_4+2x_1^2x_4x_6+4x_1^2x_3^2x_4+x_1^2x_2x_8+7x_1^2x_2x_4^2+12x_1^2x_2^2x_3^2+6x_1^2x_2^3x_4+4x_1^4x_2x_3^2+7x_1^4x_2^2x_4+2x_1^6x_2x_4,$

$\widetilde{U}_8:\ 7x_4^3+x_4x_8+4x_2^2x_8+30x_2^2x_4^2+17x_2^4x_4+x_1^4x_8+9x_1^4x_4^2+17x_1^4x_2^4+x_1^8x_4+4x_1^8x_2^2+6x_3^4+4x_6^2+26x_2^6+9x_1^4x_2^2x_4,$

$\widetilde{U}_{10}:\ x_1^3x_3x_6+x_4^3+3x_1^3x_2^3x_3+x_3x_9+x_2^3x_6+x_2^3x_3^2+x_1^3x_9+4x_1^3x_3^3+x_1^6x_2^3+x_1^9x_3+3x_3^4+x_2^6,$

$\widetilde{U}_{11}:\ 3x_1^6x_2^3+x_1^6x_3^2+x_1^6x_6+4x_2^6+3x_2^3x_3^2+3x_2^3x_6+x_3^2x_6+3x_4^3,$

$\widetilde{U}_{12}:\ 5x_1^4x_2^4+3x_1^4x_4^2+x_1^4x_8+x_1^8x_2^2+4x_2^6+3x_2^2x_4^2+x_2^2x_8+x_1^8x_4+5x_2^4x_4+2x_4^3+x_4x_8+3x_3^4+x_6^2,$

$\widetilde{U}_{13}:\ x_1^6x_2^3+x_1^6x_3^2+x_1^6x_6+x_2^3x_3^2+x_2^3x_6+x_3^2x_6,$

$\widetilde{U}_{14}:\ 3x_2^6+x_4^3+x_6^2,$

$\widetilde{U}_{16}:\ x_{12}+x_6^2+x_4^3+x_3^4+x_2^6+x_1^{12}.$

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