Tiling problems in music theory

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Abstract

In mathematical music theory we often come across various constructions on Z_n, the set of residues modulo n for $n \geq 2$. Different objects constructed on Z_n are considered to be equivalent if there exists a symmetry motivated by music which transforms one object into the other one. Usually we are dealing with cyclic, dihedral, or affine symmetry groups on Z_n. Here we will compare partitions of Z_n, sometimes also called mosaics, and rhythmic tiling canons on Z_n. Especially we will investigate regular complementary canons of maximal category in more details.

1 Introduction

In the present paper we compare two tiling problems of

$$Z_n = \{0, 1, \ldots, n-1\},$$

the set of all residues modulo n for $n \geq 2$. We discuss how to partition the set Z_n in essentially different ways, and we describe a special class of canons which also partition Z_n. When speaking about partitioning of a set X, here $X = Z_n$, we assume that there exist an integer $k \geq 1$ and nonempty subsets $P_1, \ldots, P_k$ of X such that $X = P_1 \cup \ldots \cup P_k$, and the intersection $P_i \cap P_j$ is the empty set for all $i \neq j$. Two partitions are called 'essentially different' if there is no symmetry operation of Z_n which transforms one partition into the other one. Of course this notion heavily depends on what is assumed to be a symmetry of Z_n. If the temporal shift T, retrograde inversion R and affine mappings $A_{a,b}$ are determined by

$$T\colon Z_n \to Z_n \qquad i \mapsto T(i) := i + 1$$

$$R\colon Z_n \to Z_n \qquad i \mapsto R(i) := -i$$

$$A_{a,b}\colon Z_n \to Z_n \qquad i \mapsto A_{a,b}(i) := ai + b \qquad a, b \in Z_n,$$

then usually the cyclic group $C_n := \langle T \rangle$, the dihedral group $D_n := \langle T, R \rangle$, or the group $\mathrm{Aff}_1(Z_n) := \{A_{a,b} \mid a, b \in Z_n,\ \gcd(a,n) = 1\}$ of all affine mappings on Z_n are symmetry groups on Z_n which can be motivated by music theory. (Cf. Mazzola (1990, 2002).)

Such constructions on Z_n we are interested in can be best described in the notion of discrete structures. The mathematical tool for working with symmetry operations are group actions, which will be introduced later.

In general *discrete structures* are objects constructed as

– subsets, unions, products of finite sets,

– mappings between finite sets,
– bijections, linear orders on finite sets,
– equivalence classes on finite sets,
– vector spaces over finite fields, etc.

For example it is possible to describe graphs, necklaces, designs, codes, matroids, switching functions, molecules in chemistry, spin-configurations in physics, or objects of local music theory as discrete structures.

As was indicated above, often the elements of a discrete structure are not simple objects, but they are themselves classes of objects which are considered to be equivalent. Then each class collects all those elements which are not essentially different. For instance, in order to describe mathematical objects we often need labels, but for the classification of these objects the labelling is not important. Thus all elements which can be derived by relabelling of one labelled object are collected to one class.

For example, a labelled graph is usually described by its set of vertices V and its set of edges E. (An edge connects exactly two different vertices of the graph.) If the graph has n vertices, then usually $V = \underline{n} := \{1, \ldots, n\}$ and E is a subset of the set of all 2-subsets of V. Then $\{i, j\}$ belongs to E if the two vertices with labels i and j are connected by an edge of the graph. An unlabelled graph is the set of all graphs which can be constructed by relabelling a labelled graph.

Besides relabelling also naturally motivated symmetry operations give rise to collect different objects to one class of essentially not different objects. This is for instance the case when we describe different partitions of Z_n or different canons on Z_n.

The process of *classification* of discrete structures provides more detailed information about the objects in a discrete structure. We distinguish different steps in this process:

step 1: Determine the number of different objects in a discrete structure.

step 2: Determine the number of objects with certain properties in a discrete structure.

step 3: Determine a complete list of all the elements of a discrete structure.

step 4: Generate the objects of a discrete structure uniformly at random.

In general, step 3 is the most ambitious task, it needs a lot of computing power, computing time and memory. For that reason, when the set of all elements of a discrete structure is too hard to be completely determined it is useful and makes sense to consider step 4. This approach allows to generate a huge variety of different unprejudiced objects of a discrete structure. These sets of examples can be very useful for checking certain hypotheses on them and afterwards for trying to prove the valid ones.

For example, let us have a short look at the classification of unlabelled graphs on 4 vertices:

step 1: There are 11 graphs on 4 vertices.

step 2: There exists exactly one graph with 0 edges, with 1 edge, with 5 edges or with 6 edges; two graphs with 2 or 4 edges; three graphs with 3 edges.

step 3: The unlabelled graphs on 4 vertices are given in figure 1.

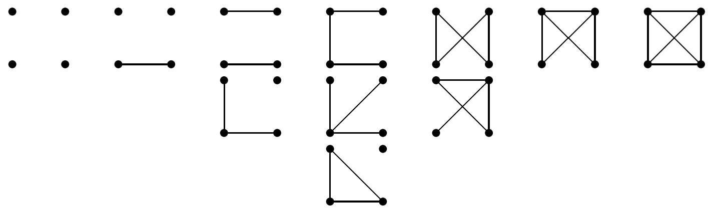

Figure 1: The unlabelled graphs on 4 vertices

The standard tool for classification of discrete structures are *group actions*. A detailed introduction to combinatorics under finite group actions can be found in Kerber (1991, 1999).

A multiplicative group G with neutral element 1 acts on a set X if there exists a mapping

$$*: G \times X \to X \qquad *\,(g,x) \mapsto g * x$$

such that

$$(g_1 g_2) * x = g_1 * (g_2 * x) \qquad g_1, g_2 \in G,\ x \in X$$

and

$$1 * x = x \qquad x \in X.$$

We usually write gx instead of $g*x$. A group action of G on X will be indicated as ${}_G X$. If G and X are finite sets, then we speak of a *finite group action*.

A group action ${}_G X$ determines a group homomorphism ϕ from G to the symmetric group $S_X := \{\sigma \mid \sigma\colon X \to X \text{ is bijective}\}$ by

$$\phi: G \to S_X, \qquad g \mapsto \phi(g) := [x \mapsto gx],$$

which is called a *permutation representation* of G on X. Usually we abbreviate $\phi(g)$ by writing $\bar{g}$, which is the permutation of X that maps x to gx. For instance $\bar{1}$ is always the identity on X. Accordingly, the image $\phi(G)$ is indicated by $\bar{G}$. It is a *permutation group* on X, i. e. a subgroup of S_X.

A group action ${}_G X$ defines the following equivalence relation on X. Two elements x_1, x_2 of X are called equivalent, we indicate it by $x_1 \sim x_2$, if there is some $g \in G$ such that $x_2 = gx_1$. The equivalence class $G(x)$ of $x \in X$ with respect to $\sim$ is the *G-orbit* of x. Hence, the orbit of x under the action of G is

$$G(x) = \{gx \mid g \in G\}.$$

The set of orbits of G on X is indicated as

$$G\backslash\backslash X := \{G(x) \mid x \in X\}.$$

In general, classification of a discrete structure means the same as describing the elements of $G\backslash\backslash X$ for a suitable group action ${}_G X$.

Theorem 1. *The equivalence classes of any equivalence relation can be represented as orbits under a suitable group action.*

If X is finite then $\bar{G}$ is a finite group since it is a subgroup of the symmetric group S_X which is of cardinality $|X|!$. For any $x \in X$ we have $G(x) = \bar{G}(x)$, whence $G\backslash\backslash X = \bar{G}\backslash\backslash X$. Hence, whenever X is finite, each group action ${}_GX$ can be described by a finite group action ${}_{\bar{G}}X$.

Let ${}_GX$ be a group action. For each $x \in X$ the *stabilizer* G_x of x is the set of all group elements which do not change x, thus

$$G_x := \{g \in G \mid gx = x\}.$$

It is a subgroup of G.

Lemma 2. *If ${}_GX$ is a group action, then for any $x \in X$ the mapping*

$$\phi\colon G/G_x \to G(x) \text{ given by } \phi(gG_x) = gx$$

is a bijection.

As a consequence we get

Theorem 3. *If ${}_GX$ is a finite group action then the size of the orbit of $x \in X$ equals*

$$|G(x)| = \frac{|G|}{|G_x|}.$$

Finally, as the last notion under group actions, we introduce the *set of all fixed points* of $g \in G$ which is denoted by

$$X_g := \{x \in X \mid gx = x\}.$$

Let ${}_GX$ be a finite group action. The main tool for determining the number of different orbits is

Theorem 4. (Cauchy Frobenius Lemma) *The number of orbits under a finite group action ${}_GX$ is the average number of fixed points:*

$$|G\backslash\backslash X| = \frac{1}{|G|}\sum_{g\in G}|X_g|.$$

Proof.

$$\begin{aligned}\sum_{g\in G}|X_g| &= \sum_{g\in G}\sum_{x:gx=x}1 = \sum_{x\in X}\sum_{g:gx=x}1 = \sum_{x\in X}|G_x| = \\ &= |G|\sum_{x\in X}\frac{1}{|G(x)|} = |G|\sum_{\omega\in G\backslash\backslash X}\sum_{x\in\omega}\frac{1}{|\omega|} = |G||G\backslash\backslash X|.\end{aligned}$$

□

The most important applications of classification under group actions can be described as symmetry types of mappings between two sets. Group actions ${}_GX$ and ${}_HY$ on the domain X and range Y induce group actions on

$$Y^X = \{f \mid f\colon X \to Y \text{ is a function}\}$$

in the following way:

G acts on Y^X by

$$G \times Y^X \to Y^X, \qquad g * f := f \circ \bar{g}^{-1}.$$

H acts on Y^X by

$$H \times Y^X \to Y^X, \qquad h * f := \bar{h} \circ f.$$

The direct product $H \times G$ acts on Y^X by

$$(H \times G) \times Y^X \to Y^X, \qquad (h, g) * f := \bar{h} \circ f \circ \bar{g}^{-1}.$$

2 Enumeration of non-isomorphic mosaics

In Isihara and Knapp (1993) it was stated that the enumeration of mosaics is an open research problem communicated by Robert Morris from the Eastman School of Music. More information about mosaics can be found in Alegant (1992). Here some results from Fripertinger (1999) are presented.

Let π be a partition of a set X. If π consists of exactly k non-empty, disjoint subsets of X, then π is called a *partition of size k*. A partition of the set Z_n is called a *mosaic*. Let Π_n denote the set of all mosaics of Z_n, and let $\Pi_{n,k}$ be the set of all mosaics of Z_n of size k.

A group action of a group G on the set Z_n induces the following group action of G on Π_n:

$$G \times \Pi_n \to \Pi_n, \qquad g * \pi := \{gP \mid P \in \pi\},$$

where $gP := \{gx \mid x \in P\}$. This action can be restricted to an action of G on $\Pi_{n,k}$. Two mosaics are called *G-isomorphic* if they belong to the same G-orbit on Π_n, in other words, $\pi_1, \pi_2 \in \Pi_n$ are isomorphic if $g\pi_1 = \pi_2$ for some $g \in G$.

As was already indicated, in music theory the groups C_n, D_n, or $\mathrm{Aff}_1(Z_n)$ are candidates for the group G.

It is well known (see de Bruijn (1964, 1979)) how to enumerate G-isomorphism classes of mosaics (i.e. G-orbits of partitions) by identifying them with $S_{\underline{n}} \times G$-orbits on the set of all functions from Z_n to $\underline{n}$. (The symmetric group of the set $\underline{n}$ is denoted by $S_{\underline{n}}$.) Furthermore, G-mosaics of size k correspond to $S_{\underline{k}} \times G$-orbits on the set of all surjective functions from Z_n to $\underline{k}$.

In Fripertinger (1999) the following theorem is proved.

Theorem 5. *Let M_k be the number of $S_{\underline{k}} \times G$-orbits on $\underline{k}^{Z_n}$. Then the number of G-isomorphism classes of mosaics of Z_n is given by M_n, and the number of G-isomorphism classes of mosaics of size k is given by $M_k - M_{k-1}$, where $M_0 := 0$.*

Using the Cauchy-Frobenius-Lemma, we have

$$M_k = \frac{1}{|S_{\underline{k}}|\,|G|} \sum_{(\sigma,g)\in S_{\underline{k}}\times G} \prod_{i=1}^{n} a_1(\sigma^i)^{a_i(\bar{g})},$$

where $a_i(\bar{g})$ or $a_i(\sigma)$ are the numbers of i-cycles in the cycle decomposition of $\bar{g}$ or σ respectively.

Finally, the number of G-isomorphism classes of mosaics of size k can also be derived by the Cauchy-Frobenius-Lemma for surjective functions by

$$\frac{1}{|S_{\underline{k}}|\,|G|} \sum_{(\sigma,g)\in S_{\underline{k}}\times G} \sum_{\ell=1}^{c(\sigma)} (-1)^{c(\sigma)-\ell} \sum_{a} \prod_{i=1}^{k} \binom{a_i(\sigma)}{a_i} \prod_{j=1}^{n} \left(\sum_{d\mid j} d \cdot a_d\right)^{a_j(\bar{g})},$$

where the inner sum is taken over the sequences $a = (a_1,\ldots,a_k)$ of nonnegative integers a_i such that $\sum_{i=1}^{k} a_i = \ell$, and where $c(\sigma)$ is the number of all cycles in the cycle decomposition of σ.

If $\pi \in \Pi_n$ consists of λ_i blocks of size i for $i \in \underline{n}$, then π is said to be of *block-type* $\lambda = (\lambda_1,\ldots,\lambda_n)$. From the definition it is obvious that $\sum_{i=1}^{n} i\lambda_i = n$. Furthermore, it is clear that π is a partition of size $\sum_{i=1}^{n} \lambda_i$. The set of mosaics of block-type λ will be indicated as Π_λ. Since the action of G on Π_n can be restricted to an action of G on Π_λ, we want to determine the number of G-isomorphism classes of mosaics of type λ. For doing that, let $\bar{\lambda}$ be a particular partition of type λ. (For instance, $\bar{\lambda}$ can be defined such that the blocks of $\bar{\lambda}$ of size 1 are given by $\{1\}$, $\{2\}$, ..., $\{\lambda_1\}$, the blocks of $\bar{\lambda}$ of size 2 are given by $\{\lambda_1+1, \lambda_1+2\}$, $\{\lambda_1+3, \lambda_1+4\}$, ..., $\{\lambda_1+2\lambda_2-1, \lambda_1+2\lambda_2\}$, and so on.) According to Kerber (1991, 1999), the stabilizer H_λ of $\bar{\lambda}$ in the symmetric group $S_{\underline{n}}$ is similar to the direct sum

$$\bigoplus_{i=1}^{n} S_{\underline{\lambda_i}}[S_{\underline{i}}]$$

of compositions of symmetric groups, which is a permutation representation of the direct product

$$\mathop{\times}_{i=1}^{n} S_{\underline{i}} \wr S_{\underline{\lambda_i}}$$

of wreath products of symmetric groups. In other words, H_λ is the set of all permutations $\sigma \in S_{\underline{n}}$, which map each block of the partition $\bar{\lambda}$ again onto a block (of the same size) of the partition.

Hence, the G-isomorphism classes of mosaics of type λ can be described as $H_\lambda \times G$-orbits of bijections from Z_n to $\underline{n}$ under the following group action:

$$(H_\lambda \times G) \times \underline{n}_{\mathrm{bij}}^{Z_n} \to \underline{n}_{\mathrm{bij}}^{Z_n}, \qquad (\sigma, g) * f := \sigma \circ f \circ \bar{g}^{-1}.$$

When interpreting the bijections from Z_n to $\underline{n}$ as permutations of the n-set $\underline{n}$, then G-mosaics of type λ correspond to double cosets (cf. Kerber (1991, 1999)) of the form

$$H_\lambda \backslash S_{\underline{n}} / G.$$

Theorem 6. *The number M_λ of G-isomorphism classes of mosaics of type λ is given by*

$$M_\lambda = \frac{1}{|H_\lambda|\,|G|} \sum_{\substack{(\sigma,g) \in H_\lambda \times G \\ z(\bar{g}) = z(\sigma)}} \prod_{i=1}^{n} a_i(\bar{g})!\, i^{a_i(\sigma)},$$

where $z(\bar{g})$ and $z(\sigma)$ are the cycle types of $\bar{g}$ and of σ respectively, given in the form $(a_i(\bar{g}))_{i \in \underline{n}}$ or $(a_i(\sigma))_{i \in \underline{n}}$. In other words, we are summing over pairs (σ, g) such that $\bar{g}$ and σ determine permutations of the same cycle type.

In conclusion, in this section we demonstrated how to classify the isomorphism classes of mosaics. We applied methods from step 1 or step 2 of the general scheme of classification of discrete structures.

3 Enumeration of non-isomorphic canons

The present concept of a canon is described in Mazzola (2002) and was presented by G. Mazzola to the present author in the following way: A *canon* is a subset $K \subseteq Z_n$ together with a covering of K by pairwise different subsets $V_i \neq \emptyset$ for $1 \leq i \leq t$, the voices, where $t \geq 1$ is the number of voices of K, in other words,

$$K = \bigcup_{i=1}^{t} V_i,$$

such that for all $i, j \in \{1, \ldots, t\}$

1. the set V_i can be obtained from V_j by a translation of Z_n,
2. there is only the identity translation which maps V_i to V_i,
3. the set of differences in K generates Z_n, i.e.

$$\langle K - K\rangle := \langle k - l \mid k, l \in K\rangle = Z_n.$$

We prefer to write a canon K as a set of its subsets V_i. Two canons $K = \{V_1, \ldots, V_t\}$ and $L = \{W_1, \ldots, W_s\}$ are called *isomorphic* if $s = t$ and if there exists a translation T^j of Z_n and a permutation σ in the symmetric group $S_{\underline{t}}$ such that $T^j(V_i) = W_{\sigma(i)}$ for $1 \le i \le t$. Then obviously T^j applied to K yields L.

Here we present some results from Fripertinger (2002). The cyclic group C_n acts on the set of all functions from Z_n to $\{0,1\}$ by

$$C_n \times \{0,1\}^{Z_n} \to \{0,1\}^{Z_n} \qquad T^j * f := f \circ T^{-j}.$$

When writing the elements $f \in \{0,1\}^{Z_n}$ as vectors $(f(0), \ldots, f(n-1))$, using the natural order of the elements of Z_n, the set $\{0,1\}^{Z_n}$ is totally ordered by the lexicographical order. As the *canonical representative* of the orbit $C_n(f) = \{f \circ T^j \mid 0 \le j < n\}$ we choose the function $f_0 \in C_n(f)$ such that $f_0 \le h$ for all $h \in C_n(f)$.

A function $f \in \{0,1\}^{Z_n}$ (or the corresponding vector $(f(0), \ldots, f(n-1))$) is called *acyclic* if $C_n(f)$ consists of n different objects. The canonical representative of the orbit of an acyclic function is called a *Lyndon word*.

As usual, we identify each subset A of Z_n with its *characteristic function* $\chi_A \colon Z_n \to \{0,1\}$ given by

$$\chi_A(i) = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise.} \end{cases}$$

Following the ideas of Fripertinger (2002) and the notion of Andreatta et al. (2001), a canon can be described as a pair (L, A), where L is the *inner* and A the *outer rhythm* of the canon. In other words, the rhythm of one voice is described by L and the distribution of the different voices is described by A, i.e. the onsets of all the voices of the canon determined by (L, A) are $a + L$ for $a \in A$. In the present situation, $L \neq 0$ is a Lyndon word of length n over the alphabet $\{0,1\}$, and A is a t-subset of Z_n. But not each pair (L, A) describes a canon. More precisely we have:

Lemma 7. *The pair (L, A) does not describe a canon in Z_n if and only if there exists a divisor $d > 1$ of n such that $L(i) = 1$ implies $i \equiv d - 1 \bmod d$ and $\chi_{A_0}(i) = 1$ implies $i \equiv d - 1 \bmod d$, where χ_{A_0} is the canonical representative of $C_n(\chi_A)$.*

An application of the *principle of inclusion and exclusion* allows to determine the number of non-isomorphic canons.

Theorem 8. *The number of isomorphism classes of canons in Z_n is*

$$K_n = \sum_{d \mid n} \mu(d)\lambda(n/d)\alpha(n/d),$$

where μ is the Möbius function, $\lambda(1) = 1$,

$$\lambda(r) = \frac{1}{r}\sum_{s \mid r} \mu(s) 2^{r/s} \ \textit{for} \ r > 1,$$

and

$$\alpha(r) = \frac{1}{r} \sum_{s|r} \varphi(s) 2^{r/s} - 1 \text{ for } r \geq 1,$$

where φ *is the Euler totient function.*

Here the description of a canon as a pair (L, A) with certain properties was used in oder to classify all canons in Z_n by methods of step 1 or step 2 of the general classification scheme.

4 Enumeration of rhythmic tiling canons

There exist more complicated definitions of canons. A canon described by the pair (L, A) of inner and outer rhythm defines a *rhythmic tiling canon* in Z_n with voices V_a for $a \in A$ if

1. the voices V_a cover entirely the cyclic group Z_n,
2. the voices V_a are pairwise disjoint.

Rhythmic tiling canons with the additional property

3. both L and A are acyclic,

are called *regular complementary canons of maximal category*.

In other words, the voices of a rhythmic tiling canon form a partition of Z_n. Hence, rhythmic tiling canons are canons which are also mosaics. More precisely, if $|A| = t$ then they are mosaics consisting of t blocks of size n/t, whence they are of block-type λ where

$$\lambda_i = \begin{cases} t & \text{if } i = n/t \\ 0 & \text{otherwise.} \end{cases}$$

This block-type will be also indicated as $\lambda = ((n/t)^t) = (|L|^{|A|})$.

So far the author did not find a characterization of those mosaics of block-type λ describing canons, which could be used in order to apply methods from step 1 or step 2 of the general classification scheme.

Applying Theorem 6, the numbers of C_n-isomorphism classes of mosaics presented in table 1 were computed. Among these there are also the isomorphism classes of canons, but many mosaics of these block types are not canons!

However, the description of the isomorphism classes of canons as pairs $(L, C_n(A))$ consisting of Lyndon words L and C_n-orbits of subsets A of Z_n with some additional properties (c.f. Lemma 7) can also be applied for the determination of complete sets of representatives of non-isomorphic canons in Z_n, as was indicated in the last part of Fripertinger (2002). These methods belong to step 3 of the general classification. There exist fast algorithms for computing all Lyndon words of length n over $\{0, 1\}$ and all C_n-orbit representatives of subsets of Z_n. For finding regular tiling canons with t voices (where t is necessarily a divisor of n), we can restrict ourselves to Lyndon words L with exactly n/t entries 1 and to representatives A_0 of the C_n-orbits of t-subsets of Z_n. Then each pair (L, A_0) must be tested whether it is a regular tiling canon. In this test we only have to test whether the voices described by (L, A_0) determine a partition on Z_n, because in this case it is obvious that (L, A_0) does not satisfy the assumptions of Lemma 7.

n	$((n/t)^t)$	M_λ
12	(6^2)	44
	(4^3)	499
	(3^4)	1306
	(2^6)	902
24	(12^2)	56450
	(8^3)	65735799
	(6^4)	4008.268588
	(4^6)	187886.308429
	(3^8)	381736.855102
	(2^{12})	13176.573910
36	(18^2)	126047906
	(12^3)	15.670055.601970
	(9^4)	24829.574426.591236
	(6^6)	103.016116.387908.956698
	(4^9)	10778.751016.666506.604919
	(3^{12})	9910.160306.188702.944292
	(2^{18})	6.156752.656678.674792
40	(20^2)	1723.097066
	(10^4)	4.901417.574950.588294
	(8^5)	1595.148844.422078.211829
	(5^8)	11.765613.697294.131102.617360
	(4^{10})	88.656304.986604.408738.684375
	(2^{20})	7995.774669.504366.055054

Table 1: Number of mosaics in Z_n of block-type $((n/t)^t)$

For finding the number of regular tiling canons, we make use of still another result concerning regular complementary canons of maximal category. First we realize that (L, A_0) is a tiling canon if and only if Z_n is the direct sum $L \oplus A_0$, i.e. $Z_n = L + A_0$ and $|Z_n| = |L| \cdot |A_0|$. In other words, for each element $x \in Z_n$ there exists exactly one pair $(x_1, x_2) \in L \times A_0$ such that $x = x_1 + x_2$.

Let G be an abelian group. A subset S of G is called *g-periodic* for $g \in G$ if $S = g + S$, and it is called *periodic* if it is g-periodic for some $g \in G$. Otherwise S is called *aperiodic*. (Subsets of Z_n are aperiodic if and only if they are acyclic.) The group G is called a *Hajós group*, or has the 2-*Hajós property*, if in each factorization of G as $S_1 \oplus S_2$ at least one factor is periodic. In Sands (1962) all finite abelian groups which are Hajós groups are classified. This classification yields the following list of cyclic Hajós groups:

Theorem 9. *The group Z_n is a Hajós group if and only if n is of the form*

$$p^k \text{ for } k \geq 0,\ p^k q \text{ for } k \geq 1,\ p^2 q^2,\ p^k qr \text{ for } k \in \{1, 2\},\ pqrs$$

for distinct primes p, q, r and s.

Corollary 10. *Z_n is a non-Hajós group if and only if n can be expressed in the form $p_1 p_2 n_1 n_2 n_3$ with p_1, p_2 primes, $n_i \geq 2$ for $1 \leq i \leq 3$, and $\gcd(n_1 p_1, n_2 p_2) = 1$.*

The smallest n for which Z_n is not a Hajós-group is $n = 72$ which is still much further than the scope of our computations.

The pair (L, A_0) is a regular complementary canon of maximal category if and only if it is a tiling canon and both L and A_0 are aperiodic. Consequently, regular complementary canons of maximal category occur only for Z_n which are non-Hajós groups. (Cf. Vuza (1991); Andreatta (1997).) Hence, we deduce that for all n such that Z_n is a Hajós-group the following is true:

Lemma 11. *If a pair (L, A_0) describes a regular tiling canon in a Hajós-group Z_n, then A_0 is not aperiodic.*

This reduces dramatically the number of pairs which must be tested for being a tiling canon, since for Hajós groups Z_n we only have to test pairs (L, A_0) where A_0 is periodic, whence A_0 is not a Lyndon word.

By an application of Theorem 8 we computed K_n, the numbers of non-isomorphic canons of length n, given in the third column of table 2.

The construction described above yields a complete list of all regular tiling canons, whence also the number of regular tiling canons of length n, indicated by T_n, given in the second column of table 2.

5 Some results on regular complementary canons of maximal category

In Vuza (1991, 1992b,a, 1993) the author showed for which n there exist regular complementary canons of maximal category. Moreover, he described a method how to construct regular complementary canons of maximal category for those Z_n which are non-Hajós groups. He proved that each pair (L, A) computed along the following algorithm is a regular complementary canon of maximal category.

Let Z_n be a non-Hajós group, whence $n = p_1 p_2 n_1 n_2 n_3$ as described in Corollary 10. Vuza presents an algorithm for constructing two aperiodic subsets L and A of Z_n such that $|L| = n_1 n_2$, $|A| = p_1 p_2 n_3$, and $L + A = Z_n$. In order to construct L, for $i = 1, 2$ let L_i be a nonperiodic set of representatives of $\frac{n}{p_i n_i} Z_n$ modulo its subgroup $\frac{n}{p_i} Z_n$. Then set $L := L_1 + L_2$. In order to determine A, for $i = 1, 2$ choose $x_i \in \frac{n}{p_i n_i} Z_n \setminus \frac{n}{p_i} Z_n$ and let

$$A_1 = \frac{n}{p_1} Z_n + \left(\frac{n}{p_2} Z_n \setminus \{0\} \cup \{x_1\} \right),$$

$$A_2 = \frac{n}{p_2} Z_n + \left(\frac{n}{p_1} Z_n \setminus \{0\} \cup \{x_2\} \right).$$

Choose a set R of representatives of Z_n modulo $n_3 Z_n$, let $A_3 := R \setminus n_3 Z_n$, and put $A := A_1 \cup (A_2 + A_3)$.

Consequently, both L or A can serve as the inner or outer rhythm of a regular complementary canon of maximal category. Moreover, as we saw there is some freedom for constructing these two sets, and each of these two sets can be constructed independently from the other one. Vuza also proves that when the pair (L, A) satisfies $L \oplus A = Z_n$, then also (kL, A), (kL, kA) have this property for all $k \in Z_n^*$.

n	T_n	K_n
2	1	1
3	1	5
4	2	13
5	1	41
6	3	110
7	1	341
8	6	1035
9	4	3298
10	6	10550
11	1	34781
12	23	117455
13	1	397529
14	13	1.370798
15	25	4.780715
16	49	16788150
17	1	59451809
18	91	212.178317
19	1	761.456429
20	149	2749.100993
21	121	9973.716835
22	99	36347.760182
23	1	133022.502005
24	794	488685.427750
25	126	1.801445.810166
26	322	6.662133.496934
27	766	24.711213.822232
28	1301	91.910318.016551
29	1	342.723412.096889
30	3952	1281.025524.753966
31	1	4798.840870.353221
32	4641	18014.401038.596400
33	5409	67756.652509.423763
34	3864	255318.257892.932894
35	2713	963748.277489.391403
36	31651	3.643801.587330.857840
37	1	13.798002.875101.582409
38	13807	52.325390.403899.973926
39	40937	198.705759.014912.561995
40	64989	755.578639.350274.265100

Table 2: Number of non-isomorphic tiling canons and canons in Z_n

A regular complementary canon of maximal category which can be constructed by Vuza's algorithm will be called *Vuza constructible canon*. The following table shows the numbers of isomorphism classes of Vuza constructible canons for some values of n: (With $\#L$, $\#A$, and $\#(L, A)$ we denote the number of possibilities to determine essentially different aperiodic sets L and aperiodic sets A, and non-isomorphic canons (L, A) by using Vuza's algorithm.)

n	p_1	p_2	n_1	n_2	n_3	$\#L$	$\#A$	$\#(L, A)$
72	2	3	2	3	2	3	6	18
120	2	3	2	5	2	16	20	320
120	2	5	2	3	2	8	6	48
144	2	3	4	3	2	6	36	216
144	2	3	2	3	4	3	2808	8424
200	2	5	2	5	2	125	20	2500
240	2	3	4	5	2	32	120	3840
240	2	5	4	3	2	16	36	576

Table 3: Number of isomorphism classes of Vuza constructible canons in Z_n

In order to determine the complete number of isomorphism classes of Vuza constructible canons in Z_n for given n, we have to determine all possibilities to decompose n as in Corollary 10 and sum up the number of isomorphism classes of Vuza constructible canons for these parameters. For instance, for $n = 72$ we have 18 isomorphism classes of canons with $|L| = 6$, and by interchanging L and A also18 isomorphism classes of canons with $|L| = 12$, whence 36 isomorphism classes of Vuza constructible canons.

In order to determine the number of Vuza constructible canons of length 144 with $|L| = 12$, we first realize that Vuza's algorithm yields 6 different sets L and 36 different possibilities for A, which also consists of 12 elements. So, when exchanging the role of L and A, we have to take care to count only non-isomorphic canons. As a matter of fact, here in this case the process of exchanging L and A yields new canons, so that in addition to the previous 216 canons (L, A) we have another 216 canons of the form (A, L). Hence, we end up with 432 Vuza constructible canons of length 144 with $|L| = |A| = 12$. Moreover, there exist 8424 Vuza constructible canons with $|L| = 6$ and 8424 Vuza constructible canons with $|L| = 24$, so that in conclusion there are 17280 Vuza constructible canons of length 144.

Finally, we want to deal with the question whether there exist regular complementary canons of maximal category which are not Vuza constructible canons. In Fripertinger (2002) for an integer $d \geq 1$ we introduced the function ψ_d defined on $\{0, 1\}$ such that $\psi_d(0)$ is the vector $(0, 0, \ldots, 0)$ consisting of d entries of 0, and $\psi_d(1) = (0, \ldots, 0, 1)$ is a vector consisting of $d - 1$ entries of 0 and 1 in the last position. We write the values of ψ_d in the form

$$\psi_d(0) = 0^d, \qquad \psi_d(1) = 0^{d-1}1.$$

If we apply ψ_d to each component of a vector $f \in \{0, 1\}^{Z_n}$ we get a vector $\psi_d(f) \in \{0, 1\}^{Z_{nd}}$ by concatenating all the vectors $\psi_d(f(0)), \ldots, \psi_d(f(n-1))$. Among other properties we showed that

1. $f_0 \in \{0,1\}^{Z_n}$ is the canonical representative of $C_n(f)$ if and only if $\psi_d(f_0)$ is the canonical representative of $C_{nd}(\psi_d(f))$.
2. $f \neq 0$ is acyclic if and only if $\psi_d(f)$ is acyclic.

The mapping ψ_d can be considered as an augmentation, mapping a rhythm in Z_n to an augmented rhythm in Z_{nd}. In correspondence with ψ_d, we define the mapping Ψ_d which can be interpreted as k-fold subdivision. It also maps a rhythm in Z_n to a rhythm in Z_{nd}. It is defined on $\{0,1\}$ by

$$\Psi_d(0) = 0^d, \qquad \Psi_d(1) = 1^d.$$

Again the value $\Psi_d(f)$ for $f \in \{0,1\}^{Z_n}$ is obtained by concatenation of the vectors $\Psi_d(f(0)), \ldots, \Psi_d(f(n-1))$. It is easy to prove that the mapping Ψ_d satisfies

1. $f_0 \in \{0,1\}^{Z_n}$ is the canonical representative of $C_n(f)$ if and only if $\Psi_d(f_0)$ is the canonical representative of $C_{nd}(\Psi_d(f))$.
2. Assume that $n > 1$. The mapping $f \in \{0,1\}^{Z_n}$ is acyclic if and only if $\Psi_d(f)$ is acyclic.

These two functions ψ_d and Ψ_d can be used in order to show that there exist regular complementary canons of maximal category which are not Vuza constructible canons.

Theorem 12. *Let $d > 1$ and assume that (L, A) is a Vuza constructible canon in Z_n. Then $(\Psi_d(L), \psi_d(A))$ is a regular complementary canon of maximal category in Z_{nd}.*

Among the 432 complementary canons of maximal category of length $2 \cdot 72 = 144$ with $|L| = 12$ we did not find a canon which was constructed in this way for $d = 2$ from the 18 canons of length 72 with $|L| = 6$.

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