Group actions and the functional equation of the mean sun

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Abstract

A generalization of the functional equation $g(s+t)x(u) = g(s)x(t+u)$ $(\forall s, t, u \in R)$ of the mean sun is studied, where a group G acts on a set X, $(R, +)$ is a not necessarily commutative group and both $x : R \to X$ and $g : R \to G$ are unknown functions, which will be determined by the equation.

Résumé

Une généralisation de l'équation fonctionnelle $g(s+t)x(u) = g(s)x(t+u)$ $(\forall s, t, u \in R)$ du soleil moyen est examinée, où une groupe agit sur un ensemble X, $(R, +)$ est un groupe, mais pas nécessairement commutatif, et $x : R \to X$ et aussi $g : R \to G$ sont des fonctions inconnues lesquelles seront déterminées par l'équation fonctionnelle.

Local solar time is measured by a sundial. When the center of the sun is on an observer's meridian, the observer's local solar time is zero hours (noon). Because the earth moves with varying speed in its orbit at different times of the year and because the plane of the earth's equator is inclined to its orbital plane, the length of the solar day is different depending on the time of year. It is more convenient to define time in terms of the average of local solar time. Such time, called mean solar time, may be thought of as being measured relative to an imaginary sun (the mean sun) that lies in the earth's equatorial plane and about which the earth orbits with constant speed. Every mean solar day is of the same length.[1]

In [5, 2] it is shown that the mean sun satisfies the functional equation

$$M(\lambda + t, \phi)^T y(s) = M(\lambda, \phi)^T y(s + t) \qquad \forall s, t, \lambda \in \mathbb{R},\ -\pi/2 < \phi < \pi/2$$

where $y(s)$ is a vector of length 1 which is the direction from the center of the earth to the sun at the time s (one day corresponds to 2π) expressed in a geocentric coordinate

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[1] http://www.infoplease.com/ce6/society/A0845838.html

system. As a basis of this system we can choose two orthogonal vectors in the equatorial plane and one vector along the axis of the earth. $M(\lambda, \phi)$ is the matrix

$$M(\lambda, \phi) = \begin{pmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{pmatrix}.$$

Then $M(\lambda, \phi)y(s)$ is the direction from the earth to the sun expressed in a local coordinate system on the surface of the earth in the point of longitude λ and latitude ϕ.

In the present paper we investigate a generalization of this equation for fixed ϕ. To be more precise we will deal with the following problem:

Let $(G, \cdot)$ be a group acting on a set X (cf. [1, 3, 4, 6]) and let $(R, +)$ be a not necessarily commutative group. Find all functions $x : R \to X$ and $g : R \to G$ which satisfy

$$g(s + t)x(u) = g(s)x(t + u), \qquad \forall s, t, u \in R. \tag{1}$$

The group R is a generalization of $\mathbb{R}$, the matrices expressing the change of the coordinate system are now elements of the group G and X represents a generalization of the set of all vectors in $\mathbb{R}^3$.

To begin with we will collect some properties of the functions g and x. Later we determine all solutions g and x of (1). In a first step we replace g by another function $h : R \to G$ defined by

$$h(r) := g(0)^{-1}g(r).$$

It is easy to prove that $h(0) = 1 \in G$ and

$$h(s)x(u) = x(s + u) \qquad \forall s, u \in R. \tag{2}$$

Lemma 1. *If the functions $x : R \to X$ and $h : R \to G$ satisfy (2) then for arbitrary $g_0 \in G$ the function $g : R \to G$ defined by $g(r) := g_0h(r)$ satisfies (1).*

Proof. $g(s+t)x(u) = g_0h(s+t)x(u) = g_0x(s+t+u) = g_0h(s)x(t+u) = g(s)x(t+u)$. $\square$

Furthermore the function h satisfies

$$h(s + t)x(u) = h(s)h(t)x(u) \qquad \forall s, t, u \in R, \tag{3}$$

since $h(s + t)x(u) = g(0)^{-1}g(s + t)x(u) = g(0)^{-1}g(s)x(t + u) = h(s)h(t)x(u)$.

For the rest of the paper we will work with h instead of g. For $x \in X$ let G_x denote the stabilizer of x, i.e.

$$G_x := \{g \in G \mid gx = x\},$$

which is a subgroup of G. From (3) we deduce that $(h(s)h(t))^{-1}\, h(s + t) \in G_{x(u)}$ for all $u \in R$ and all $s, t \in R$. In other words

$$(h(s)h(t))^{-1}\, h(s + t) \in \bigcap_{u\in R} G_{x(u)} =: \hat{G}.$$

Using this for $t = -s$ we see that there exists $g_s \in \hat{G}$ such that $h(s)^{-1} = h(-s)g_s$ and for $s = -t$ there is $g'_t \in \hat{G}$ such that $h(t)^{-1} = g'_t h(-t)$.

Let $H := \langle h(R) \rangle$ and $\tilde{G} := \hat{G} \cap H$ then the following lemma holds.

Lemma 2. *The subgroup $\tilde{G}$ of H is normal.*

Proof. It is clear that $\tilde{G}$ is a subgroup of H. We only have to prove that it is a normal subgroup. From the definition of H we know that

$$H = \left\{ \prod_{i=1}^{n} h(r_i)^{j_i} \mid n \in \mathbb{N},\ r_i \in R,\ j_i \in \{1, -1\} \right\}.$$

So it is enough to prove that $h(r)\tilde{G}h(r)^{-1} \leq \tilde{G}$ and $h(r)^{-1}\tilde{G}h(r) \leq \tilde{G}$ for all $r \in R$. Let $r, u \in R$ and $g \in \tilde{G}$ then there is a $g'_r \in \hat{G}$ such that $h(r)gh(r)^{-1}x(u) = h(r)gg'_r h(-r)x(u) = h(r)gg'_r x(-r+u) = h(r)x(-r+u) = x(r-r+u) = x(u)$ since $gg'_r \in \hat{G}$ stabilizes each element of the form $x(t)$. This means, since g was an arbitrary element of $\tilde{G}$, that

$$h(r)\tilde{G}h(r)^{-1} \leq G_{x(u)} \qquad \forall u \in R$$

so $h(r)\tilde{G}h(r)^{-1} \leq \hat{G} \cap H = \tilde{G}$. For the second part of the proof similar arguments can be used. □

This permits to define a function φ from R to the factor group $H/\tilde{G}$ by

$$\varphi(r) := h(r)\tilde{G} =: \overline{h(r)}.$$

Lemma 3. *The mapping φ is a surjective group homomorphism.*

Proof. For $s, t \in R$ we know from (3) that $h(s)h(t) \in h(s+t)\tilde{G}$. So

$$\varphi(s+t) = h(s+t)\tilde{G} = h(s)h(t)\tilde{G} = h(s)\tilde{G}h(t)\tilde{G} = \varphi(s)\varphi(t).$$

In order to prove that φ is surjective let

$$y := \left(\prod_{i=1}^{n} h(r_i)^{j_i} \right) \tilde{G} = \overline{\prod_{i=1}^{n} h(r_i)^{j_i}} \in H/\tilde{G}.$$

Then

$$y = \prod_{i=1}^{n} \overline{h(r_i)^{j_i}} = \prod_{i=1}^{n} \overline{h(r_i)}^{j_i} = \prod_{i=1}^{n} \varphi(r_i)^{j_i} = \prod_{i=1}^{n} \varphi(j_i \cdot r_i) = \varphi\left(\sum_{i=1}^{n} j_i \cdot r_i \right)$$

and $\sum_{i=1}^{n} j_i \cdot r_i \in R$. □

Even the following result is true.

Lemma 4. *If a subgroup N of $G_{x(0)}$ is a normal subgroup of H then N is a subgroup of $\tilde{G}$.*

Proof. It is enough to prove that N is a subgroup of $G_{x(u)}$ for all $u \in R$, because then N is a subgroup of $\hat{G}$. By assumption $N \leq H$, so $N \leq \tilde{G}$. From (2) it is clear that $x(u) = h(u)x(0)$ for all $u \in R$, so $G_{x(u)} = G_{h(u)x(0)} = h(u)G_{x(0)}h(u)^{-1}$. Since N is a normal subgroup of H it is obvious that $N = h(u)Nh(u)^{-1} \leq h(u)G_{x(0)}h(u)^{-1} = G_{x(u)}$. $\square$

So far we derived necessary conditions for solutions of (2). Conversely consider a group G acting on a set X. Let H be a subgroup of G, x_0 an arbitrary element of X and $\tilde{G}$ a normal subgroup of H such that $\tilde{G}$ is a subgroup of the stabilizer G_{x_0}. Then the factor group $H/\tilde{G}$ acts on the orbit $H(x_0) := \{hx_0 \mid h \in H\}$ in the following way:

$$H/\tilde{G} \times H(x_0) \to H(x_0) \qquad (\bar{h}, kx_0) \mapsto (hk)x_0. \tag{4}$$

In order to prove that this action is well defined consider an arbitrary $g \in \tilde{G}$. Since $\tilde{G}$ is a normal subgroup of H there exists $g' \in \tilde{G}$ such that $gk = kg'$. From

$$(hg)kx_0 = h(gk)x_0 = h(kg')x_0 = (hk)g'x_0 = (hk)x_0$$

we derive that the action of $\bar{h}$ on $H(x_0)$ does not depend on the special choice of the representative of $\bar{h}$. Furthermore it is clear that $\bar{1}kx_0 = 1kx_0 = kx_0$ and $(\bar{h}_1\bar{h}_2)kx_0 = \overline{h_1h_2}kx_0 = (h_1h_2)kx_0 = h_1(h_2k)x_0 = \bar{h}_1(h_2kx_0) = \bar{h}_1(\bar{h}_2kx_0)$ for all $\bar{h}_1, \bar{h}_2 \in H/\tilde{G}$. Moreover $\tilde{G}$ is a subgroup of all the stabilizers G_{hx_0} for all $h \in H$ since

$$\tilde{G} = h\tilde{G}h^{-1} \leq hG_{x_0}h^{-1} = G_{hx_0}.$$

Lemma 5. *Let $\varphi : R \to H/\tilde{G}$ be a homomorphism. When defining the two functions x and h by $x(r) := \varphi(r)x_0$, and $h(r)$ being an arbitrary element in the coset $\varphi(r)$ for $r \in R$ then h and x satisfy (2).*

Proof. $h(s)x(u) = \varphi(s)\varphi(u)x_0 = \varphi(s+u)x_0 = x(s+u)$ for all $s, u \in R$. $\square$

These results are summarized in the following

Theorem 6. *The functions $x : R \to X$ and $h : R \to G$ satisfy (2) if and only if there exist $x_0 \in X$, a subgroup H of G, a normal subgroup $\tilde{G}$ of H which is a subgroup of the stabilizer G_{x_0} and a homomorphism $\varphi : R \to H/\tilde{G}$ such that*

$$x(r) = \varphi(r)x_0 \textit{ and } h(r) \in \varphi(r) \qquad \forall r \in R$$

where the natural action of the factor group $H/\tilde{G}$ the orbit $H(x_0)$ is described by (4).

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