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The functional equation of the mean sun written as a group action

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Abstract

A generalization $g(s+t)x(u) = g(s)x(t+u)$ $(\forall s, t, u \in R)$ of the functional equation of the mean sun is studied, where a group G acts on a set X, $(R, +)$ is a not necessarily commutative group and both $x : R \to X$ and $g : R \to G$ are unknown functions, which will be determined by the equation.

Local solar time is measured by a sundial. When the center of the sun is on an observer's meridian, the observer's local solar time is zero hours (noon). Because the earth moves with varying speed in its orbit at different times of the year and because the plane of the earth's equator is inclined to its orbital plane, the length of the solar day is different depending on the time of year. It is more convenient to define time in terms of the average of local solar time. Such time, called mean solar time, may be thought of as being measured relative to an imaginary sun (the mean sun) that lies in the earth's equatorial plane and about which the earth orbits with constant speed. Every mean solar day is of the same length.[1]

In [5, 2] it is shown that the mean sun satisfies the functional equation

$$M(\lambda + t, \phi)^T y(s) = M(\lambda, \phi)^T y(s + t) \qquad \forall s, t, \lambda \in \mathbb{R},\ -\pi/2 < \phi < \pi/2,$$

where $y(s)$ is a vector of length 1 which is the direction from the center of the earth to the sun at the time s (one day corresponds to 2π) expressed in a geocentric coordinate system. As a basis of this system we can choose two orthogonal vectors in the equatorial plane and one vector along the axis of the earth. $M(\lambda, \phi)$ is the matrix

$$M(\lambda, \phi) = \begin{pmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{pmatrix}.$$

Then $M(\lambda, \phi)y(s)$ is the direction from the earth to the sun expressed in a local coordinate system on the surface of the earth in the point of longitude λ and latitude ϕ. This

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[1] http://www.infoplease.com/ce6/society/A0845838.html

local coordinate frame is given by the unit vectors indicating the directions East, North, and to the zenith.

In the present paper we investigate a generalization of this equation for fixed ϕ. To be more precise, we will deal with the following problem:

Let $(G,\cdot)$ be a group acting on a set X (cf. [1, 3, 4, 6]) and let $(R,+)$ be a not necessarily commutative group. Find all functions $x : R \to X$ and $g : R \to G$, which satisfy

$$g(s+t)x(u) = g(s)x(t+u), \qquad \forall s,t,u \in R. \tag{1}$$

The group R is a generalization of $\mathbb{R}$, the matrices expressing the change of the coordinate system are now elements of the group G, and X represents a generalization of the set of all vectors in $\mathbb{R}^3$.

First we want to give some basic definitions about group actions. A group action of the (multiplicative) group G on the set X is given by a mapping

$$G \times X \to X, \qquad (\gamma, x) \mapsto \gamma x,$$

which satisfies $1x = x$ and $(\gamma_2\gamma_1)x = \gamma_2(\gamma_1 x)$ for all $x \in X$ and $\gamma_1, \gamma_2 \in G$, where 1 is the identity element in G. In other words, a group action describes a homomorphism from G to the set of all bijections on X, which is a group together with the composition of functions as the multiplication in the group. Conversely, each homomorphism of this kind determines a group action of G on X.

A group action of G on X defines an equivalence relation on X. Two elements x_1, x_2 of x are called G-equivalent, $x_1 \sim_G x_2$, if and only if there is some $\gamma \in G$ such that $x_2 = \gamma x_1$. The equivalence classes $G(x)$ with respect to $\sim_G$ are called orbits of G on X, i.e.

$$G(x) = \{\gamma x \mid \gamma \in G\}.$$

For each $x \in X$ the stabilizer G_x of x, which is the set

$$G_x := \{\gamma \in G \mid \gamma x = x\}$$

is a subgroup of G.

Two elements x and y of the same orbit have conjugate stabilizers. To be more precise, if $y = \gamma x$ for $\gamma \in G$, then $G_y = \gamma G_x \gamma^{-1}$.

Coming back to the functional equation (1), we start with collecting some properties of the functions g and x. Later we determine all solutions g and x of (1). In a first step we replace g by another function $h : R \to G$, defined by

$$h(r) := g(0)^{-1} g(r).$$

The properties of h are described in

Lemma 1. *Assume that (g, x) is a solution of (1). Then the function h, defined above, satisfies $h(0) = 1 \in G$,*

$$h(s)x(u) = x(s + u), \qquad \forall s, u \in R, \tag{2}$$

and

$$h(s + t)x(u) = h(s)h(t)x(u), \qquad \forall s, t, u \in R. \tag{3}$$

Proof. The first statement is clear from the definition of h. Then $h(s)x(u) = g(0)^{-1}g(s)x(u) = g(0)^{-1}g(0 + s)x(u) = g(0)^{-1}g(0)x(s + u) = x(s + u)$ for all $s, u \in R$. Finally $h(s + t)x(u) = g(0)^{-1}g(s + t)x(u) = g(0)^{-1}g(s)x(t + u) = h(s)h(t)x(u)$ for all $s, t, u \in R$. □

It is also possible to determine g by h.

Lemma 2. *If the functions $x : R \to X$ and $h : R \to G$ satisfy (2), then for arbitrary $g_0 \in G$ the function $g : R \to G$ defined by $g(r) := g_0 h(r)$ together with x satisfies (1).*

Proof. $g(s+t)x(u) = g_0 h(s+t)x(u) = g_0 x(s+t+u) = g_0 h(s)x(t+u) = g(s)x(t+u)$. □

For the rest of the paper we will work with h instead of g. As was indicated earlier, for $x \in X$ let G_x denote the stabilizer of x. From (3) we deduce that $(h(s)h(t))^{-1} h(s+t) \in G_{x(u)}$ for all $u \in R$ and all $s, t \in R$. In other words

$$(h(s)h(t))^{-1} h(s + t) \in \bigcap_{u \in R} G_{x(u)} =: \hat{G}.$$

Using this for $t = -s$, we see that there exists $\gamma_s \in \hat{G}$, such that $h(s)^{-1} = h(-s)\gamma_s$. And for $s = -t$ there is $\gamma'_t \in \hat{G}$, such that $h(t)^{-1} = \gamma'_t h(-t)$.

Let $H := \langle h(R) \rangle$ and $\tilde{G} := \hat{G} \cap H$, then the following lemma holds.

Lemma 3. *The subgroup $\tilde{G}$ of H is normal.*

Proof. It is clear that $\tilde{G}$ is a subgroup of H. We only have to prove that it is a normal subgroup. From the definition of H we know that

$$H = \left\{ \prod_{i=1}^{n} h(r_i)^{j_i} \mid n \in \mathbb{N},\ r_i \in R,\ j_i \in \{1, -1\} \right\}.$$

So it is enough to prove that $h(r)\tilde{G}h(r)^{-1} \leq \tilde{G}$ and $h(r)^{-1}\tilde{G}h(r) \leq \tilde{G}$ for all $r \in R$. Let $r, u \in R$ and $\gamma \in \tilde{G}$, then there is a $\gamma'_r \in \hat{G}$, such that $h(r)\gamma h(r)^{-1}x(u) = h(r)\gamma\gamma'_r h(-r)x(u) = h(r)\gamma\gamma'_r x(-r + u) = h(r)x(-r + u) = x(r - r + u) = x(u)$, since $\gamma\gamma'_r \in \hat{G}$ stabilizes each element of the form $x(t)$. This means, since γ was an arbitrary element of $\tilde{G}$, that

$$h(r)\tilde{G}h(r)^{-1} \leq G_{x(u)}, \qquad \forall u \in R,$$

so $h(r)\tilde{G}h(r)^{-1} \leq \hat{G} \cap H = \tilde{G}$. For the second part of the proof similar arguments can be used. □

This permits to define a function φ from R to the factor group $H/\tilde{G}$ by

$$\varphi(r) := h(r)\tilde{G} =: \overline{h(r)}.$$

Lemma 4. *The mapping φ is a surjective group homomorphism.*

Proof. For $s, t \in R$ we know from (3) that $h(s)h(t) \in h(s+t)\tilde{G}$. So

$$\varphi(s+t) = h(s+t)\tilde{G} = h(s)h(t)\tilde{G} = h(s)\tilde{G}h(t)\tilde{G} = \varphi(s)\varphi(t).$$

In order to prove that φ is surjective let

$$y := \left(\prod_{i=1}^{n} h(r_i)^{j_i}\right)\tilde{G} = \overline{\prod_{i=1}^{n} h(r_i)^{j_i}} \in H/\tilde{G}.$$

Then

$$y = \prod_{i=1}^{n} \overline{h(r_i)^{j_i}} = \prod_{i=1}^{n} \overline{h(r_i)}^{j_i} = \prod_{i=1}^{n} \varphi(r_i)^{j_i} = \prod_{i=1}^{n} \varphi(j_i \cdot r_i) = \varphi\left(\sum_{i=1}^{n} j_i \cdot r_i\right)$$

and $\sum_{i=1}^{n} j_i \cdot r_i \in R$. □

Even the following result is true.

Lemma 5. *If a subgroup N of $G_{x(0)}$ is a normal subgroup of H, then N is a subgroup of $\tilde{G}$.*

Proof. It is enough to prove that N is a subgroup of $G_{x(u)}$ for all $u \in R$, because then N is a subgroup of $\hat{G}$. By assumption $N \leq H$, so $N \leq \tilde{G}$. From (2) it is clear that $x(u) = h(u)x(0)$ for all $u \in R$, so $G_{x(u)} = G_{h(u)x(0)} = h(u)G_{x(0)}h(u)^{-1}$. Since N is a normal subgroup of H, it is obvious that $N = h(u)Nh(u)^{-1} \leq h(u)G_{x(0)}h(u)^{-1} = G_{x(u)}$. □

So far we derived necessary conditions for solutions of (2). Before describing sufficient conditions we prove a general result about group actions.

Lemma 6. *Consider a group G acting on a set X. Let S be a subgroup of G, x_0 an arbitrary element of X, and N a normal subgroup of S, such that N is a subgroup of the stabilizer G_{x_0}. Then the factor group S/N acts on the orbit $S(x_0)$ in the following way:*

$$S/N \times S(x_0) \to S(x_0) \qquad (\bar{\eta}, \kappa x_0) \mapsto (\eta\kappa)x_0. \tag{4}$$

Proof. In order to prove that this action is well defined, consider an arbitrary $\nu \in N$. Since N is a normal subgroup of S, for each $\kappa \in S$ there exists $\nu' \in N$, such that $\nu\kappa = \kappa\nu'$. From

$$(\eta\nu)\kappa x_0 = \eta(\nu\kappa)x_0 = \eta(\kappa\nu')x_0 = (\eta\kappa)\nu' x_0 = (\eta\kappa)x_0,$$

we derive that the action of $\bar{\eta}$ on $H(x_0)$ does not depend on the special choice of the representative of $\bar{\eta}$. Furthermore, it is clear that $\bar{1}\kappa x_0 = 1\kappa x_0 = \kappa x_0$, and $(\bar{\eta}_1\bar{\eta}_2)\kappa x_0 = \overline{\eta_1\eta_2}\kappa x_0 = (\eta_1\eta_2)\kappa x_0 = \eta_1(\eta_2\kappa)x_0 = \bar{\eta}_1(\eta_2\kappa x_0) = \bar{\eta}_1(\bar{\eta}_2\kappa x_0)$ for all $\bar{\eta}_1, \bar{\eta}_2 \in S/N$. □

Under the assumptions of the last lemma, N is a subgroup of each stabilizer $G_{\eta x_0}$ for $\eta \in S$, since

$$N = \eta N \eta^{-1} \leq \eta G_{x_0} \eta^{-1} = G_{\eta x_0}.$$

Lemma 7. *Let G, x_0, S and N be given as in Lemma 6 and let $\psi : R \to S/N$ be a homomorphism. When defining the two functions x and h by $x(r) := \psi(r)x_0$, and $h(r)$ being an arbitrary element in the coset $\psi(r)$ for $r \in R$, then h and x satisfy (2).*

Proof. $h(s)x(u) = \psi(s)\psi(u)x_0 = \psi(s+u)x_0 = x(s+u)$ for all $s, u \in R$. □

Finally all these results are summarized in

Theorem 8. *The functions $x : R \to X$ and $h : R \to G$ satisfy (2) if and only if there exist $x_0 \in X$, a subgroup S of G, a normal subgroup N of S, which is a subgroup of the stabilizer G_{x_0}, and a homomorphism $\psi : R \to S/N$, such that*

$$x(r) = \psi(r)x_0 \text{ and } h(r) \in \psi(r), \qquad \forall r \in R,$$

where the natural action of the factor group S/R on the orbit $S(x_0)$ is described by (4).

Proof. If x and h satisfy (2), then choose $x_0 = x(0)$. Moreover, $S = H$, $N = \tilde{G}$, and $\psi = \varphi$ satisfy the above conditions by construction, Lemma 3 and Lemma 4. Finally, by construction it is clear that $h(r) \in \varphi(r)$, and from (2) it follows that $\varphi(r)x_0 = \bar{h}(r)x(0) = x(r)$ for all $r \in R$.

The rest of the present theorem follows immediately from Lemma 7. □

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