On covariant embeddings of a linear functional equation with respect to an analytic iteration group

Joint work with Ludwig Reich

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for the unknown series $\varphi(x) \in \mathbb{C}\llbracket x \rrbracket$.

L. Reich introduced the following notion:

The linear functional equation (L) has a covariant embedding with respect to the analytic iteration group $(\pi(s,x))_{s\in\mathbb{C}}$ of $p(x)$, if there exist families $(\alpha(s,x))_{s\in\mathbb{C}}$ and $(\beta(s,x))_{s\in\mathbb{C}}$ of formal power series with entire coefficient functions $\alpha_n(s)$ and $\beta_n(s)$ for all $n \geq 0$ such that the following equations

$$\varphi(\pi(s,x)) = \alpha(s,x)\varphi(x) + \beta(s,x) \qquad (Ls)$$

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$$\alpha(1,x) = a(x) \qquad \beta(1,x) = b(x) \qquad (B2)$$

hold for all $s,t \in \mathbb{C}$ and for all solutions $\varphi(x)$ of (L) in $\mathbb{C}[\![x]\!]$.

Covariant embeddings were studied in a much more general setting by Z. Moszner and for real-valued functions by G. Guzik.

Motivation

Computing the natural iterates of $p(x)$

$$p^n(x) := \begin{cases} x, & n = 0 \\ p(p^{n-1}(x)), & n > 0 \\ (p^{-1})^{-n}(x), & n < 0 \end{cases}$$

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we derive that each solution $\varphi(x)$ of (L) satisfies

$$\varphi(p^n(x)) = \alpha(n,x)\varphi(x) + \beta(n,x) \qquad (Ln)$$

for all $n \in \mathbb{Z}$, where $\alpha(n,x)$ and $\beta(n,x)$ are given by

$$\alpha(n,x) := \begin{cases} \prod_{r=0}^{n-1} a(p^r(x)), & n \geq 0 \\ \frac{1}{\alpha(-n,p^n(x))}, & n < 0 \end{cases}$$

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and

$$\beta(n,x) := \begin{cases} \alpha(n,x) \sum_{r=0}^{n-1} \frac{b(p^r(x))}{\prod_{j=0}^{r} a(p^j(x))}, & n \geq 0 \\ -\alpha(n,x)\beta(-n,p^n(x)), & n < 0. \end{cases}$$

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They satisfy $(B1)$ and $(B2)$ and the following system of equations

$$\alpha(n+m,x) = \alpha(m,x)\alpha(n,p^m(x)) \tag{C1}$$

$$\beta(n+m,x) = \beta(m,x)\alpha(n,p^m(x)) + \beta(n,p^m(x)) \tag{C2}$$

for all $n, m \in \mathbb{Z}$.

Analytic iteration groups

A family $\pi := (\pi(s,\cdot))_{s\in\mathbb{C}}$ of series of order 1 is called an analytic iteration group, if all the coefficient functions $\pi_n(s)$ are entire functions and if the translation equation

$$\pi(t+s,x) = \pi(t,\pi(s,x)) \tag{T}$$

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There exist three different types of analytic iteration groups:

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1. $\pi(s,x) = S^{-1}(e^{\lambda s}S(x))$ for all $s \in \mathbb{C}$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $S(x) = x + s_2x^2 + \dots$. Each iteration group of this type is simultaneously conjugate to the iteration group $(e^{\lambda s}x)_{s\in\mathbb{C}}$.

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2. $\pi(s,x) = x + c_k s x^k + P_{k+1}^{(k)}(s)x^{k+1} + \ldots$ for all $s \in \mathbb{C}$, where $c_k \neq 0$, $k \geq 2$ and $P_r^{(k)}(s)$ are certain polynomials in s for $r > k$.

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1. $p(x) = x$ can trivially be embedded.
2. $p(x) \neq x$ and $p(x) = \rho x + c_2 x^2 + \ldots$, where $\rho \neq 0$. If ρ is **not** a complex root of 1, then let λ be a logarithm $\ln \rho$. There exists exactly one analytic iteration group π of type 1 such that $\pi(s, x) = e^{\lambda s} x + \ldots$.

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3. If ρ is a complex root of 1 and $\rho\neq 1$, then the series $p(x)$ need not have an analytic embedding. But if such a $p(x)$ has an analytic embedding, then it is of the first type. In this situation the embedding need not be unique.

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4. If $p(x) = x + c_k x^k + \dots$, with $c_k \neq 0$ and $k \geq 2$, then there exists exactly one analytic embedding of $p(x)$ in an iteration group of the second type.

Reduction of the problem for iteration groups of type 1

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If $\pi(s,x) = S^{-1}(e^{\lambda s}S(x))$, then we change to the corresponding system for the series:

$\tilde{\varphi} := \varphi \circ S^{-1}$,
$\tilde{a} := a \circ S^{-1}$,
$\tilde{b} := b \circ S^{-1}$,
$\tilde{\alpha}(s,y) := \alpha(s, S^{-1}(y))$,
$\tilde{\beta}(s,y) := \beta(s, S^{-1}(y))$
and $\tilde{\pi}(s,x) = e^{\lambda s}x$.

Solving the cocycle equation $(Co1)$

Lemma 1. Writing $\alpha(s,x)$ as $\alpha_0(s)\hat{\alpha}(s,x)$ with $\hat{\alpha}(s,x) = 1 + \hat{\alpha}_1(s)x + \ldots$, then α_0 is an exponential function, and $\hat{\alpha}$ is also a solution of $(Co1)$. Moreover $\alpha_0(s) = e^{\mu s}$ for some $\mu \in \mathbb{C}$.

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Theorem 1. The family $(\alpha(s,x))_{s\in\mathbb{C}}$ is a solution of $(Co1)$ if and only if there exist $\mu \in \mathbb{C}$ and $K(y) \in \mathbb{C}\llbracket y \rrbracket$, $\operatorname{ord} K(y) \geq 1$ such that

$$\alpha(s,x) = e^{\mu s} \exp \int_0^s K(\pi(\sigma,x))d\sigma.$$

Proof. Dividing $\alpha(s,x)$ by $\alpha_0(s) = e^{\mu s}$ and using the formal logarithm $(Co1)$ is equivalent to

$$\tilde{\alpha}(t+s,x) = \tilde{\alpha}(s,x) + \tilde{\alpha}(t,\pi(s,x)), \qquad (Co1')$$

for $\tilde{\alpha}(s,x) = \ln(\alpha(s,x)/e^{\mu s})$.

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for $\tilde{\alpha}(s,x) = \ln(\alpha(s,x)/e^{\mu s})$. If $\tilde{\alpha}$ is a solution of $(Co1')$, then coefficientwise differentiation of $(Co1')$ with respect to the variable t and the chain rule for this differentiation yields

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For $t = 0$ we get $\tilde{\alpha}'(s,x) = \tilde{\alpha}'(0,\pi(s,x))$.

Putting $K(y) := \tilde{\alpha}'(0,y)$, we obtain $\operatorname{ord} K(y) \geq 1$ and $\tilde{\alpha}'(s,x) = K(\pi(s,x))$. By coefficientwise integration it follows that

$$\tilde{\alpha}(s,x) = \int_0^s K(\pi(\sigma,x))d\sigma.$$

Solving the cocycle equation $(Co2)$

Now we assume that α satisfies $(Co1)$. Since α has an inverse with respect to multiplication, it is possible to define

$$\gamma(s,x) := \frac{\beta(s,x)}{\alpha(s,x)} \qquad \forall s \in \mathbb{C}.$$

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Lemma 2. The families α and β satisfy the system $((Co1),(Co2))$ if and only if α satisfies $(Co1)$ and γ is a solution of

$$\gamma(t+s,x) = \gamma(s,x) + \frac{\gamma(t,\pi(s,x))}{\alpha(s,x)}. \qquad (Co2')$$

Theorem 2. α and β are a solution of $(Co2)$ if and only if there exists a series $L(y) \in \mathbb{C}\llbracket y \rrbracket$ such that

$$\beta(s,x) = \alpha(s,x) \int_0^s \frac{L(\pi(\sigma,x))}{\alpha(\sigma,x)} d\sigma,$$

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Now we will describe a different form of representing the general solution of $(Co1)$ and of the system $((Co1),(Co2))$, involving as few integrals as possible.

Solutions of $(Co1)$

Theorem 3. If π is an iteration group of type 1, then α is a solution of $(Co1)$ if and only if there exist $\mu \in \mathbb{C}$ and a formal power series $E(x) = 1 + e_1 x + \ldots \in \mathbb{C}[\![x]\!]$ such that

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If π is an iteration group of type 2, then α is a solution of $(Co1)$ if and only if there exist $\mu \in \mathbb{C}$ and a formal power series $E(x) = 1 + e_1 x + \ldots \in \mathbb{C}[\![x]\!]$ such that

$$\alpha(s,x) = e^{\mu s} \underbrace{\prod_{n=1}^{k-1} \left(\exp \int_0^s \pi(\sigma,x)^n d\sigma \right)^{\kappa_n}}_{=:P(s,x)} \frac{E(\pi(s,x))}{E(x)},$$

with arbitrary $\kappa_n \in \mathbb{C}$. In both cases the series $E(x)$ is uniquely determined by α.

Solutions of $(Co2)$

Theorem 4. Let $E(x) := 1 + e_1 x + \ldots \in \mathbb{C}[\![x]\!]$, $e_0 \neq 0$, and assume that $F(x) \in \mathbb{C}[\![x]\!]$ and $\mu \in \mathbb{C}$.

If π is of type 1, then the family

$$\beta(s,x) := e^{\mu s} E(e^{\lambda s} x)[\ell_{n_0} s x^{n_0} + F(x) - e^{-\mu s} F(e^{\lambda s} x)]$$

together with α given in Theorem 3 is the general solution of $(Co2)$. (The summand $\ell_{n_0} s x^{n_0}$ occurs only if $\mu = n_0 \lambda$ for $n_0 \in \mathbb{N}_0$.)

If π is of type 2, then β defined by

$$\beta(s,x) := e^{\mu s} P(s,x) E(\pi(s,x))[F(x) - e^{-\mu s} \frac{F(\pi(s,x))}{P(s,x)}]$$

together with α given in the second part of Theorem 3 is the general solution of $(Co2)$.

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Under certain conditions, namely $\mu = 0$,
$n_0 := \min\{n \mid 1 \leq n \leq k-1,\ \kappa_n \neq 0\} = k-1$, and
$\kappa_{k-1} = n_1 c_k$ $(n_1 \in \mathbb{N})$ then

$$\beta(s,x) = E(\pi(s,x)) P(s,x) \cdot \left[F(x) - \frac{F(\pi(s,x))}{P(s,x)} + \right.$$

$$\left. + \ell''_{n_1+n_0} \int_0^s \frac{\pi(\sigma,x)^{n_1+n_0}}{P(\sigma,x)} d\sigma \right].$$

Solutions which satisfy the boundary conditions

We assume that $a(x)$ and $b(x)$ are given formal power series. Equation $(B1)$ is always satisfied. We only have to consider $(B2)$ for further investigations.

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If $\rho := e^{\lambda}$ is **not** a complex root of 1, then there exists exactly one series $E(x) = 1 + \ldots$ such that α satisfies both $(Co1)$ and $(B2)$.

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If $\rho := e^{\lambda}$ is **not** a complex root of 1, then there exists exactly one series $E(x) = 1 + \ldots$ such that α satisfies both $(Co1)$ and $(B2)$.
Otherwise, $J = J(\lambda) := \{n \in \mathbb{N} \mid n\lambda \in 2\pi i\mathbb{Z}\}$ is not empty. Then there exist formal power series $E(x)$ such that α satisfies both $(Co1)$ and $(B2)$ if and only if for $n \in J$ the coefficients a_n satisfy

$$a_n = -\sum_{r=1}^{n-1} a_r e_{n-r}.$$

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Theorem 6. If $\rho = e^{\lambda}$ is not a complex root of 1. Then there exists a suitable α, which satisfies $(Co1)$ and the two boundary conditions, such that there is exactly one β which satisfies together with α the cocycle equation $(Co2)$ and also the boundary conditions $(B1)$ and $(B2)$.

2. Let $\pi(s,x) = x + c_k s x^k + \ldots$ with $k \geq 2$ and $c_k \neq 0$ be an iteration group of type 2.
There exist $\mu \in \mathbb{C}$ and uniquely determined $P(s,x)$ and $E(x) = 1 + e_1 x + \ldots$ such that α satisfies $(Co1)$ and $(B2)$.
Assume that α is a solution of $(Co1)$ and $(B2)$.
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If $a_0 \neq 1$, then there exists exactly one β satisfying $(Co2)$ and $(B2)$.

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If $n_0 = k$, there exist solutons β, iff $b_n = 0$ for $0 \leq n < k$.
If $n_0 \neq k-1$, or $\kappa_{k-1} - nc_k \neq 0$ for all $n \in \mathbb{N}$, then there exists exactly one β iff $b_n = 0$ for all $0 \leq n < n_0$.

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If $n_0 = k$, there exist solutons β, iff $b_n = 0$ for $0 \leq n < k$.
If $n_0 \neq k-1$, or $\kappa_{k-1} - nc_k \neq 0$ for all $n \in \mathbb{N}$, then there exists exactly one β iff $b_n = 0$ for all $0 \leq n < n_0$.
If $n_0 = k-1$ and $\kappa_{k-1} = n_1 c_k$ for $n_1 \in \mathbb{N}$, two cases:
If $\mu \neq 0$, then there exist solutions β, iff $b_n = 0$ for all $0 \leq n < n_0$ and b_{n_1+k-1} satisfies a further condition.

2. Let $\pi(s,x) = x + c_k s x^k + \dots$ with $k \geq 2$ and $c_k \neq 0$ be an iteration group of type 2.
There exist $\mu \in \mathbb{C}$ and uniquely determined $P(s,x)$ and $E(x) = 1 + e_1 x + \dots$ such that α satisfies $(Co1)$ and $(B2)$.
Assume that α is a solution of $(Co1)$ and $(B2)$.
If $a_0 \neq 1$, then there exists exactly one β satisfying $(Co2)$ and $(B2)$.

Assume $a_0 = 1$. If $a(x) = x$, let $m_0 = k$, otherwise let $m_0 := \min\{n \in \mathbb{N} \mid a_n \neq 0\}$ and let $n_0 := \min\{m_0, k\}$.
If $n_0 = k$, there exist solutons β, iff $b_n = 0$ for $0 \leq n < k$.
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If $\mu = 0$, then there exist solutions β, iff $b_n = 0$ for all $0 \leq n < k-1$.
In these last two cases, however, β is not uniquely determined.

Solving the embedding problem

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First there exists a general method:

Theorem 9. Assume that the linear functional equation (L) has a solution $\varphi(x) \in \mathbb{C}[\![x]\!]$, and let $(\pi(s,x))_{s\in\mathbb{C}}$ be an analytic iteration group of $p(x)$. Furthermore, assume that α satisfies $(Co1)$ and the two boundary conditions $(B1)$ and $(B2)$. If there exists exactly one β, which also satisfies $(B1)$ and $(B2)$, such that (α, β) is a solution of $(Co2)$, then there exists an embedding of (L) with respect to the iteration group $(\pi(s,x))_{s\in\mathbb{C}}$.

Corollary.

1. If $\pi(s,x) = e^{\lambda s}x$ is an analytic iteration group of the first type, and e^{λ} is **not** a complex root of 1, then there exists an embedding of (L) with respect to the iteration group π.

Corollary.

1. If $\pi(s,x) = e^{\lambda s}x$ is an analytic iteration group of the first type, and e^{λ} is **not** a complex root of 1, then there exists an embedding of (L) with respect to the iteration group π.

2. If $\pi(s,x) = x + c_k s x^k + \ldots$ with $k \geq 2$ and $c_k \neq 0$ is an analytic iteration group of the second type, and

Corollary.

1. If $\pi(s,x) = e^{\lambda s}x$ is an analytic iteration group of the first type, and e^{λ} is **not** a complex root of 1, then there exists an embedding of (L) with respect to the iteration group π.

2. If $\pi(s,x) = x + c_k s x^k + \ldots$ with $k \geq 2$ and $c_k \neq 0$ is an analytic iteration group of the second type, and
$a_0 \neq 1$ or
$n_0 < k-1$ or
$n_0 = k-1$ and $a_{k-1} \neq n c_k$ for all $n \in \mathbb{N}$,
then there exists an embedding of (L) with respect to the iteration group π.

The non generic cases

So far the structure of the set of solutions of (L) did not play an explicit role. For the remaining situations it will be of importance. Detailed investigation of the following cases showed:

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1. Assume that π is an analytic embedding of $p(x) = \rho x + \ldots$ of type 1, where ρ is a complex root of 1 of order j_0. If $K = \emptyset$, then there exists a covariant embedding.

Assume that $K \neq \emptyset$. If

$$B(x) := \sum_{k=0}^{j_0-1} \frac{b(\rho^k x)}{\prod_{j=0}^{k} a(\rho^j x)}$$

is equal to 0, then (L) has solutions, but there is no covariant embedding of (L).

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1. Assume that π is an analytic embedding of $p(x) = \rho x + \dots$ of type 1, where ρ is a complex root of 1 of order j_0. If $K = \emptyset$, then there exists a covariant embedding.

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is equal to 0, then (L) has solutions, but there is no covariant embedding of (L). If $B(x) \neq 0$, then (L) has no solutions, but it is possible to find suitable $\alpha(s,x)$ and $\beta(s,x)$ satisfying the cocycle equations $(Co1)$ and $(Co2)$ and the boundary conditions $(B1)$ and $(B2)$. Hence, there is a covariant embedding of (L).

2. Assume that π is of type 2, $a(x) = 1 + a_{k-1}x^{k-1} + \dots$ with $a_{k-1} = n_1 c_k$ for $n_1 \in \mathbb{N}$.

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If $\mu \neq 0$, then there is no covariant embedding of (L) with respect to π.

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If $\mu \neq 0$, then there is no covariant embedding of (L) with respect to π.

If $\mu = 0$, then there exists a covariant embedding of (L) with respect to π.

3. Assume that π is of type 2, $a(x) = 1$, or $a(x) = 1 + a_k x^k + \ldots$. Similarly as in the second case depending on μ there exists or does not exist a covariant embedding of (L) with respect to π.

4. Using similar methods, we finally solved the problem of covariant embeddings also for the improper functional equation

$$\varphi(x) = a(x)\varphi(x) + b(x)$$

for $p(x) = x$.

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