The formal translation equation for iteration groups of type II

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Formal functional equations for iteration groups of type I see H.F. and L. Reich: *The formal translation equation and formal cocycle equations for iteration groups of type I*, Aequationes Math., 76: 54-91, 2008.

The study of iteration groups of type II is much more complicate and interesting than the study of iteration groups of type I.

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(T) implies $c_1(s+t) = c_1(s)c_1(t)$, $s,t \in \mathbb{C}$.

If $c_1 \neq 1$, then $F(s,x)$ is of type I.

If $F(s,x) \neq x$ and if $c_1 = 1$, then $F(s,x)$ is of type II.

$\exists k \geq 2$: $F(s,x) = x + c_k(s)x^k + \sum_{n>k} c_n(s)x^n$

c_k is additive, $c_n(s) = P_n(c_k(s))$, $P_n(y) \in \mathbb{C}[y]$.

The formal translation equation

If a is a nontrivial additive function, then

a takes infinitely many values and

$P(x_1,x_2) \in \mathbb{C}[x_1,x_2]$, $P(a(s),a(t)) = 0$ for all $s,t \in \mathbb{C} \Longrightarrow P = 0$.

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Formal translation equation in $(\mathbb{C}[y,z])[[x]]$:

$$G(y+z,x) = G\big(y, G(z,x)\big) \qquad (\mathrm{T}_{\mathrm{formal}})$$

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n, \quad P_n(y) \in \mathbb{C}[y], \quad n > k \geq 2,$$

$$G(0,x) = x. \qquad (\mathrm{B})$$

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Theorem. $F(s,x) = x + c_k(s)x^k + \sum_{n>k} P_n(c_k(s))x^n$ is a solution of (T) if and only if $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n$ is a solution of $(\mathrm{T}_{\mathrm{formal}})$ and (B).

Differentiation in $(\mathbb{C}[y])[[x]]$

In $\mathbb{C}[y]$ we have the formal derivation with respect to y.

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Differentiation is now a purely algebraic process!

We are looking for relations between the solutions $G(y,x)$ of $(\mathrm{T}_{\mathrm{formal}})$ and the infinitesimal generator $H(x)$ of G

$$\frac{\partial}{\partial y} G(y,x)|_{y=0} = x^k + \sum_{n>k} h_n x^n = H(x).$$

Here $h_k := 1$. Notice that in the situation of an analytic iteration group the coefficient of x^k in $H(x)$ may be different from 1.

Three equations derived from $(\mathrm{T}_{\mathrm{formal}})$

Differentiation of $(\mathrm{T}_{\mathrm{formal}})$ with respect to y yields

$$\frac{\partial}{\partial t} G(t,x)\big|_{t=y+z} = \frac{\partial}{\partial y} G\big(y, G(z,x)\big).$$

For $y = 0$ we get

$$\frac{\partial}{\partial z} G(z,x) = H\big(G(z,x)\big). \qquad (\mathrm{D}_{\mathrm{formal}})$$

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Differentiation with respect to z together with the mixed chain rule

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Aczél–Jabotinsky differential equation

$$H(x) \frac{\partial}{\partial x} G(y,x) = H\big(G(y,x)\big). \qquad (\mathrm{AJ}_{\mathrm{formal}})$$

The differential equation $(\mathrm{PD}_{\mathrm{formal}})$ and (B)

1. For any generator $H(x) = x^k + \sum_{n>k} h_n x^n$ the differential equation $(\mathrm{PD}_{\mathrm{formal}})$ together with (B) has exactly one solution. It is given by

$$G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n \in (\mathbb{C}[y])[[x]].$$

2. The polynomials P_n, $n \geq k$, have a formal degree $\lfloor (n-1)/(k-1) \rfloor$ and they are of the form

$$P_n(y) = \begin{cases} h_n y & \text{if } k \leq n < 2k-1 \\ h_{2k-1} y + \frac{k}{2} y^2 & \text{if } n = 2k-1 \\ h_n y + \frac{n+1}{2} h_{n-k+1} y^2 + \Phi_n(y, h_{k+1}, \dots, h_{n-k}) & \text{if } n \geq 2k, \end{cases}$$

where Φ_n are polynomials in y and in the coefficients $h_{k+1}, \dots, h_{n-k}$. They satisfy $\Phi_n(0, h_{k+1}, \dots, h_{n-k}) = 0$.

The polynomial Φ_{2k} is of the form

$$\Phi_{2k}(y) = \begin{cases} \Phi_4(y) = y^3 & \text{if } k = 2 \\ 0 & \text{if } k > 2. \end{cases}$$

For $n \geq 2k$ a formal degree of Φ_n as a polynomial in y is $\lfloor (n-1)/(k-1) \rfloor$.

3. Each solution $G(y,x)$ of $(\mathrm{PD}_{\mathrm{formal}})$ and (B) is a solution of $(\mathrm{T}_{\mathrm{formal}})$.

We prove that

$$U(y,z,x) := G(y+z,x)$$
$$V(y,z,x) := G(z,G(y,x))$$

satisfy the system

$$\frac{\partial}{\partial y} f(y,z,x) = H(x)\frac{\partial}{\partial x} f(y,z,x)$$
$$f(0,z,x) = G(z,x),$$

and we further prove that this system has a unique solution in $(\mathbb{C}[y,z])[[x]]$. This shows that G satisfies $(\mathrm{T}_{\mathrm{formal}})$.

The differential equation $(\mathrm{D}_{\mathrm{formal}})$ and (B)

The same result for $(\mathrm{D}_{\mathrm{formal}})$ and (B).

Computations are more involved since we have to determine $H(G(y,x))$.

Lemma. $\nu \in \mathbb{N}$, then

$$[G(z,x)]^{\nu} = x^{\nu} + \nu P_k(z) x^{\nu+k-1} + \sum_{n > \nu+k-1} \left(\nu P_{n-\nu+1}(z) + Q_n^{(\nu)}(z) \right) x^n$$

for

$$Q_n^{(\nu)}(z) = \sum_{\substack{(j_1, j_k, \ldots, j_{n-\nu}) \in \mathbb{N}_0^{n-\nu-k+2} \\ \sum j_i = \nu \\ \sum i j_i = n}} \binom{\nu}{j_1\, j_k \ldots j_{n-\nu}} \prod_{i=k}^{n-\nu} P_i(z)^{j_i}.$$

The Aczél–Jabotinsky equation $(\mathrm{AJ}_{\mathrm{formal}})$ and (B)

More solutions for $(\mathrm{AJ}_{\mathrm{formal}})$ and (B).

Theorem. For any polynomial $P_k(y) \in \mathbb{C}[y]$ with $P_k(0) = 0$ the differential equation $(\mathrm{AJ}_{\mathrm{formal}})$ together with (B) has exactly one solution of the form

$$G(y,x) = x + P_k(y)x^k + \sum_{n>k} P_n(y)x^n \in (\mathbb{C}[y])[[x]].$$

The polynomials $P_n(y)$ for $n > k$ depend on $P_k(y)$ as described above.

For $P_k(y) = y$ we obtain the same solutions as of $(\mathrm{PD}_{\mathrm{formal}})$ (or $(\mathrm{D}_{\mathrm{formal}})$) and (B).

Computations are even more involved.

Reordering the summands

Solution of $(\mathrm{T}_{\mathrm{formal}})$: $G(y,x) = x + yx^k + \sum_{n>k} P_n(y)x^n \in (\mathbb{C}[y])[[x]]$

$$P_n(y) = \sum_{j=1}^{d_n} P_{n,j} y^j \in \mathbb{C}[y], \quad d_n = \lfloor \frac{n-1}{k-1} \rfloor, \qquad n \geq k,$$

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$$G(y,x) = \sum_{n\geq 0} \phi_n(x)y^n \in (\mathbb{C}[[x]])[[y]]$$

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$$G(y,x) = \sum_{n\geq 0} \phi_n(x)y^n \in (\mathbb{C}[[x]])[[y]]$$

$$\phi_n(x) = \sum_{r\geq k} P_{r,n}x^r = \sum_{r\geq n(k-1)+1} P_{r,n}x^r, \qquad n \geq 1.$$

$(\phi_n(x))_{n\geq 0}$ and $(\phi_n(x)y^n)_{n\geq 0}$ are summable families.

This allows us to rewrite $(\mathrm{PD}_{\mathrm{formal}})$ and (B) as

$$\sum_{n\geq 1} n\phi_n(x)y^{n-1} = H(x)\sum_{n\geq 0} \phi_n'(x)y^n \tag{1}$$

$$\phi_0(x) = x \tag{2}$$

(1) is satisfied if and only if

$$\phi_{n+1}(x) = \frac{1}{n+1} H(x)\phi_n'(x) \qquad (1_n)$$

holds true for all $n \geq 0$.

$$\begin{aligned}
\phi_1(x) &= H(x), \\
\phi_2(x) &= H(x)H'(x)/2, \\
\phi_3(x) &= H(x)\big(H(x)H'(x)\big)'/6 = \big(H(x)H'(x)^2 + H(x)^2H''(x)\big)/6. \qquad (3)
\end{aligned}$$

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Theorem. For any generator $H(x) = \sum_{n\geq k} h_n x^n$, $k \geq 2$, $h_k \neq 0$, the system $((1),(2))$ has a unique solution. The order of $\phi_n(x)$ is equal to $n(k-1)+1$ and $\phi_n(0) = 0$ for all $n \geq 0$.

Some results

1. The coefficients $P_{r,2}$, $r \geq 2k-1$, of $\phi_2(x)$ are of the form

$$P_{2k-1,2} = \frac{k}{2} h_k^2$$

$$P_{2k,2} = \frac{2k+1}{2} h_k h_{k+1}$$

$$P_{r,2} = \begin{cases} \frac{r+1}{2} \left(h_k h_{r+1-k} + \sum_{\nu=k+1}^{r/2} h_\nu h_{r+1-\nu} \right) & \text{if } r \equiv 0 \ (2) \\ \frac{r+1}{2} \left(h_k h_{r+1-k} + \sum_{\nu=k+1}^{(r-1)/2} h_\nu h_{r+1-\nu} + \frac{1}{2} h^2_{(r+1)/2} \right) & \text{if } r \equiv 1 \ (2). \end{cases}$$

2. Two coefficients of $\phi_{n+1}(x)$:

$$P_{(n+1)(k-1)+1,n+1} = \frac{h_k^{n+1}}{(n+1)!}\prod_{j=1}^{n}\big(j(k-1)+1\big), \qquad n \geq 0,$$

and

$$P_{(n+1)(k-1)+2,n+1} = \frac{h_k^n h_{k+1}}{(n+1)!}\sum_{r=1}^{n+1}\prod_{s=r+1}^{n+1}\big((s-1)(k-1)+2\big)\prod_{j=1}^{r-1}\big(j(k-1)+1\big)$$

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3. For $n \geq 1$ we have $\phi_n(x) =$

$$\frac{1}{n!}\sum_{r \geq n(k-1)+1}\left(\sum_{(\nu_1,\ldots,\nu_{n-1})}^{*r}\prod_{s=1}^{n-1}h_{\nu_s}\left(r+s-\sum_{t=1}^{s}\nu_t\right)h_{r+(n-1)-\sum_{t=1}^{n-1}\nu_t}\right)x^r$$

In $\sum_{(\nu_1,\ldots,\nu_{n-1})}^{*r}$ we are taking the sum over all $(n-1)$-tuples $(\nu_1,\ldots,\nu_{n-1})$ of integers, such that $k \leq \nu_s \leq r-(n-s)k+(n-1)-\sum_{t=1}^{s-1}\nu_t$.

4. A generalization of (3):

$$\phi_n(x) = \frac{1}{n!} \sum_{i \in I_n} K(i) \prod_{j=0}^{n-1} \left[H^{(j)}(x) \right]^{i_j}, \qquad n \geq 1,$$

$$I_n = \left\{ (i_j)_{j \geq 0} \,\middle|\, i_j \in \mathbb{Z},\ i_j \geq 0,\ i_0 \geq 1,\ \sum_{j=0}^{n-1} i_j = n,\ \sum_{j=1}^{n-1} j i_j = n-1 \right\},$$

$$K(1,0,0,\ldots) := 1, \quad K(v) := \sum_{\substack{u \in I_{n-1} \\ u \prec v}} \tilde{K}(u,v) K(u), \quad v \in I_n,\ n > 1,$$

$$\tilde{K}(u,v) := \begin{cases} u_0 & \text{if } (\prec_1) \text{ is applied} \\ u_{s-1} & \text{if } (\prec_2) \text{ is applied,} \end{cases}$$

$u \prec v$ if either $(\prec_1)$ or $(\prec_2)$, where

$$u_0 = v_0,\ u_1 = v_1 - 1,\ u_j = v_j \text{ for } j > 1 \qquad (\prec_1)$$

$$u_0 = v_0 - 1,\ \exists s > 1 : u_{s-1} = v_{s-1} + 1,\ u_s = v_s - 1,\ u_j = v_j,\ j \notin \{0, s-1, s\} \qquad (\prec_2)$$

5. Solution as a Lie–Gröbner-series:

$$G(y,x) := \sum_{n\geq 0} \frac{1}{n!} D^n(x) y^n,$$

where

$$D\colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad D(f(x)) := H(x) f'(x).$$

Normal forms

$$\Gamma_1 := \{f(x) \in \mathbb{C}[[x]] \mid f(x) \equiv x \bmod x^2\}$$

Theorem. For each $G(y,x)$ with generator $H(x) = \sum_{n\geq k} h_n x^n$, there exist some $S(x) \in \Gamma_1$ and exactly one $h \in \mathbb{C}$, so that

$$x^k + hx^{2k-1}$$

is the generator of $S^{-1}(G(y,S(x)))$.

We say $S^{-1}(G(y,S(x)))$ is in normal form.

We try to study the dependence of iteration groups of type II on the parameter h.

We assume that h is an indeterminate and we write

$$G(y,x) = \sum_{r\geq 0} G_r(y,x)h^r \in (\mathbb{C}[[x,y]])[[h]].$$

Normal forms and $(\mathrm{PD}_{\mathrm{formal}})$ and (B)

Consider $H(x) = x^k + hx^{2k-1}$, then

$$G(y,x) = \sum_{r\geq 0} G_r(y,x)h^r$$

where

$$G_r(y,x) = \sum_{n\geq r} \sum_{\substack{(j_1,\dots,j_r)\\ 1\leq j_1\\ j_s\geq j_{s-1}+2,\ s\geq 2\\ j_r\leq n+r-1}} \frac{\prod_{i=1}^{n+r-1}[i]}{\prod_{s=1}^{r}[j_s]} \frac{x^{[n+r]}}{n!} y^n, \qquad r\geq 0,$$

and

$$[r] := r(k-1)+1.$$

Normal forms and $(\mathrm{D}_{\mathrm{formal}})$ and (B)

Theorem. Consider $H(x) = x^k + hx^{2k-1}$, then

$$G(y,x) = \sum_{r\geq 0} G_r(y,x)h^r$$

where

$$G_r(y,x) = x^{[r]}(1-(k-1)yx^{k-1})^{-[r]/(k-1)}P_r(\ln(1-(k-1)yx^{k-1})), \qquad r \geq 0,$$

$[r] := r(k-1)+1$ and P_r are polynomials of degree r.
Moreover $P_0 = 1$ and $P_1(z) = -z/(k-1)$.

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$[r] := r(k-1)+1$ and P_r are polynomials of degree r.
Moreover $P_0 = 1$ and $P_1(z) = -z/(k-1)$.

$G_0(y,x) = x(1-(k-1)yx^{k-1})^{-1/(k-1)}$ is the only remaining term of $G(y,x)$ if $h = 0$. These functions play an important role in the theory of reversible formal power series (see J. Haneczok: *Conjugacy type problems in the ring of formal power series*, to appear in Grazer Math. Ber.)

Contents