On a functional equation by Th. Rassias

Harald Fripertinger
Karl-Franzens-Universität Graz
joint work with Jens Schwaiger

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The equation

$$rf\left(\frac{1}{r}\sum_{j=1}^{d}x_j\right)+\sum_{S\in\binom{\underline{d}}{\ell}} rf\left(\frac{1}{r}\left(\sum_{j\notin S}x_j-\sum_{j\in S}x_j\right)\right)$$
$$=\left(\binom{d-1}{\ell}-\binom{d-1}{\ell-1}+1\right)\sum_{j=1}^{d}f(x_j) \qquad (1)$$

$f\colon X\to Y$, X, Y Banach spaces ($\mathbb{K}$-vector spaces, $\mathbb{K}$ of characteristic 0)
$r\in\mathbb{Q}$, ℓ,d integers satisfying $1<\ell<d/2$,
$x_1,\ldots,x_d\in X$,
$\binom{\underline{d}}{\ell}$ is the set of all ℓ-subsets of $\underline{d}=\{1,\ldots,d\}$.

In [1] it is shown that if f is an odd solution, then f is additive. Moreover the Hyers–Ulam stability of this equation is studied. Actually, the equation itself is not so interesting for us as the applied method (based on a Theorem of L. Székelyhidi [4, Theorem 9.5, p. 73]) and the obtained results. We also investigate the situation $r \notin \mathbb{Q}$.

In [1] it is shown that if f is an odd solution, then f is additive. Moreover the Hyers–Ulam stability of this equation is studied. Actually, the equation itself is not so interesting for us as the applied method (based on a Theorem of L. Székelyhidi [4, Theorem 9.5, p. 73]) and the obtained results. We also investigate the situation $r \notin \mathbb{Q}$.

If we omit f in (1) we have

$$r\left(\frac{1}{r}\sum_{j=1}^{d} x_j\right) + \sum_{S\in\binom{d}{\ell}} r\left(\frac{1}{r}\left(\sum_{j\notin S} x_j - \sum_{j\in S} x_j\right)\right)$$
$$= \left(\binom{d-1}{\ell} - \binom{d-1}{\ell-1} + 1\right)\sum_{j=1}^{d} x_j$$

which is an identity in X.

Theorem 1. Let X be a $\mathbb{K}$-vector space.

1. Consider $a_{i,j} \in \mathbb{K}$, $r_i \in \mathbb{K}^*$, $i \in I$, $j \in J$, I and J finite. Then

$$\sum_{i\in I} r_i \sum_{j\in J} \frac{1}{r_i} a_{i,j} x_j = 0, \qquad \forall x_j \in X,\ \forall j \in J,$$

if and only if

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2. If Y is also a $\mathbb{K}$-vector space, $r_i \in \mathbb{Q}^*$ and $\sum_{i\in I} a_{i,j} = 0$ for all $j \in J$, then any additive $f : X \to Y$ satisfies

$$\sum_{i\in I} r_i f\left(\sum_{j\in J} \frac{1}{r_i} a_{i,j} x_j\right) = 0.$$

Value of $f(0)$

Substituting $x_1 = \cdots = x_d = 0$ in (1) we obtain

$$\left[r \left(1 + \binom{d}{\ell} \right) - \left(\binom{d-1}{\ell} - \binom{d-1}{\ell-1} + 1 \right) d \right] f(0) = 0.$$

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$$\left[r\left(1+\binom{d}{\ell}\right)-\left(\binom{d-1}{\ell}-\binom{d-1}{\ell-1}+1\right)d\right]f(0)=0.$$

Theorem 2. Let f be a solution of (1).

1. If $r \neq \frac{\left(\binom{d-1}{\ell}-\binom{d-1}{\ell-1}+1\right)d}{\left(1+\binom{d}{\ell}\right)}$, then $f(0)=0$.
2. If $r = \frac{\left(\binom{d-1}{\ell}-\binom{d-1}{\ell-1}+1\right)d}{\left(1+\binom{d}{\ell}\right)}$, then $f+y$, $y \in Y$, is a solution of (1).

Solutions with $f(0)=0$

Substitute $x_1 = x$, $x_2 = y$, and $x_3 = \cdots = x_d = 0$ in (1), then we obtain

$$\begin{aligned}&\left(1+\binom{d-2}{\ell}\right) rf(\frac{1}{r}(x+y)) + \binom{d-2}{\ell-1} rf(\frac{1}{r}(x-y))\\&+\binom{d-2}{\ell-1} rf(\frac{1}{r}(-x+y)) + \binom{d-2}{\ell-2} rf(\frac{1}{r}(-x-y))\\&=\left(\binom{d-1}{\ell}-\binom{d-1}{\ell-1}+1\right)(f(x)+f(y)).\end{aligned} \tag{2}$$

Thus

$$\begin{aligned}&f(x)+c_1 f(\frac{1}{r}(x+y))+c_2 f(\frac{1}{r}(x-y))+c_3 f(\frac{1}{r}(-x+y))\\&+c_4 f(\frac{1}{r}(-x-y))+c_5 f(y)=0.\end{aligned}$$

Theorem by L.Székelyhidi

Theorem [4, Theorem 9.5, p. 73] Let X, Y be $\mathbb{K}$-vector spaces, $\varphi_i, \psi_i\colon X \to X$ homomorphisms such that $\varphi_i(X) \subseteq \psi_i(X)$ for $1 \leq i \leq n+1$. If $f, f_1, \ldots, f_{n+1}\colon X \to Y$ satisfy

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0, \qquad x, y \in X,$$

then f is a generalized polynomial of degree at most n.

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Corollary 1. The solutions of (2) are generalized polynomials of degree at most 4.

The solutions f of (1) with $f(0) = 0$ are generalized polynomials of degree at most 4.

Generalized polynomials

A mapping $f : X \to Y$ is called a generalized polynomial homogeneous of degree k ($k \in \mathbb{N}_0$), if there exists some symmetric, k-additive mapping $F : X^k \to Y$ so that $f(x) = F(x, \dots, x)$, $x \in X$. (This F is additive or $\mathbb{Q}$-linear in each component.) The set of all generalized polynomials homogeneous of degree k is indicated by $\mathscr{P}_k^{\hom}(X, Y)$.
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A mapping $f: X \to Y$ is called a generalized polynomial of degree at most n ($n \in \mathbb{N}_0$) if $f = f_0 + f_1 + \dots + f_n$, where $f_k \in \mathscr{P}_k^{\mathrm{hom}}(X, Y)$, $0 \le k \le n$. The set of all generalized polynomials of degree at most n is the direct sum $\bigoplus_{k=0}^n \mathscr{P}_k^{\mathrm{hom}}(X, Y)$.

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Exactly the same method was described by A. Gilányi in his talk “Computer assisted methods for functional equations” during the last ISFE. Cf. also [2].

Solutions of (2)

$f = f_0 + f_1 + f_2 + f_3 + f_4 \in \bigoplus_{k=0}^{4} \mathscr{P}_k^{\mathrm{hom}}(X,Y)$ is a solution of (2) if and only if f_k is a solution of (2) for $0 \le k \le 4$.

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From [1] we know that each f_1 is a solution of (1) and no $f_3 \neq 0$ is a solution of (1).

We have to check whether there exist generalized polynomials homogeneous of degree 2 or 4 solving (2).

Quadratic solutions of (2)

If $f(x) = f_2(x) = F(x,x)$ is a quadratic function, then (2) can be replaced by

$$\left[\frac{1}{r}\left(1+\binom{d}{\ell}\right)-\left(\binom{d-1}{\ell}-\binom{d-1}{\ell-1}+1\right)\right](f(x)+f(y))$$
$$+\frac{2}{r}\left[1+\binom{d-2}{\ell}-2\binom{d-2}{\ell-1}+\binom{d-2}{\ell-2}\right]F(x,y)=0.$$

There exists a non-zero solution f if and only if both square brackets are equal to 0.

$1+\binom{d-2}{\ell}-2\binom{d-2}{\ell-1}+\binom{d-2}{\ell-2}=0$ if and only if $1+\binom{d}{\ell}=4\binom{d-2}{\ell-1}$. Under the assumption $2\leq\ell<d/2$ this is equivalent to $d=6$ and $\ell=2$.

If $d=6$ and $l=2$ and

$$r=\frac{\binom{d-1}{\ell}-\binom{d-1}{\ell-1}+1}{1+\binom{d}{\ell}}=\frac{8}{3},$$

both square brackets are 0.

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both square brackets are 0.

Theorem 3. If $r=8/3$, $d=6$, and $\ell=2$ any quadratic function is a solution of (1).

Otherwise there are no non-zero quadratic solutions of (1).

Solutions of (2) homogeneous of degree 4

If $f(x) = f_4(x) = F(x,x,x,x)$ is a generalized polynomial of degree 4, then (2) can be replaced by

$$\left[\frac{1}{r^3}\left(1+\binom{d}{\ell}\right) - \left(\binom{d-1}{\ell} - \binom{d-1}{\ell-1} + 1\right)\right](f(x)+f(y))$$
$$+\frac{6}{r^3}\left(1+\binom{d}{\ell}\right)F(x,x,y,y)$$
$$+\frac{4}{r^3}\left[1+\binom{d-2}{\ell} - 2\binom{d-2}{\ell-1} + \binom{d-2}{\ell-2}\right](F(x,x,x,y)+F(x,y,y,y))$$
$$=0.$$

Since the coefficient of $F(x,x,y,y)$ is different from 0, $f=0$.

Solutions of (2) homogeneous of degree 4

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$$+\frac{6}{r^3}\left(1+\binom{d}{\ell}\right)F(x,x,y,y)$$

$$+\frac{4}{r^3}\left[1+\binom{d-2}{\ell} - 2\binom{d-2}{\ell-1} + \binom{d-2}{\ell-2}\right](F(x,x,x,y)+F(x,y,y,y))$$

$$= 0.$$

Since the coefficient of $F(x,x,y,y)$ is different from 0, $f = 0$.

Theorem 4. There are no nonzero generalized polynomials of degree 4 solving (1).

Rational r

Corollary 2.

The solutions of (1) are of the form
$f = f_0 + f_1 + f_2 + f_3 + f_4 \in \bigoplus_{k=0}^{4} \mathscr{P}_k^{\mathrm{hom}}(X,Y)$.

If $r = \frac{\left(\binom{d-1}{\ell} - \binom{d-1}{\ell-1} + 1\right)d}{\left(1 + \binom{d}{\ell}\right)}$, then any f_0 is a solution of (1), otherwise $f_0 = 0$.

Any f_1 is a solution of (1).

If $d = 6$, $\ell = 2$, and $r = 8/3$, then any f_2 is a solution of (1), otherwise $f_2 = 0$.

There are no non-zero solutions of (1) which are generalized polynomials of degree 3 or 4.

Non-rational r

Again by Székelyhidi's Theorem the solutions of (1) are of the form $f = f_0 + f_1 + f_2 + f_3 + f_4 \in \bigoplus_{k=0}^{4} \mathscr{P}_k^{\mathrm{hom}}(X,Y)$.

Theorem 5.

f_0, f_3, f_4 are solutions of (1) if and only if $f_0 = 0$, $f_3 = 0$, $f_4 = 0$, respectively.

f_1 is a solution of (1) if and only if $rf_1(x) = f_1(rx)$, $x \in X$.

If $d = 6$ and $\ell = 2$, any f_2 satisfying $f_2(rx) = \frac{8}{3} r f_2(x)$, $x \in X$, is a solution. Otherwise $f_2 = 0$ is the only quadratic solution of (1).

Additive functions with $f(rx) = rf(x)$

Two elements $a, A \in \mathbb{K}$ are called conjugate if either, they both have the same minimal polynomial over $\mathbb{Q}$, or they both are transcendental over $\mathbb{Q}$.

A special case of [3, Theorem 4.12.1] is

Theorem. If a and A are conjugate, then there exists an additive $\alpha\colon \mathbb{K} \to \mathbb{K}$, $\alpha \neq 0$, so that $\alpha(ax) = A\alpha(x)$, $x \in \mathbb{K}$.

Corollary 3. There exist non-zero additive solutions f of (1) with $f(rx) = rf(x)$, $x \in X$, when r is not rational.

Quadratic functions with $f(rx) = crf(x)$, $c \in \mathbb{Q}^*$

Corollary 4. If r is transcendental over $\mathbb{Q}$, then there exist quadratic functions $f\colon X \to Y$, $f \neq 0$, so that $f(rx) = crf(x)$, $x \in X$.

Proof. In $\mathbb{R}$: If $cr > 0$, then both $a = r$ and $A = \sqrt{cr}$ are transcendental over $\mathbb{Q}$, so there exists an additive $\alpha\colon \mathbb{Q}(a) \to \mathbb{Q}(A)$, $\alpha \neq 0$, so that $\alpha(ax) = A\alpha(x)$. Then $f = \alpha^2$ is quadratic, non-zero, and $f(rx) = \alpha(ax)^2 = (\sqrt{cr}\alpha(x))^2 = crf(x)$, $x \in \mathbb{Q}(a)$.

If $cr < 0$, let $a = r$ and $A = \sqrt{-cr}$, then there exist non-zero additive functions $\alpha, \beta\colon \mathbb{Q}(a) \to \mathbb{Q}(A)$, so that $\alpha(ax) = A\alpha(x)$ and $\beta(ax) = -A\beta(x)$. Then $f = \alpha\beta$ is quadratic, non-zero, and $f(rx) = \alpha(ax)\beta(ax) = -A^2\alpha(x)\beta(x) = crf(x)$, $x \in \mathbb{Q}(a)$.

In $\mathbb{C}$: Let A be a square root of cr. Construct a, α and f as in the first case.

For $r \notin \mathbb{Q}$ algebraic over $\mathbb{Q}$ we have only partial results.

If $r \in \mathbb{R}$ satisfies $r^n = a_0 \in \mathbb{Q}$ for some integer $n \geq 2$, then there are no non-zero quadratic functions satisfying $f(rx) = crf(x)$.

If the minimal polynomial of $r \in \mathbb{R}$ over $\mathbb{Q}$ has degree 2, there are no non-zero quadratic functions satisfying $f(rx) = crf(x)$.

For $r = \frac{c}{2}(-1 + \mathrm{i}\sqrt{3}) \in \mathbb{C}$, there exist non-zero quadratic functions $f\colon \mathbb{Q}(r) \to \mathbb{Q}(r)$ so that $f(rx) = crf(x)$.

In general, let $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be the minimal polynomial of r over $\mathbb{Q}$. We try to define a symmetric, 2-additive mapping $F\colon \mathbb{Q}(r)^2 \to \mathbb{Q}(r)$ by determining its values on the basis $\{(r^i, r^j) \mid 0 \le i, j < n\}$ of $\mathbb{Q}(r)^2$:

$f(r^i) = F(r^i, r^i) = c^i r^i$, $0 \le i < n$, and
$F(r^i, r^j) = c^i r^i F(1, r^{j-i})$ for $i < j < n$.

Then the values $F(1, r^j)$ for $1 \le j < n$ must be determined so that the following system of n equations is satisfied:

$$f(r^n) = c^n r^n = \sum_{i=0}^{n-1} a_i^2 c^i r^i + 2\sum_{i<j} a_i a_j c^i r^i F(1, r^{j-i})$$

and

$$F(r^i, r^n) = c^i r^i F(1, r^{n-i}) = \sum_{j=0}^{i-1} -a_j c^j r^j F(1, r^{i-j}) + \sum_{j=i}^{n-1} -a_j c^i r^i F(1, r^{j-i})$$

for $1 \le i \le n-1$.

For $n = 2$ these are only two equations,

$$c^2 r^2 = a_0^2 + a_1^2 cr + 2a_0 a_1 F(1,r)$$
$$crF(1,r) = -a_0 F(1,r) - a_1 cr.$$

From the latter we obtain

$$F(1,r) = \frac{-a_1 cr}{cr + a_0}.$$

Substituting this into the first equation we obtain conditions on the coefficients a_i of the minimal polynomial, namely $a_1^2 = a_0$ and $a_1 = c$, thus the minimal polynomial of r is

$$x^2 + cx + c^2$$

which has the two complex roots

$$\frac{c}{2}(-1 \pm \mathrm{i}\sqrt{3}).$$

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