Methods and Results in Musical Enumeration Theory

Harald Fripertinger
Karl-Franzens-Universität Graz

Third International Seminar on Mathematical Music Theory and Music Informatics
University Zürich, October 24 – October 26, 2002

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— basic facts about classification under group actions,
— some applications in music theory,
— some results on the enumeration of canons.

Discrete Structures

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Examples: graphs, necklaces, designs, codes, matroids, switching functions, molecules in chemistry, spin-configurations in physics, objects of local music theory.

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Often the elements of a discrete structure are themselves classes of objects which are considered to be equivalent. These classes collect all those elements which are not essentially different. (Relabellings of labelled structures, or otherwise naturally motivated equivalence relations.)

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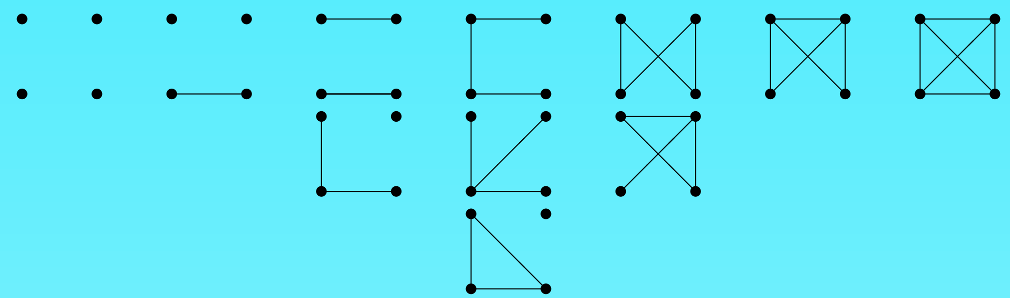

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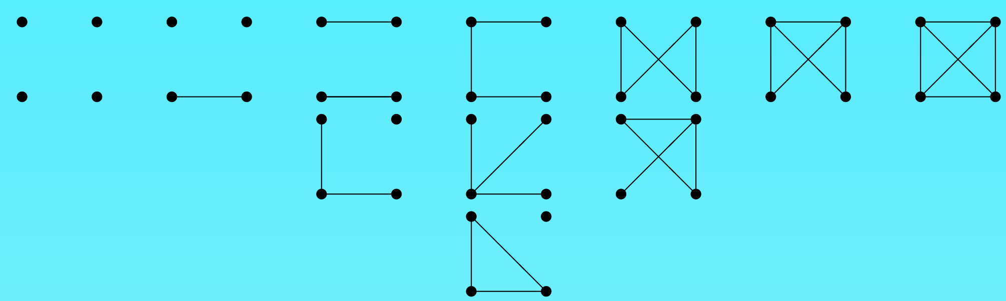

The standard tool for the classification of discrete structures are group actions.

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$$*\colon G \times X \longrightarrow X \qquad *(g,x) \mapsto g * x$$

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If G and X are finite sets, then we speak of a ***finite group action***.

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Theorem. *The equivalence classes of any equivalence relation can be represented as orbits under a suitable group action.*

Stabilizers and Fixed Points

Let ${}_GX$ be a group action. For each $x \in X$ the ***stabilizer*** G_x of x is the set of all group elements which do not change x, in other words

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Finally, the ***set of all fixed points*** of $g \in G$ is denoted by

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— Then the direct product $H \times G$ acts on Y^X by

$$(H \times G) \times Y^X \to Y^X, \qquad ((h,g), f) \mapsto \bar{h} \circ f \circ \bar{g}^{-1}.$$

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The set of all orbits $n\mathbb{Z}\backslash\backslash\mathbb{Z}$ will be indicated as Z_n. It consists of exactly n objects. With the naturally motivated addition and multiplication, $(Z_n, +, \cdot)$ is a commutative ring with 1, the ***residue class ring modulo*** n.

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— $\mathrm{Aff}_1(Z_n) := \{\pi_{a,b} \mid a \in Z_n^*,\ b \in Z_n\}$ is the group of all ***affine mappings*** from Z_n to Z_n, with $\pi_{a,b}(i) := ai + b$.

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$G \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	6	19	43	66	80	66	43	19	6	1	1

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C_{12}	1	6	19	43	66	80	66	43	19	6	1	1
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Extensions:

Intervals, Chords and Rhythms

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Extensions:

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— Determination of the interval structure of non-equivalent chords.

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When speaking about motives we first have to find all possible combinations of beats in an m-bar Z_m and pitch-classes in an n-scale Z_n. The set of all these combinations is the product $Z_m \times Z_n$.

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For $n = m = 12$ the numbers of k-motives for small values of k are the coefficients of z^k in

$$1 + z + 5z^2 + 26z^3 + 216z^4 + 2024z^5 + 27806z^6 + 417209z^7 + 6345735z^8 + 90590713z^9 + 1190322956z^{10} + \ldots.$$

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For $n \geq 3$ a ***tone-row*** in Z_n is a bijective mapping $f\colon Z_n \to Z_n$ where $f(i)$ is the tone which occurs in i-th position in the tone-row.

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Usually two tone-rows f_1, f_2 are considered to be similar if f_1 can be constructed by transposing, inversion and retrograde inversion R of f_2. Thus the similarity classes of tone-rows are the $D_n \times \langle R \rangle$ orbits on the set of all bijections on Z_n.

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$$\begin{cases} \frac{1}{4}\Big((n-1)! + (n-1)!!\Big) & \text{if } n \equiv 1 \bmod 2 \\ \frac{1}{4}\Big((n-1)! + (n-2)!!(\frac{n}{2}+1)\Big) & \text{if } n \equiv 0 \bmod 2. \end{cases}$$

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Especially there are 9985920 classes of tone-rows in 12-tone music.

Mosaics

A ***partition*** π of Z_n is a collection of subsets of Z_n, such that the empty set is not an element of π and such that for each $i \in Z_n$ there is exactly one $P \in \pi$ with $i \in P$.

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$$G \times \Pi_n \to \Pi_n, \qquad (g, \pi) \mapsto g\pi := \{gP \mid P \in \pi\},$$

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Number of G-mosaics consisting of k blocks.

$G \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12
C_{12}	1	179	7254	51075	115100	110462	52376	13299	1873	147	6	1
D_{12}	1	121	3838	26148	58400	56079	26696	6907	1014	96	6	1
$\mathrm{Aff}_1(Z_{12})$	1	87	2155	13730	30121	28867	13835	3667	571	63	5	1

In conclusion there are 351773 C_{12}-mosaics, 179307 D_{12}-mosaics and 93103 $\mathrm{Aff}_1(Z_{12})$-mosaics in twelve tone music.

Canons

A ***canon*** is a subset $K \subseteq Z_n$ together with a covering of K by pairwise different subsets $V_i \neq \emptyset$ for $1 \leq i \leq t$, the voices, where $t \geq 1$ is the number of voices of K, in other words

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such that for all $i, j \in \{1, \ldots, t\}$

1. the set V_i can be obtained from V_j by a translation of Z_n,
2. there is only the identity translation which maps V_i to V_i,
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We prefer to write a canon K as a set of its subsets V_i.

Two canons $K = \{V_1, \ldots, V_t\}$ and $L = \{W_1, \ldots, W_s\}$ are called ***isomorphic*** if $s = t$ and if there exists a translation T of Z_n and a permutation π in the symmetric group S_t such that $T(V_i) = W_{\pi(i)}$ for $1 \leq i \leq t$. Then obviously $T(K) = L$.

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Theorem. *The number of isomorphism classes of canons in Z_n is*

$$K_n := \sum_{d|n} \mu(d)\lambda(n/d)\alpha(n/d),$$

where μ is the Moebius function, $\lambda(1) = 1$,

$$\lambda(r) = \frac{1}{r}\sum_{s|r} \mu(s) 2^{r/s} \text{ for } r > 1,$$

$$\alpha(r) = \frac{1}{r}\sum_{s|r} \varphi(s) 2^{r/s} - 1 \text{ for } r \geq 1,$$

where φ is the Euler totient function.

Rhythmic Tiling Canons

A canon is a ***rhythmic tiling canon*** if
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Classification of rhythmic tiling canons by computing complete lists of representatives. (T_n or K_n are the numbers of rhythmic tiling canons or canons respectively.)

n	T_n	K_n
2	1	1
3	1	5
4	2	13
5	1	41
6	3	110
7	1	341
8	6	1035
9	4	3298

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10	6	10550
11	1	34781
12	23	117455
13	1	397529
14	13	1.370798
15	25	4.780715
16	49	16788150
17	1	59451809

Regular Complementary Canons of Maximal Category

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Z_n is not a Hajós group if and only if n can be expressed in the form $p_1 p_2 n_1 n_2 n_3$ with p_1, p_2 primes, $n_i \geq 2$ for $1 \leq i \leq 3$, and $\gcd(n_1 p_1, n_2 p_2) = 1$.

Vuza's Algorithm

If Z_n is not a Hajós group, Vuza presents an algorithm for constructing two aperiodic subsets L and A of Z_n, such that $|L| = n_1 n_2$, $|A| = p_1 p_2 n_3$, and $L + A = Z_n$.

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Moreover, it is important to mention that there is some freedom for constructing these two sets, and each of these two sets can be constructed independently from the other one.

He also proves that when L and A describe an RCMC-canon, then also (kL, A), (kL, kA) have this property for all $k \in Z_n^*$.

Enumeration of Vuza Canons

A ***Vuza canon*** is a regular complementary canon of maximal category which can be constructed by his algorithm.

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Enumeration of non-equivalent Vuza canons by construction:

p_1	p_2	n_1	n_2	n_3	$\#L$	$\#A$	$\#$
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2	3	4	5	2	34	120	4080

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2	5	2	3	2	9	6	54

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Among the 216 RCMC-canons of length $2 \cdot 72 = 144$ with $|L| = 12$ we did not find a canon which was constructed in this way from the 18 canons of length 72.

From L equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]

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0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]

and A equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,1,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,1]

From L equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]

and A equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,1,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,1]

we get the canon with L' equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1]

From L equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]

and A equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,1,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,1]

we get the canon with L' equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1]

and A' equal to

[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,1,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,1]

From L equal to
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1]
and A equal to
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,1,0,0,0,0,1,0,0,0,0,0,1,0,
0,0,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,1,0,0,1,1]
we get the canon with L' equal to
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,
0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1]
and A' equal to
[0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,1,0,0,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,
0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,1]

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